LECTURE 2: RECURSIVELY DEFINED PROGRAMS

In lecture 1 we proposed that

* semantic domains are (pointed) posets;
* computable functions are monotonic.

In this lecture we consider the semantics of recursively defined programs, and find that these propositions must be strengthened to

* semantic domains are cpo's (complete posets);
* computable functions are continuous.

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RECURSION AND FIXPOINTS

Observation: every recursively defined function f can be expressed in the form

$$f = \Phi(f)$$

where Φ is a non-recursive (higher-order) function.

$$f = \overbrace{\lambda x. \ldots f \ldots f \ldots}^{\mathbf{E}}$$

$$\Big\Downarrow \ \text{define } \Phi(f) = \mathbf{E}$$

$$f = \Phi(f)$$

Defn: Let $\Phi : A \to A$. Then $x \in A$ is called a fixed-point (or fixpoint) of Φ if $\Phi(x) = x$.

Hence the problem of giving recursively defined programs a semantics can be cast as the problem of finding fixpoints of non-recursive functions.

SOME PROBLEMS...

* not all monotonic fns on posets have fixpoints;
* some have many fixpoints; which do we choose?

One possible solution is predicted by the well-known Knaster-Tarski fixpoint theorem:

> Every monotonic $f : A \to A$ on a complete lattice A (a poset with lubs and glbs of all subsets) has a complete lattice of fixpoints.

But intuitively, recursive programs are executed by "unfolding". The KT theorem ensures least fixpoints exist, but does not provide a way to compute them using an "unfolding-like" process.

Solution:

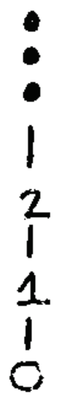
* complete lattices are too strong: cpos are sufficient;
* monotonicity is too weak: continuity is necessary.

A FIRST STEP TO CPO'S : CHAINS

A subset $Y \subseteq X$ of a poset X is a chain iff

$$\forall x, y \in Y.\ x \sqsubseteq y \text{ or } y \sqsubseteq x$$

Example: the naturals $\mathbb{N}$ form a chain under $\leq$



note: chains can be infinite in either (or both) directions

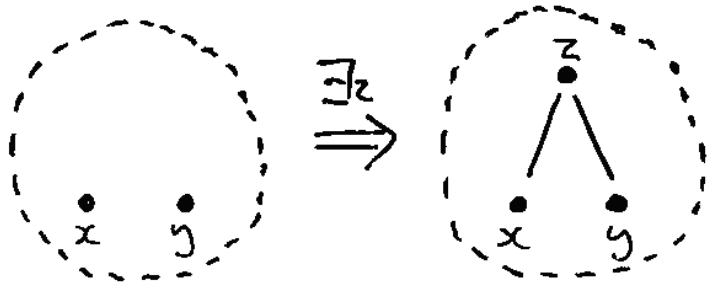
In domain theory, we usually only consider chains with a countable number of elements, i.e. ω-chains.

"omega"

GENERALISING CHAINS: DIRECTED SETS

A non-empty subset $Y \subseteq X$ of a poset X is directed iff

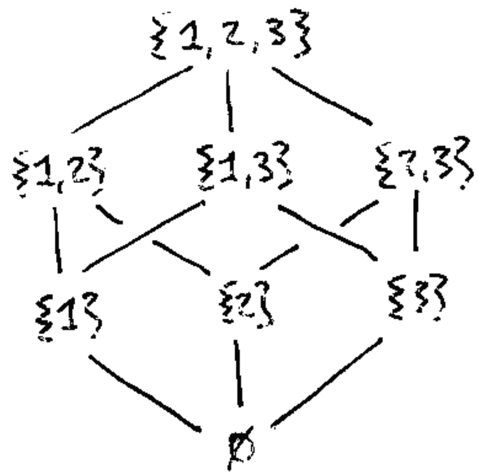
$$\forall x, y \in Y.\ \exists z \in Y.\ x \sqsubseteq z \text{ and } y \sqsubseteq z.$$



Intuition: directed sets are "going somewhere". Given two elements we can always find a bigger one.

