In lecture 1 we proposed that

* semantic domains are (pointed) posets;
* computable functions are monotonic.

In this lecture we consider the semantics of recursively defined programs, and find that these propositions must be strengthened to

* semantic domains are cpo's (complete posets);
* computable functions are continuous.
**RECURSION AND FIXPOINTS**

**Observation:** every recursively defined function $f$ can be expressed in the form

$$f = \Phi(f)$$

where $\Phi$ is a non-recursive (higher-order) function.

$$f = \lambda x. ... f ... f ...$$

define $\Phi(f) = E$

$$f = \Phi(f)$$

**Defn:** Let $\Phi : A \to A$. Then $x \in A$ is called a **fixed-point** (or fixpoint) of $\Phi$ if $\Phi(x) = x$.

Hence the problem of giving recursively defined programs a semantics can be cast as the problem of finding fixpoints of non-recursive functions.
SOME PROBLEMS...

* not all monotonic fins on posets have fixpoints;
* some have many fixpoints; which, do we choose?

One possible solution is predicted by the well-known Knaster-Tarski fixpoint theorem:

Every monotonic $f : A \rightarrow A$ on a complete lattice $A$ (a poset with lubs and glbs of all subsets) has a complete lattice of fixpoints.

But intuitively, recursive programs are executed by "unfolding". The KT theorem ensures least fixpoints exist, but does not provide a way to compute them using an "unfolding-like" process.

Solution:

* complete lattices are too strong: cpos are sufficient;
* monotonicity is too weak: continuity is necessary.
A FIRST STEP TO CPO'S: CHAINS

A subset $Y \subseteq X$ of a poset $X$ is a chain iff

$\forall x, y \in Y. \ x \leq y \text{ or } y \leq x$

Example: the naturals $\mathbb{N}$ form a chain under $\leq$

\[0 \quad 1 \quad 2 \quad 3 \quad \cdots\]

note: chains can be infinite in either (or both) directions

In domain theory, we usually only consider chains with a countable number of elements, i.e. $\omega$-chains.

"omega"
GENERALISING CHAINS: DIRECTED SETS

A non-empty subset \( Y \subseteq X \) of a poset \( X \) is directed iff

\[
\forall x, y \in Y. \exists z \in Y. \ x \leq z \text{ and } y \leq z.
\]

Intuition: directed sets are "going somewhere". Given two elements we can always find a bigger one.

Example: any (non-empty) chain is directed.

Example: the powerset \( \mathcal{P}(X) \) of any \( X \) is directed under \( \subseteq \).
AN EQUIVALENT DEFINITION

A subset \( Y \subseteq X \) of a poset \( X \) is consistent iff

\[ \exists x \in X . \quad \forall y \in Y . \quad y \leq x. \]

Such an \( x \) is called an "upper bound" for \( Y \).

Hence a (non-empty) set is directed iff every pair of values has an upper bound in the set.

Lemma: a subset \( Y \subseteq X \) of a poset \( X \) is directed iff every finite subset \( Y' \subseteq Y \) has an upper bound in \( Y \).

(this definition is sometimes more useful in proofs.)
Proof ($\Leftarrow$). Trivial.

(But note that requiring an upper bound of $\emptyset \subseteq Y$ means that $Y$ must have a $\bot$. Hence this defn. implies directed sets are non-empty.)

Proof ($\Rightarrow$). Let $Y = \{a_0, \ldots, a_n\}$ be finite/non-empty. We define a function $f(x)$ by induction on $x \in 0 \ldots n$:

$$f(x) = \begin{cases} a_0 & \text{if } x = 0 \\ \text{an upper bound of } f(x-1) \text{ and } a_x & \text{otherwise} \\ \text{(which exists since } Y \text{ is directed.)} \end{cases}$$

A simple inductive proof shows that $f(n)$ is an upper bound for $Y$. Since $n$ is finite, $f(n)$ terminates.  

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Corollary: A finite set $Y$ is directed iff it has a top $\top$.

($\top$ is the required upper bound for $Y$ itself.)
LEAST UPPER BOUND

Let $X \subseteq Y$ be a subset of a poset $Y$. An element $y \in Y$ is a "least upper bound" (lub) for $X$ iff

1. $X \subseteq y$ (i.e. $\forall x \in X. x \leq y$), "upper bound"
2. $\forall y' \in Y. (X \subseteq y' \Rightarrow y \leq y')$, "least"

Fact: the lub is unique if it exists, and written $\bigcup X$.

(we normally write $x \cup y$ rather than $\bigcup \{x, y\}$)

Intuition: "$\bigcup X$ combines the information content of all the elements in $X$, but doesn't add extra info."

