# LECTURE 3: CONSTRUCTIONS ON CPO'S

In lecture 2 we proposed that

* semantic domains are cpo's,
* computable functions are continuous,

and showed how the "CPO fixpoint theorem" allows us to give semantics to recursively defined programs.

—— * ——

In this lecture we introduce natural constructions on cpo's, corresponding to the familiar type-forming operators $\times$ (product), $\rightarrow$ (function space), and $+$ (sum).

Hence the fixpoint approach of lecture 2 applies uniformly to programs that manipulate compound values.

# PRODUCTS

Recall (lecture 2): if $A$ and $B$ are cpo's, then

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

is a cpo under the ordering

$$(a,b) \sqsubseteq (a',b') \quad \text{iff} \quad a \sqsubseteq a' \text{ and } b \sqsubseteq b'$$

with the cpo structure given by

$$\perp_{A \times B} = (\perp_A, \perp_B)$$

$$\bigsqcup X = \left(\bigsqcup \pi_0 X, \bigsqcup \pi_1 X\right) \qquad (\text{for directed } X \subseteq A \times B)$$

———— * ————

Now we look at products in more detail...

# PRIMITIVE "FUNCTIONS ON PRODUCTS"

## ① Destructing pairs

$$\pi_0 : A \times B \to A \qquad \text{"project left"}$$

$$\pi_1 : A \times B \to B \qquad \text{"project right"}$$

are defined by

$$\pi_0\,(a,b) = a \quad \text{and} \quad \pi_1\,(a,b) = b.$$

## ② Constructing pairs

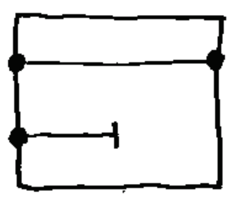
If $f : A \to B$ and $g : A \to C$ then

$$\langle f, g \rangle : A \to B \times C \qquad \text{"split"}$$

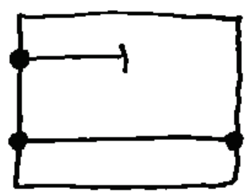
is defined by

$$\langle f, g \rangle\, a = (f a, g a).$$

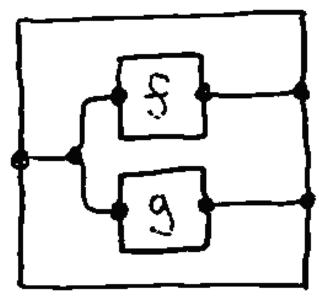
Note: "picturing" these primitives can be helpful



$\pi_0$



$\pi_1$




$\langle f, g \rangle$

<u>Fact</u> : $\pi_0$ and $\pi_1$ are continuous.

<u>Fact</u> : if $f$ and $g$ are continuous, so is $\langle f, g \rangle$.

Proof (in two parts):

① <u>$\langle f, g \rangle$ is monotonic.</u>

Let $x \sqsubseteq y$ in $A$. Then

$$
\begin{aligned}
\langle f, g \rangle\, x &= (f x, g x) && [\text{defn. } \langle -,- \rangle] \\
&\sqsubseteq (f y, g y) && [\text{defn. } \sqsubseteq \text{ on pairs, } f, g \text{ are continuous}] \\
&= \langle f, g \rangle\, y && [\text{defn } \langle -,- \rangle]
\end{aligned}
$$

□

② <u>$\langle f, g \rangle$ preserves lubs of directed sets.</u>

Let $X \subseteq A$ be directed. Then

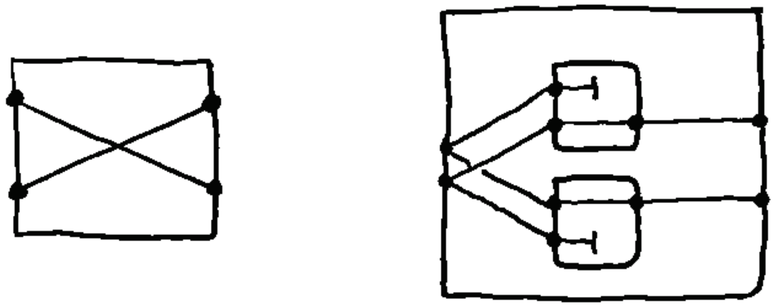
$$
\begin{aligned}
\langle f, g \rangle \sqcup X &= (f \sqcup X, g \sqcup X) && [\text{defn. } \langle -,- \rangle] \\
&= (\sqcup f X, \sqcup g X) && [f, g \text{ are continuous}] \\
&= \sqcup \{ (f x, g x) \mid x \in X \} && [\text{defn. } \sqcup \text{ on pairs}] \\
&= \sqcup \langle f, g \rangle X && [\text{defn. } \langle -,- \rangle]
\end{aligned}
$$