Example: any (non-empty) chain is directed.

Example: the powerset $\mathcal{P}(X)$ of any X is directed under $\subseteq$.

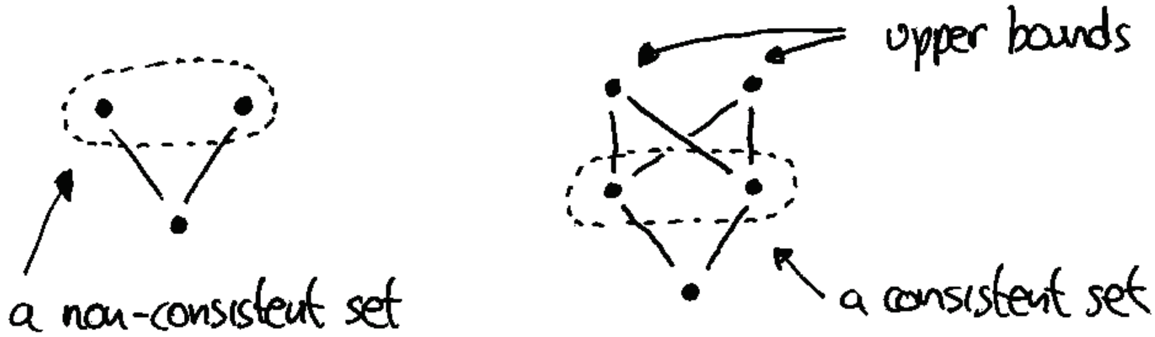


AN EQUIVALENT DEFINITION

A subset $Y \subseteq X$ of a poset X is <u>consistent</u> iff

$$\exists x \in X.\ \forall y \in Y.\ y \sqsubseteq x.$$

Such an x is called an "<u>upper bound</u>" for Y.



Hence a (non-empty) set is directed iff every pair of values has an upper bound <u>in the set</u>.

<u>Lemma</u>: a subset $Y \subseteq X$ of a poset X is directed iff every <u>finite</u> subset $Y' \subseteq Y$ has an upper bound in Y.

(this definition is sometimes more useful in proofs.)

<u>Proof</u> ($\Leftarrow$). Trivial.

(But note that requiring an upper bound of $\emptyset \subseteq Y$ means that Y must have a $\perp$. Hence this defn. implies directed sets are non-empty.) □

<u>Proof</u> ($\Rightarrow$). Let $Y' = \{a_0, \ldots, a_n\}$ be finite/non-empty. We define a function $f(x)$ by induction on $x \in 0..n$:

$$f(x) = \begin{cases} a_0 & \text{if } x = 0 \\ \text{an upper bound of } f(x-1) \text{ and } a_x & \text{otherwise} \\ \text{(which exists since } Y \text{ is directed.)} & \end{cases}$$

A simple inductive proof shows that $f(n)$ is an upper bound for Y'. Since n is finite, $f(n)$ terminates. □

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<u>Corollary</u>: a finite set Y is directed iff it has a top $\top$.

($\top$ is the required upper bound for Y itself.)

LEAST UPPER BOUND

Let $X \subseteq Y$ be a subset of a poset Y. An element $y \in Y$ is a "least upper bound" (lub) for X iff

1. $X \sqsubseteq y$ (i.e. $\forall x \in X.\ x \sqsubseteq y$), "upper bound"
2. $\forall y' \in Y.\ (X \sqsubseteq y' \Rightarrow y \sqsubseteq y')$. "least"

Fact: the lub is unique if it exists, and written $\bigsqcup X$.

(we normally write $x \sqcup y$ rather than $\bigsqcup\{x, y\}$.)

Intuition: "$\bigsqcup X$ combines the information content of all the elements in X, but doesn't add extra info."