Example: (in $\mathbb{P}_\perp \times \mathbb{P}_\perp$)

$(\perp, \text{false}) \cup (\text{true}, \perp) = (\text{true}, \text{false})$

$(\perp, \text{false}) \cup (\perp, \perp) = (\perp, \text{false})$

$(\perp, \text{false}) \cup (\text{true}, \text{true})$ does not exist.
**Familiar Examples of LUBs.**

* For the poset \( \text{false} \sqsubseteq \text{true} \) (18 ordered by \( \Rightarrow \)), \( x \lor y = x \lor y \).

\[
\begin{align*}
\text{false} \lor \text{false} &= \text{false} \\
\text{false} \lor \text{true} &= \text{true} \\
\text{true} \lor \text{false} &= \text{true} \\
\text{true} \lor \text{true} &= \text{true}
\end{align*}
\]

* For any set \( X \), the powerset \( \mathcal{P}(X) \) is a poset ordered by \( \subseteq \), with lubs given by union \( \cup \).

![Diagram of a partial order](image)
SOME USEFUL FACTS ABOUT LUBS

* a poset has a bottom iff \(\bigcup\emptyset\) exists. \((\bigcup\emptyset = \bot)\)

* by defn., a poset \(A\) has all binary lubs iff there
  is an operator \(\sqcup: A \times A \to A\) such that

  \[x \sqcup y \text{ and } y \sqcup x \text{ are } \text{ "upper bound" }\]

  \((x \sqsubseteq z \text{ and } y \sqsubseteq z) \implies x \sqcup y \sqsubseteq z. \text{ "least" }\)

An equiv. defn. (very useful in proofs) is

\((x \sqsubseteq z \text{ and } y \sqsubseteq z) \iff x \sqcup y \sqsubseteq z.\)

* if all binary lubs exist, then \(\sqcup\) satisfies

  \[x \sqcup x = x\]  \hspace{1cm} \text{idempotent}

  \[x \sqcup y = y \sqcup x\]  \hspace{1cm} \text{symmetric}

  \[x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z\]  \hspace{1cm} \text{associative}

  \[x \sqsubseteq y \iff x \sqcup y = y\]  \hspace{1cm} \text{extremal}

(Asside: these 4 properties are in fact an equiv. axiomatisation for posets with binary lubs!)
**COMPLETE PARTIAL ORDERS**

A cpo (complete partial order) is a poset $A$ s.t

1. $A$ has a bottom $\bot$
2. $\bigcup X$ exists for all directed $X \subseteq A$

"painted"  "directed complete"

**Intuition**: directed sets are "going somewhere".

![Diagram](image)

In a cpo the directed sets "get there" in that $\bigcup X$ exists (although it need not be in $X$ itself.)

**Note**: directed completeness $\equiv$ chain completeness

(this is a non-trivial result; note that if we restrict to $\omega$-chains, we only get an implication $\Rightarrow$)

* chain completeness is "simpler" and technically sufficient
* BUT directed completeness results in a simpler treatment of "finiteness" properties in Lecture 4.
Example: any finite (pointed) poset is a cpo.

Any finite directed set has a top (see slide 2.6), which is by definition the lub for the set.

Example: the set $\mathbb{N}$ of naturals is not a cpo.

There is a directed subset ($\mathbb{N}$ itself) with no lub. (But adding a top makes $\mathbb{N}$ a cpo.)

\[ \cdots \quad \begin{array}{c}
\vdots \\
2 \\
1 \\
0 \\
\hline
\end{array} \quad \begin{array}{c}
\vdots \\
\uparrow \\
\uparrow \\
\downarrow \\
\end{array} \quad \begin{array}{c}
\vdots \\
2 \\
1 \\
0 \\
\hline
\end{array} \quad \begin{array}{c}
\vdots \\
\uparrow \\
\uparrow \\
\downarrow \\
\end{array} \]

Example: the set $\mathbb{Q}$ of rationals is not a cpo.

Not only does $\mathbb{Q}$ lack a lub for $\mathbb{Q}$ itself, but it lacks for example $\sqrt{2}$, which can be expressed as the lub of an infinite sequence of rational approximations. (But $[0,1] \subseteq \mathbb{R}$ is a cpo.)

thesis: semantic domains are cpo's.
**CONTINUOUS FUNCTIONS**

A function \( f : A \to B \) between cpo's \( A \) and \( B \) is continuous iff for all directed \( X \subseteq A \),

1. \( \text{L}X(fX) \) exists, (where \( fX = \{ f(x) \mid x \in X \} \))
2. \( f(\text{L}X) = \text{L}(fX) \). "preserves lubs"

**Intuition:** "nothing is suddenly invented at infinity"

**Fact:** continuity \( \Rightarrow \) monotonicity

Let \( x \leq y \). Then \( X = \{ x, y \} \) is directed. Expanding (2) gives \( f_{xy} = f_x \circ f_y \), which is equiv. to \( fX \leq fY \). \( \square \)
Fact: monotonicity \(\Rightarrow\) condition 1.