□

# DERIVED "FUNCTIONS ON PRODUCTS"

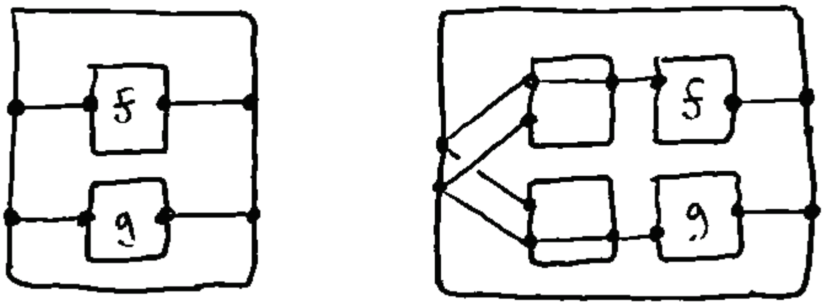
For example ...

swap : $A \times B \to B \times A$



$$\text{swap} = \langle \pi_1, \pi_0 \rangle$$

$f \times g : A \times C \to B \times D$ (for $f : A \to B$ and $g : C \to D$)

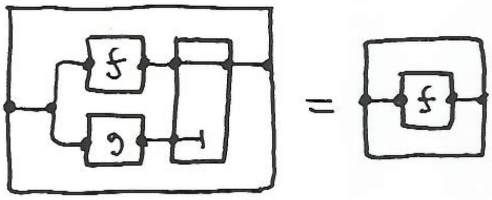



$$f \times g = \langle f \circ \pi_0, g \circ \pi_1 \rangle$$

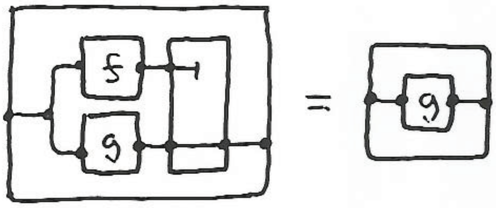
**Note**: $\times$ is now overloaded, e.g. $A \times C$ is a cpo, while $f \times g$ is a continuous function

# UNIVERSALITY

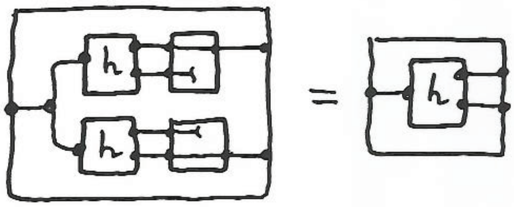
$\pi_0 \circ \langle f, g \rangle = f$




$\pi_1 \circ \langle f, g \rangle = g$




$\langle \pi_0 \circ h, \pi_1 \circ h \rangle = h$




Fact: these are together equivalent to

$$(\pi_0 \circ h = f \text{ and } \pi_1 \circ h = g) \text{ iff } h = \langle f, g \rangle .$$

This is the universal property or unique extension property (UEP) for $\langle -, - \rangle$ and says that "$\langle f, g \rangle$ is the unique $h : A \to B \times C$ such that $\pi_0 \circ h = f$ and $\pi_1 \circ h = g$."

Note: the UEP rule is easy to remember as a "commuting diagram":

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[d, dashed, "\langle f, g \rangle"] \arrow[dr, "g"] & \\
B & B \times C \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & C
\end{tikzcd}$$

# DERIVED PROPERTIES

Fact: the UEP rule for $\langle f, g \rangle$ can be used to prove all the familiar properties of products, e.g.

$$\pi_0 \circ (f \times g) = f \circ \pi_0$$

$$\pi_1 \circ (f \times g) = g \circ \pi_1$$

$$(f \times g) \circ \langle h, i \rangle = \langle f \circ h, g \circ i \rangle$$

$$\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$$

$$id_A \times id_B = id_{A \times B} \qquad (*)$$

$$(f \circ g) \times (h \circ i) = (f \times h) \circ (g \times i) \qquad (**)$$

(thinking "in pictures" illuminates these equations)

A categorical aside:

* Categorically, (*) and (**) together with the typing rule for $\times$ express that $\times$ is a bifunctor on the category CPO with cpo's as objects and continuous functions as arrows.