Example: (in $\mathbb{B}_\perp \times \mathbb{B}_\perp$)

$$(\perp, \text{false}) \sqcup (\text{true}, \perp) = (\text{true}, \text{false})$$

$$(\perp, \text{false}) \sqcup (\perp, \perp) = (\perp, \text{false})$$

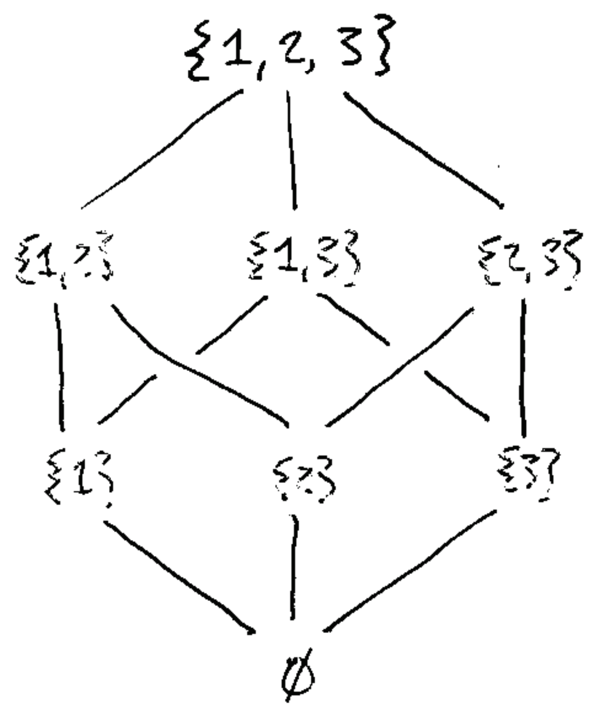
$(\perp, \text{false}) \sqcup (\text{true}, \text{true})$ does not exist.

FAMILIAR EXAMPLES OF LUBS.

* For the poset $\begin{array}{c}\text{true}\\ | \\ \text{false}\end{array}$ ($\mathbb{B}$ ordered by $\Rightarrow$), $x \sqcup y = x \text{ or } y$.

$$\begin{array}{lcl}
\text{false} \sqcup \text{false} & = & \text{false} \\
\text{false} \sqcup \text{true} & = & \text{true} \\
\text{true} \sqcup \text{false} & = & \text{true} \\
\text{true} \sqcup \text{true} & = & \text{true}
\end{array}$$

* For any set X, the powerset $\mathbb{P}(X)$ is a poset ordered by $\subseteq$, with lubs given by union $\bigcup$.



SOME USEFUL FACTS ABOUT LUBS

* a poset has a bottom iff $\bigsqcup \emptyset$ exists. $(\bigsqcup \emptyset = \bot)$

* by defn., a poset A has all binary lubs iff there is an operator $\sqcup : A \times A \to A$ such that

  $x \sqsubseteq x \sqcup y$ and $y \sqsubseteq x \sqcup y$, "upper bound"

  $(x \sqsubseteq z$ and $y \sqsubseteq z)$ implies $x \sqcup y \sqsubseteq z$. "least"

  An equiv. defn. (very useful in proofs) is

  $(x \sqsubseteq z$ and $y \sqsubseteq z)$ iff $x \sqcup y \sqsubseteq z$.

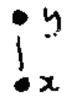
* if all binary lubs exist, then $\sqcup$ satisfies

  $x \sqcup x = x$ idempotent

  $x \sqcup y = y \sqcup x$ symmetric

  $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ associative

  $x \sqsubseteq y$ iff $x \sqcup y = y$



(Aside: these 4 properties are in fact an equiv. axiomatisation for posets with binary lubs!)