It suffices (since \(B\) is a cpo) to show that 
\(f_X\) is directed for all directed \(X\).

Let \(f_X, f_Y \in f_X\) (where \(x, y \in X\)). Since \(X\) is directed, there exists \(z \in X\) s.t. \(x \leq z\) and \(y \geq z\).

Now since \(f\) is monotonic, \(f_z \leq f_X\) is such that 
\(f_z \leq f_z\) and \(f_y \leq f_z\). Hence \(f_X\) is directed. \(\square\)

Combining the two facts, \(f: A \to B\) is continuous iff

1. \(f\) is monotonic,
2. \(f(LX) = L(fX)\) for all directed \(X \leq A\).

(this defn. of continuity is normally more useful in practice.)
Example: (continuity is only interesting for infinite cpos)

If \( f: A \to B \) is monotonic (where \( A, B \) are cpos) and \( A \) is finite, the \( f \) is continuous.

Example: (monotonicity \( \not\Rightarrow \) continuity.)

The function \( f: \mathbb{N}^\omega \to 2 \) defined by

\[
  f(x) = \begin{cases} 
    \bot & \text{if } x \in \mathbb{N} \\
    T & \text{otherwise}
  \end{cases}
\]

is monotonic, but not continuous.

(take \( X = \mathbb{N}^\omega \), then \( f(LX) = fT = T \),
but \( L(fX) = L \varnothing \bot 3 = \bot \).

* * *

thesis: computable functions are continuous
RECURSIVE PROGRAMS

Recall (slide 2.1) : "the problem of giving recursively defined programs a semantics can be cast as the problem of finding fixpoints of non-recursive functions."

Theorem : (the CPO Fixpoint theorem)

Every continuous function \( f : A \rightarrow A \) on a cpo \( A \) has a least fixpoint \( \text{fix}(f) \), which can be computed as the limit of the \( \omega \)-chain \( \bot \in f(\bot) \in f(f(\bot)) \ldots \), i.e

\[
\text{fix}(f) = \bigsqcup \{ f^n(\bot) \mid n \in \mathbb{N} \}.
\]

Proof (in three parts)

1. \( \bigsqcup \{ f^n(\bot) \mid n \in \mathbb{N} \} \) exists.

A simple inductive proof shows \( \forall n \in \mathbb{N} : f^n(\bot) \in f^{n+1}(\bot) \), i.e that \( \{ f^n(\bot) \mid n \in \mathbb{N} \} \) is a chain, and hence directed.
2. $\text{fix}(S)$ is a fixpoint of $S$.

\[ S(\text{fix}(S)) = \bigcup \{ f^n(1) \mid n \in \mathbb{N} \} \]  
\[ = \bigcup \{ f^{n+1}(1) \mid n \in \mathbb{N} \} \]  
\[ = \bigcup \{ f^n(1) \mid n \in \mathbb{N} \} \]  
\[ = \text{fix}(S) \]  

[defn fix(S)]

(3) $\text{fix}(S)$ is the least fixpoint of $S$.

Let $x \in X$ be a fixpoint of $S$, i.e. $S(x) = x$. A simple inductive proof shows that $\forall n \in \mathbb{N}, S^n(1) \leq x$, i.e. that $x$ is an upper bound for $\{ S^n(1) \mid n \in \mathbb{N} \}$.

Now since by defn. $\text{LUB}(X)$ is the least upper bound of a set $X$, then $\text{fix}(S) = \bigcup \{ S^n(1) \mid n \in \mathbb{N} \} \leq x$.  

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EXAMPLES

* what is \( \text{fix}(\text{id}) \), where \( \text{id} = \) ?

\[
\text{fix}(\text{id}) = \bigcup \{ \text{id}^n(\perp) \mid n \in \mathbb{N} \}
\]
\[
= \bigcup \{ \perp, \text{id}(\perp), \text{id}(\text{id}(\perp)), \ldots \}
\]
\[
= \bigcup \{ \perp, \perp, \perp, \perp, \ldots \}
\]
\[
= \perp.
\]

* what is \( \text{fix}(K_F) \), where \( K_F = \) ?