* The UEP rule expresses that CPO has "binary products".

# SOME USEFUL ISO'S ON PRODUCTS

Sets $X$ and $Y$ are <u>isomorphic</u> ($X \cong Y$) iff there are functions $f : X \to Y$ and $g : Y \to X$ such that

$$g \circ f = id_X \quad \text{and} \quad f \circ g = id_Y . \qquad (\text{i.e. } f = g^{-1})$$

($X \cong Y$ means that $X$ and $Y$ are in one-to-one correspondance.)

<u>Fact</u>: cpo's form a commutative monoid <u>"up to iso"</u> under $\times$ and $\mathbb{1}$, i.e for all cpo's $A, B, C$,

$$A \times \mathbb{1} \cong A,$$
$$A \times B \cong B \times A,$$
$$A \times (B \times C) \cong (A \times B) \times C.$$

<u>Fact</u>: pairs of continuous functions $A \to B$ and $A \to C$ are in 1-1 correspondance with continuous functions $A \to B \times C$,

$$[A \to B] \times [A \to C] \cong [A \to (B \times C)]$$

<u>Proof</u>: take $f = \langle -,- \rangle$ and $g = \lambda h. (\pi_0 \circ h, \pi_1 \circ h)$ then the iso follows immediately from the UEP rule for $\langle -,- \rangle$.

Categorically: the existance of an operator $\times$ on cpo's satisfying (*) expresses that CPO has binary products.

# FUNCTIONS

**Theorem:** if $A$ and $B$ are cpo's then

$$A \to B = \{ f : A \to B \mid f \text{ is continuous} \}$$

is a cpo under the ordering

$$f \sqsubseteq g \quad \text{iff} \quad \forall a \in A.\ fa \sqsubseteq ga.$$

**Intuition:** "the information content of a function is increased by increasing the information content of the result value for any (or many) argument value(s)."

**Proof:** We must show two things.

① $A \to B$ has a least element

$\bot_{A\to B}$ is the "constant bottom" function $\lambda a.\bot$. □

② $\bigsqcup X$ exists for all directed $X \subseteq A \to B$

We will show that $\bigsqcup X = \lambda a. \bigsqcup \{ fa \mid f \in X \}$...

We must show four things now:

(2.1) $\bigsqcup\{fa \mid f \in X\}$ exists in $B$

Let $ga, ha \in \{fa \mid f \in X\}$. Since $X$ is directed, there exists a function $k \in X$ s.t. $g \sqsubseteq k$ and $h \sqsubseteq k$. Now (using the defn. of $\sqsubseteq$ on functions), $ka \in \{fa \mid f \in X\}$ is such that $ga \sqsubseteq ka$ and $ha \sqsubseteq ka$. Hence $\{fa \mid f \in X\}$ is directed and its lub exists since $B$ is a cpo. $\square$

(2.2) $\Phi = \lambda a. \bigsqcup\{fa \mid f \in X\}$ is continuous.

Let $Y \subseteq A$ be directed. Then

$$
\begin{aligned}
\Phi(\bigsqcup Y) &= \bigsqcup\{f(\bigsqcup Y) \mid f \in X\} && [\text{defn } \Phi] \\
&= \bigsqcup\{\bigsqcup\{fy \mid y \in Y\} \mid f \in X\} && [f \text{ is continuous}] \\
&= \bigsqcup\{\bigsqcup\{fy \mid f \in X\} \mid y \in Y\} && [\text{lubs}] \\
&= \bigsqcup\{\Phi(y) \mid y \in Y\} && [\text{defn } \Phi]
\end{aligned}
$$

(Note: monotonicity of $\Phi$ is easy too.) $\square$

(2.3) $\Phi$ is an upper bound for $X \subseteq A \to B$.