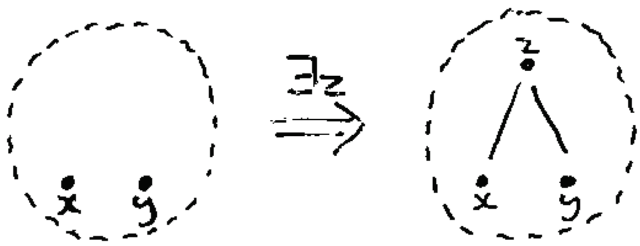
COMPLETE PARTIAL ORDERS

A <u>cpo</u> (<u>complete</u> <u>partial</u> <u>order</u>) is a poset A s.t

① A has a bottom $\perp$ "pointed"

② $\bigsqcup X$ exists for all directed $X \subseteq A$ "directed complete"

<u>Intuition</u>: directed sets are "going <u>somewhere</u>",



In a cpo the directed sets "<u>get there</u>" in that $\bigsqcup X$ exists (although it need not be in X itself.)

<u>Note</u>: directed completeness $\iff$ chain completeness
(this is a non-trivial result; note that if we restrict to ω-chains, we only get an implication $\Rightarrow$.)

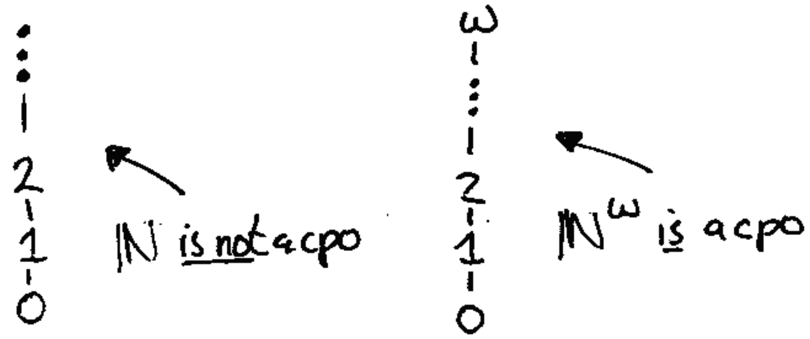
* chain completeness is "simpler" and technically sufficient
* <u>BUT</u> directed completeness results in a simpler treatement of "finiteness" properties in Lecture 4.

Example: any finite (pointed) poset is a cpo.

Any finite directed set has a top (see slide 2.6), which is by definition the lub for the set.

Example: the set $\mathbb{N}$ of naturals is not a cpo.

There is a directed subset ($\mathbb{N}$ itself) with no lub. (But adding a top makes $\mathbb{N}$ a cpo.)



Example: the set $\mathbb{Q}$ of rationals is not a cpo.

Not only does $\mathbb{Q}$ lack a lub for $\mathbb{Q}$ itself, but it lacks for example $\sqrt{2}$, which can be expressed as the lub of an infinite sequence of rational approximations. (But $[0,1] \subseteq \mathbb{R}$ is a cpo.)

$$\boxed{\text{thesis: semantic domains are cpo's.}}$$

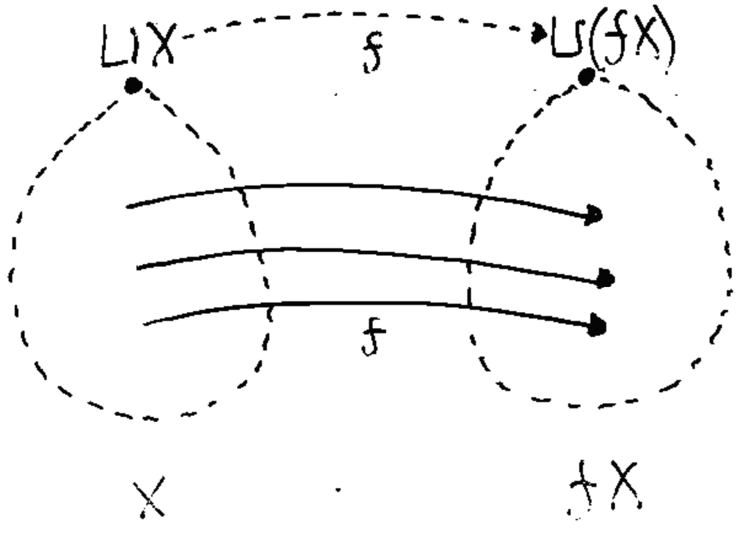
CONTINUOUS FUNCTIONS

A function $f : A \to B$ between cpo's A and B is <u>continuous</u> iff for all directed $X \subseteq A$,