\[
\text{fix}(K_F) = \bigcup \{ K_F^n(\perp) \mid n \in \mathbb{N} \}
\]
\[
= \bigcup \{ \perp, K_F(\perp), K_F(K_F(\perp)), \ldots \}
\]
\[
= \bigcup \{ \perp, \perp, \perp, \perp, \ldots \}
\]
\[
= \perp.
\]

* what is \( \text{fix}(s) \), where \( s = 1 : x \) ? \( (s \in \mathcal{N}_L \Rightarrow (\mathcal{N}_L)) \)

\[
\text{fix}(s) = \bigcup \{ s^n(\perp) \mid n \in \mathbb{N} \}
\]
\[
= \bigcup \{ \perp, 1 : \perp, 1 : 1 : \perp, \ldots \}
\]
\[
= 1 : 1 : 1 : \ldots \quad (\text{an infinite list of ones}.)
\]

Note: \( \text{fix}(s) \) is the semantics of the rec.defn. \( x = 1 : x. \)
**EXAMPLE:** what is \( \text{fix}(\Phi) \), where the higher-order function \( \Phi \in (\mathbb{N}_1 \to \mathbb{N}_1) \to (\mathbb{N}_1 \to \mathbb{N}_1) \) is defined by

\[
\Phi(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \times \Phi(n-1) & \text{otherwise} \end{cases}
\]

\[
\Phi^0(1) = \lambda n. \bot \quad (\bot \in \mathbb{N}_1 \to \mathbb{N}_1)
\]

\[
\Phi^1(1) = \lambda n. \begin{cases} 1 & \text{if } n = 0 \\ \bot & \text{otherwise} \end{cases}
\]

\[
\Phi^2(1) = \lambda n. \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ n \times (\Phi(n-1) = 0 \text{ then } 1 \text{ else } \bot) & \text{otherwise} \end{cases}
\]

\[
\Phi^3(1) = \lambda n. \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ n \times (\Phi(n-1) = 0 \text{ then } 1 \text{ else } \bot) & \text{otherwise} \end{cases}
\]

\[
\vdots
\]

\[
\Phi^m(1) = \lambda n. \begin{cases} 1 \times 2 \times \ldots \times n & \text{if } n \leq (m-1) \\ \bot & \text{otherwise} \end{cases}
\]

The \( m \)th approximation \( \Phi^m(1) \) to \( \text{fix}(\Phi) \) is the function that gives \( 1 \times \cdots \times n \) for all \( n \leq (m-1) \), and diverges on all other arguments. Hence the limit \( \text{fix}(\Phi) \) is the factorial function on \( \mathbb{N}_1 \).
EXERCISES

1. Give an example of a poset $A$ and a monotonic function $f : A \rightarrow A$ st. $f$ doesn't have a fixpoint. (Hint: $A$ must be infinite.)

2. What is the lub operator on subsets $X \subseteq IN$ of the poset $(IN, \leq)$ more commonly known as?

3. Show that if $f : A \rightarrow B$ is monotone ($A$ and $B$ are cpo's) and $A$ is finite, then $f$ is continuous. (Hint: Finite directed sets contain their lub.)

4a. Show that if $A, B$ are posets and $X \subseteq A \times B$ is directed, then the subsets $\Pi_0(X) \subseteq A$ and $\Pi_1(X) \subseteq B$ (defined below) are also directed.

$$\Pi_0(X) = \{ a \in A \mid \exists b \in B. (a, b) \in X \}$$

$$\Pi_1(X) = \{ b \in B \mid \exists a \in A. (a, b) \in X \}$$
4b. Give an example of a set $X \subseteq 2 \times 2$ such that $\pi_0(X)$ and $\pi_2(X)$ are directed, but $X$ is not.

4c. Show that if $A, B$ are cpos and $X \subseteq A \times B$ is directed, then $\bigcup X = (\bigcup \pi_0(X), \bigcup \pi_2(X))$.

(Note: together with $\bot_{A \times B} = (\bot_A, \bot_B)$ this shows that the Cartesian prod. of two cpos is a cpo.)

5. Write down — in the manner of slide 2.18 — the first few approximations to $\text{fix}(\Phi)$, where the function $\Phi : (\mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \rightarrow \mathbb{Z}_+)$ is defined by

$$\Phi(f) = \lambda n. n \cdot f(n+1).$$

What is $\Phi^m(1)$? What is $\text{fix}(\Phi)$?

6. Repeat 5 for $\widetilde{\Phi} : (\mathbb{Z}_+ \rightarrow \mathbb{Z}_+) \rightarrow (\mathbb{Z}_+ \rightarrow \mathbb{Z}_+)$ defined by

$$\widetilde{\Phi}(f) = \lambda n. \text{if } n = 0 \text{ then } 0 \text{ else } f(n-1).$$