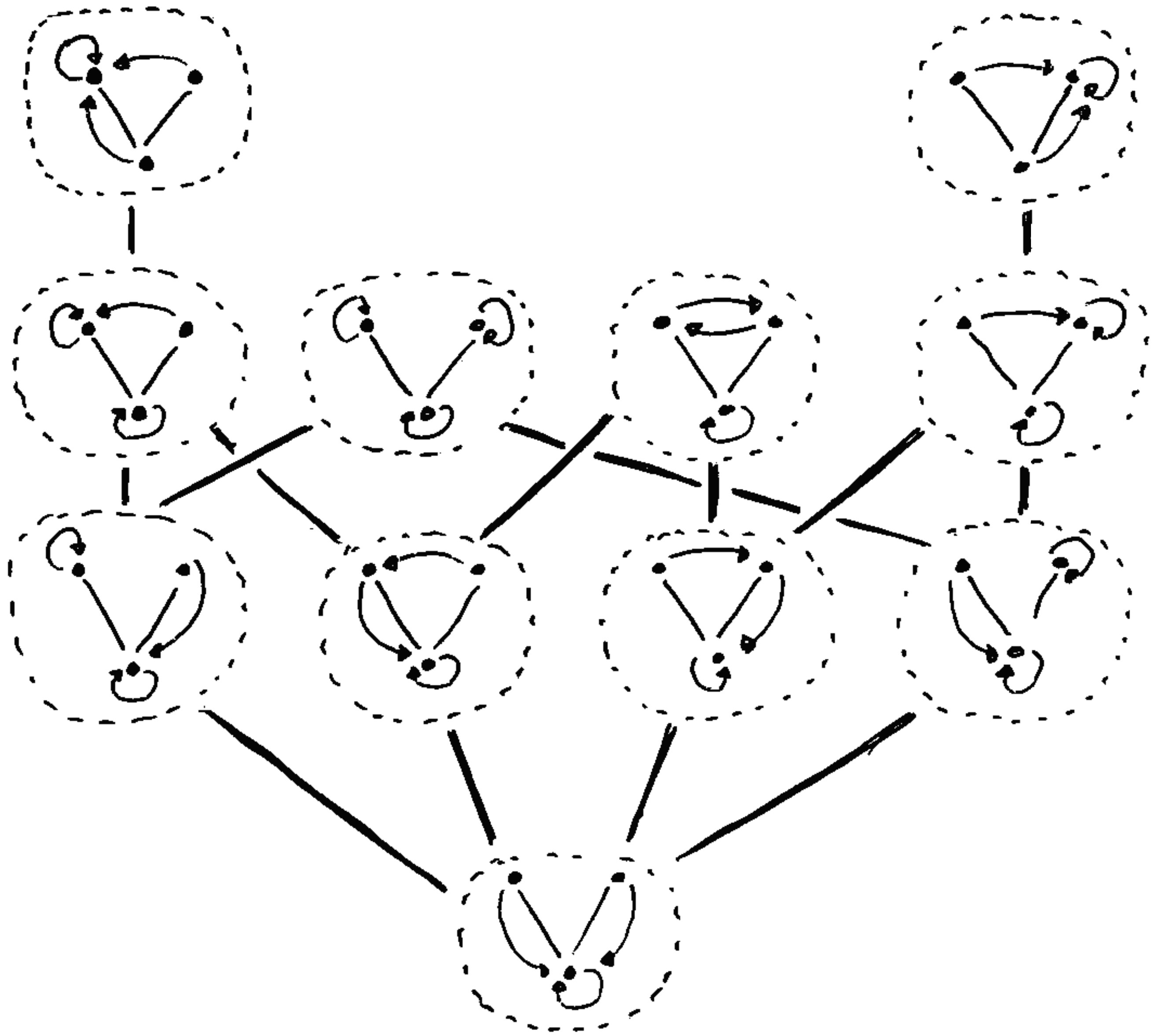
Let $g \in X$. Then for any $a \in A$ we have $ga \in \{fa \mid f \in X\}$. Now (by 2.1 and lubs) we have $ga \sqsubseteq \bigsqcup \{fa \mid f \in X\}$, which using the defn. for $\Phi$ reads $ga \sqsubseteq \Phi a$. Hence (using $\sqsubseteq$ on functions) we have $g \sqsubseteq \Phi$. □

(2.4) $\Phi$ is the least upper bound for $X$

Let $g : A \to B$ be an upper bound for $X \subseteq A \to B$. Then (using $\sqsubseteq$ on functions) $ga$ is an upper bound for $\{fa \mid f \in X\}$ for all $a \in A$. Now (using 2.1 and lubs) we have $\bigsqcup \{fa \mid f \in X\} \sqsubseteq ga$, which using the defn for $\Phi$ reads $\Phi a \sqsubseteq ga$. Hence (using $\sqsubseteq$ on functions) we have $\Phi \sqsubseteq g$. □

* 

# EXAMPLE $(\mathbb{B}_\perp \to \mathbb{B}_\perp)$



Note: at each step up in this cpo we move one arrow-tip one step up in the cpo $\mathbb{B}_\perp$.