① $\bigsqcup(fX)$ exists, (where $fX = \{fx \mid x \in X\}$.)

② $f(\bigsqcup X) = \bigsqcup(fX)$. "preserves lubs"

<u>Intuition</u>: "nothing is suddenly invented at infinity"



<u>Fact</u>: continuity $\Rightarrow$ monotonicity

Let $x \sqsubseteq y$. Then $X = \{x, y\}$ is directed. Expanding ② gives $fy = fx \sqcup fy$, which is equiv. to $fx \sqsubseteq fy$. □

<u>Fact</u>: monotonicity $\Rightarrow$ condition ①.

It suffices (since B is a cpo) to show that fX is directed for all directed X.

Let $fx, fy \in fX$ (where $x, y \in X$). Since X is directed, there exists $z \in X$ s.t. $x \sqsubseteq z$ and $y \sqsubseteq z$. Now since f is monotonic, $fz \in fX$ is such that $fx \sqsubseteq fz$ and $fy \sqsubseteq fz$. Hence fX is directed. ▫

Combining the two facts, $f : A \to B$ is continuous iff

① f is monotonic,

② $f(\bigsqcup X) = \bigsqcup(fX)$ for all directed $X \subseteq A$.

(this defn. of continuity is normally more useful in practice.)

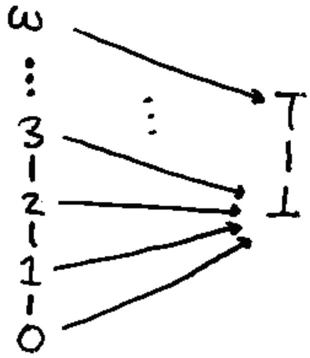
Example: (continuity is only "interesting" for infinite cpo's)

If $f : A \to B$ is monotonic (where A, B are cpo's) and A is finite, the f is continuous.

Example: (monotonicity $\not\Rightarrow$ continuity.)

The function $f : \mathbb{N}^{\top} \to \mathbb{2}$ defined by

$$f(x) = \begin{cases} \bot & \text{if } x \in \mathbb{N} \\ \top & \text{otherwise} \end{cases}$$



is monotonic, but not continuous.

(take $X = \mathbb{N}$, then $f(\bigsqcup X) = f\top = \top$,
but $\bigsqcup(fX) = \bigsqcup\{\bot\} = \bot$.)

$\boxed{\text{thesis: computable functions are continuous}}$

RECURSIVE PROGRAMS

Recall (slide 2.1): "the problem of giving recursively defined programs a semantics can be cast as the problem of finding fixpoints of non-recursive functions."

<u>Theorem</u>: (the CPO fixpoint theorem)

Every continuous function $f : A \to A$ on a cpo A has a <u>least</u> fixpoint $fix(\cancel{A}\,f)$, which can be computed as ← the limit of the ω-chain $\bot \sqsubseteq f(\bot) \sqsubseteq f(f(\bot)) \ldots$, i.e

$$fix(f) = \bigsqcup \{ f^n(\bot) \mid n \in \mathbb{N} \}.$$

<u>Proof</u> (in three parts)

① <u>$\bigsqcup \{ f^n(\bot) \mid n \in \mathbb{N} \}$ exists.</u>

A simple inductive proof shows $\forall n \in \mathbb{N}, f^n(\bot) \sqsubseteq f^{n+1}(\bot)$, i.e that $\{ f^n(\bot) \mid n \in \mathbb{N} \}$ is a chain, and hence directed. □