# PRIMITIVE "FUNCTIONS ON FUNCTIONS"

① "Destructing" functions

$$\text{apply} : (A \to B) \times A \to B$$
$$\text{apply}(f, a) = f(a)$$

Fact: apply is continuous

② "Constructing" functions

If $f : A \times B \to C$ then

$$\text{curry}(f) : A \to (B \to C)$$

is defined by

$$\text{curry}(f)\, a = \lambda b.\, f(a,b)$$

Fact: if $f$ is continuous, so is curry$(f)$.

Intuition: curry$(f)$ has the same behaviour as $f$ except it takes its arguments "one at a time" rather than together.

# FACTS ABOUT apply and curry

* They can be used to define $\rightarrow$ on functions:

  If $f : A \to B$ and $g : C \to D$ then $f \to g : (B \to C) \to (A \to D)$ is defined by $f \to g = \text{curry}\,(g \circ \text{apply} \circ id \times f)$.

  (operationally: $f \to g = \lambda h.\ g \circ h \circ f$.)

* curry satisfies a universal property:

$$
\begin{tikzcd}
A \times B \arrow[r, "f"] \arrow[d, dotted, "\text{curry}(f) \times id_B"'] & C \\
(B \to C) \times B \arrow[ur, "\text{apply}"'] &
\end{tikzcd}
$$

* the UEP can be used to prove properties of $\rightarrow$, e.g.

$$id_A \to id_B = id_{A \to B}$$
$$(f \circ g) \to (h \circ i) = (g \to h) \circ (h \to i)$$

* the UEP establishes an important iso:

$$[A \times B \to C] \cong [A \to (B \to C)]$$

Categorically: $\rightarrow$ is a functor $CPO^{op} \times CPO \to CPO$, CPO has exponentials.

# SUMS

**Theorem**: if $A$ and $B$ are cpo's then

$$A+B = \{(0,a) \mid a \in A\} \cup \{(1,b) \mid b \in B\} \cup \{\underbrace{\bot_{A+B}}_{\text{a new } \bot}\}$$

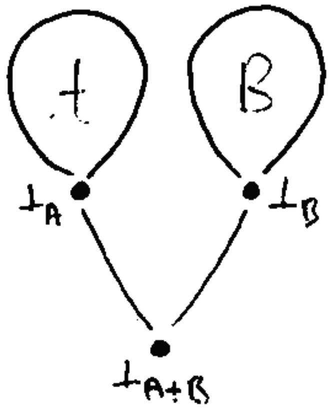
"tags"

is a cpo under the ordering

$$x \sqsubseteq y \text{ iff } \begin{cases} \text{true} & \text{if } x = \bot_{A+B}, \\ a \sqsubseteq a' & \text{if } x = (0,a) \text{ and } y = (0,a'), \\ b \sqsubseteq b' & \text{if } x = (1,b) \text{ and } y = (1,b'), \\ \text{false} & \text{otherwise.} \end{cases}$$

**Intuition**: the ordering on $A+B$ is just as in $A$ and $B$ separately, except that $\bot_{A+B}$ is less than anything.

**Note**: the cpo $A+B$ can be pictured as



# FUNCTIONS ON SUMS

## ① Constructing sums

$\mathrm{inl} : A \to A+B \qquad \mathrm{inl}\, a = (0,a) \qquad$ "inject left"

$\mathrm{inr} : B \to A+B \qquad \mathrm{inr}\, b = (1,b) \qquad$ "inject right"

## ② Destructing sums

If $f : A \to C$ and $g : B \to C$ then

$$[f,g] : A+B \to C \qquad \text{"case"}$$

is defined by

$$[f,g]\, x = \begin{cases} f a & \text{if } x = (0,a), \\ g b & \text{if } x = (1,b), \\ \perp_C & \text{if } x = \perp_{A+B}. \end{cases}$$

## ③ Defining + on functions

If $f : A \to B$ and $g : C \to D$ then

$$f+g : A+C \to B+D$$

is defined by

$$f+g = [\mathrm{inl} \circ f, \mathrm{inr} \circ g].$$

# FACTS ABOUT SUMS

* $[f,g] \circ inl = f$ and $[f,g] \circ inr = g$, i.e.

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] \arrow[r, "inl"'] & A+B \arrow[u, "{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "inr"] \end{tikzcd}$$

* BUT in general, $[f,g]$ is only weakly universal:

$$(h \circ inl = f \text{ and } h \circ inr = g) \text{ iff } [f,g] \sqsubseteq h.$$

For example,

$$\begin{tikzcd} & \mathbb{2} & \\ \mathbb{1} \arrow[ur, "\lambda x.T"] \arrow[r, "inl"'] & \mathbb{1}+\mathbb{1} \arrow[u, "h"] & \mathbb{1} \arrow[ul, "\lambda x.T"'] \arrow[l, "inr"] \end{tikzcd}$$

$h = \lambda x.T$ makes this commute, but $h \neq [\lambda x.T, \lambda x.T]$

* ... hence many "expected" iso's do not hold; e.g.

$$A + (B+C) \not\cong (A+B) + C$$

$$[A \to C] \times [B \to C] \not\cong [A+B \to C]$$

**Categorically:** $+$ is a bifunctor on CPO, but CPO only has local binary sums.

# STRICT CPO CONSTRUCTIONS

When giving semantics to programs in a call-by-name (or lazy) language, $\to$, $\times$ and $+$ are just what we want:

* functions can be non-strict, i.e. $f \bot \neq \bot$ is OK;
* pairing is non-strict, e.g. $(\mathit{false}, \bot) \in \mathbb{B}_\bot \times \mathbb{B}_\bot \neq \bot$;
* constructors are non-strict, e.g. $(0, \bot) \in \mathbb{B}_\bot + \mathbb{B}_\bot \neq \bot$.

To properly capture strictness properties (e.g. in a call-by-value language) we also need strict versions of $\to$, $\times$ and $+$.

Theorem: if $A$ and $B$ are cpos then so are

$$A \multimap B = \{ f \in A \to B \mid f \bot = \bot \},$$

$$A \otimes B = \{ (a,b) \in A \times B \mid a \neq \bot \text{ and } b \neq \bot \} \cup \{ \bot_{A \otimes B} \},$$

$$A \oplus B = \{ (t,x) \in A + B \mid x \neq \bot \} \cup \{ \bot_{A \oplus B} \}.$$

(the orderings are the evident ones.)

Note: $\oplus$ and $\otimes$ are often called "smash" sum and product.

(two $\bot$'s are smashed together to give a single $\bot$)

Fact: the strict cpo constructions satisfy

$$[A \circ\!\!\rightarrow B] \times [A \circ\!\!\rightarrow C] \cong [A \circ\!\!\rightarrow B \otimes C],$$

$$[A \otimes B \circ\!\!\rightarrow C] \cong [A \circ\!\!\rightarrow (B \circ\!\!\rightarrow C)],$$

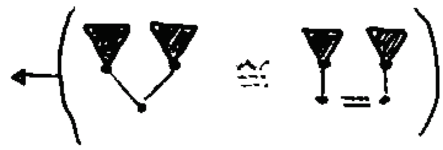
$$[A \circ\!\!\rightarrow C] \times [B \circ\!\!\rightarrow C] \cong [A \oplus B \circ\!\!\rightarrow C].$$

Categorically: the category $CPO_\bot$ of cpo's and strict continuous functions has products, exponentials, and sums.

Fact: the lazy/strict constructions are related in useful ways,

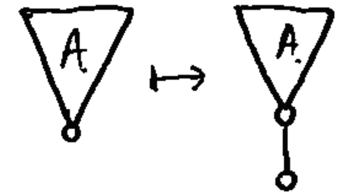
$$A \longrightarrow B \cong A_\bot \circ\!\!\rightarrow B$$

$$A + B \cong A_\bot \oplus B_\bot$$



$$(A \times B)_\bot \cong A_\bot \otimes B_\bot$$

where "lifting" $(-)_\bot$ adds a new $\bot$ to a cpo, i.e.



# EXERCISES

① Show using the UEP rule that $\langle f,g\rangle \circ h = \langle f\circ h, g\circ h\rangle$.

② Draw the cpo $\mathbb{3}\to\mathbb{3}$.

(see exercise 1.2 and slide 3.11.)

③ Express $\lambda(a,(b,c)).((a,b),c)$ using combinators.

④ What is the common name for $\mathrm{apply}\circ f\times id$ ?

⑤ Show that apply is continuous.

(You may omit the proof of monotonicity, and make use of the following useful result:

Fact: a function $f : A\times B\to C$ is continuous iff it is continuous in each argument separately, i.e.

$\forall a\in A.\ f(a,-) : B\to C$ is continuous; and

$\forall b\in B.\ f(-,b) : A\to C$ is continuous. )