(2) fix(f) is a fixpoint of f.

$$\begin{aligned}
f(\mathrm{fix}(f)) &= f(\bigsqcup\{f^n(\bot) \mid n \in \mathbb{N}\}) && [\text{defn fix}(f)] \\
&= \bigsqcup\{f^{n+1}(\bot) \mid n \in \mathbb{N}\} && [f \text{ is continuous}] \\
&= \bigsqcup\{f^n(\bot) \mid n \in \mathbb{N}\} && [f^0(\bot) = \bot, \text{ and } \bot \text{ does not affect } \bigsqcup.] \\
&= \mathrm{fix}(f) && [\text{defn. fix}(f).]
\end{aligned}$$

$\square$

(3) fix(f) is the least fixpoint of f.

Let $x \in A$ be a fixpoint of f, i.e. $f(x) = x$. A simple inductive proof shows that $\forall n \in \mathbb{N}.\ f^n(\bot) \sqsubseteq x$, i.e that x is an upper bound for $\{f^n(\bot) \mid n \in \mathbb{N}\}$.

Now since by defn. $\bigsqcup X$ is the least upper bound of a set X, then $\mathrm{fix}(f) = \bigsqcup\{f^n(\bot) \mid n \in \mathbb{N}\} \sqsubseteq x$. $\square$

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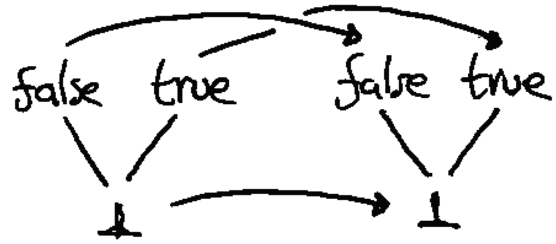
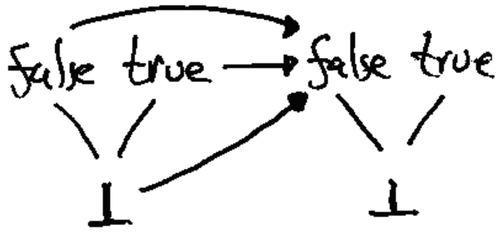
EXAMPLES

* <u>what is</u> $\underline{fix}(\underline{id})$, where id =  ?

$$\begin{aligned} fix(id) &= \bigsqcup \{ id^n(\bot) \mid n \in \mathbb{N} \} \\ &= \bigsqcup \{ \bot, id(\bot), id(id(\bot)), \ldots \} \\ &= \bigsqcup \{ \bot, \bot, \bot, \ldots \} \\ &= \bot. \end{aligned}$$

* <u>what is</u> $\underline{fix}(\underline{K_F})$, where K_F =  ?

$$\begin{aligned} fix(K_F) &= \bigsqcup \{ K_F^{\,n}(\bot) \mid n \in \mathbb{N} \} \\ &= \bigsqcup \{ \bot, K_F(\bot), K_F(K_F(\bot)), \ldots \} \\ &= \bigsqcup \{ \bot, false, false, \ldots \} \\ &= false. \end{aligned}$$

* <u>what is</u> $\underline{fix}(\underline{f})$, where $f\,x = 1 : x$? $\quad (f \in [N_\bot] \to [N_\bot])$

$$\begin{aligned} fix(f) &= \bigsqcup \{ f^n(\bot) \mid n \in \mathbb{N} \} \\ &= \bigsqcup \{ \bot, 1:\bot, 1:1:\bot, \ldots \} \\ &= 1 : 1 : 1 : \ldots \quad \text{(an infinite list of ones.)} \end{aligned}$$

<u>Note</u>: $fix(f)$ is the semantics of the rec. defn. $x = 1 : x$.

EXAMPLE: what is $fix(\Phi)$, where the higher-order function $\Phi \in (\mathbb{N}_\perp \to \mathbb{N}_\perp) \to (\mathbb{N}_\perp \to \mathbb{N}_\perp)$ is defined by

$$\Phi(f) = \lambda n.\ \text{if } n = 0 \text{ then } 1 \text{ else } n * f(n-1) \ ?$$

$$\Phi^0(\perp) = \lambda n. \perp \qquad (\perp \text{ in } \mathbb{N}_\perp \to \mathbb{N}_\perp)$$

$$\Phi^1(\perp) = \lambda n.\ \text{if } n = 0 \text{ then } 1 \text{ else } \perp$$

$$\Phi^2(\perp) = \lambda n.\ \text{if } n = 0 \text{ then } 1 \text{ else } n * (\text{if } (n-1) = 0 \text{ then } 1 \text{ else } \perp)$$

$$\Phi^3(\perp) = \lambda n.\ \text{if } n = 0 \text{ then } 1 \text{ else } n * (\text{if } (n-1) = 0 \text{ then } 1 \text{ else } (n-1) * (\text{if } (n-2) = 0 \text{ then } 1 \text{ else } \perp))$$

$$\vdots$$

$$\Phi^m(\perp) = \lambda n. \begin{cases} 1 * 2 * \dots * n & \text{if } n \in 0..(m-1) \\ \perp & \text{otherwise} \end{cases}$$

The mth approximation $\Phi^m(\perp)$ to $fix(\Phi)$ is the function that gives $!x$ for all $x \in 0..(m-1)$, and diverges on all other arguments. Hence the limit $fix(\Phi)$ is the factorial function on $\mathbb{N}_\perp$!

EXERCISES

① Give an example of a poset A and a monotonic function $f : A \to A$ s.t. f doesn't have a fixpoint.

(Hint: A must be infinite.)

② What is the lub operator on subsets $X \subseteq \mathbb{N}$ of the poset $(\mathbb{N}, \leqslant)$ more commonly known as?

③ Show that if $f : A \to B$ is monotone (A and B are cpo's) and A is finite, then f is continuous.

(Hint: finite directed sets contain their lub.)

(4a) Show that if A, B are posets and $X \subseteq A \times B$ is directed, then the subsets $\pi_0(X) \subseteq A$ and $\pi_1(X) \subseteq B$ (defined below) are also directed.

$$\pi_0(X) = \{ a \in A \mid \exists b \in B.\ (a,b) \in X \}$$

$$\pi_1(X) = \{ b \in B \mid \exists a \in A.\ (a,b) \in X \}$$

(4b) Give an example of a set $X \subseteq \mathbb{2} \times \mathbb{2}$ such that $\pi_0(X)$ and $\pi_1(X)$ are directed, but X is not.

(4c) Show that if A, B are cpo's and $X \subseteq A \times B$ is directed, then $\bigsqcup X = (\bigsqcup \pi_0(X), \bigsqcup \pi_1(X))$.

(Note: together with $\perp_{A \times B} = (\perp_A, \perp_B)$ this shows that the Cartesian prod. of two cpo's is a cpo.)

(5) Write down — in the manner of slide 2.18 — the first few approximations to $fix(\Phi)$, where the function $\Phi : (\mathbb{Z}_\perp \to [\mathbb{Z}_\perp]) \to (\mathbb{Z}_\perp \to [\mathbb{Z}_\perp])$ is defined by

$$\Phi(f) = \lambda n.\ n : f(n+1).$$

What is $\Phi^m(\perp)$? What is $fix(\Phi)$?

(6) Repeat (5) for $\Phi : (\mathbb{Z}_\perp \to \mathbb{Z}_\perp) \to (\mathbb{Z}_\perp \to \mathbb{Z}_\perp)$ defined by

$$\Phi(f) = \lambda n.\ \text{if } n = 0 \text{ then } 0 \text{ else } f(n-2)$$

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