LECTURE 4 : SCOTT DOMAINS

In this lecture we formalise the idea that

"a programs meaning can be viewed as the limit of a sequence of finite approximations."

by strengthening "semantic domains are cpo's" to

* semantic domains are Scott domains.

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Technical benefits:

(1) a theory of finite approximations can be used to develop a theory of computability for semantic domains.

(2) finiteness restrictions are necessary for cpo's to be closed under powerdomains (lecture 6).

Note: Scott domains are not technically necessary to solve (most) recursive domain equations (lecture 5).
For most purposes in semantics, cpo's are sufficient.

FINITE ELEMENTS

We begin by formalising the notion of an element in a cpo representing a finite amount of info.

Intuition : there are two kinds of directed sets :

(1) Boring - contain their lub (e.g. finite directed sets are boring ; but boring $\not\Rightarrow$ finite set.)

(2) Interesting - don't contain their lub (e.g. $\mathbb{N}$ in the chain $\mathbb{N}^\omega$ is interesting ; interesting $\Rightarrow$ infinite set.)

Basic idea : an element x in a cpo is

infinite iff "x is the lub of an interesting directed set"

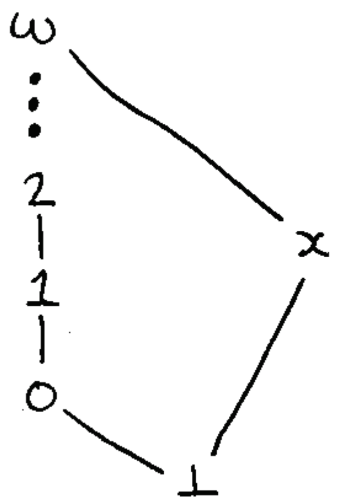
i.e. $\exists X. \, (x = \bigsqcup X$ and $x \notin X)$;

finite iff "x is not infinite"

i.e. $\forall X. \, (x = \bigsqcup X$ implies $x \in X)$.

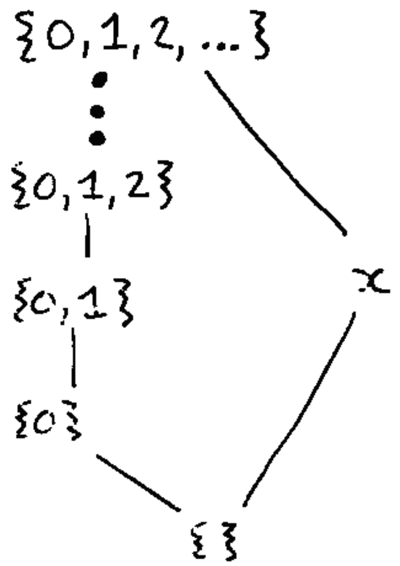
BUT... this notion of "finite element" is too weak.

Example: consider the cpo,



Using the above definitions, all elements in this cpo are finite, except for ω. ($\omega = \bigsqcup \mathbb{N}$, but $\omega \notin \mathbb{N}$).

Now consider the isomorphic cpo,



If x were a finite set, it would approximate one of the finite sets in the infinite chain $\{\} \subseteq \{0\} \subseteq \{0,1\} \subseteq \dots$.
Hence it is appropriate to call x "infinite"

We need a stronger definition of "finite element":

<u>Defn</u>: Let A be a cpo. Then $x \in A$ is <u>finite</u> (or <u>compact</u>) iff for all directed $X \subseteq A$,

$$x \sqsubseteq \bigsqcup X \text{ implies } \exists y \in X.\ \cancel{y \sqsubseteq x}\ x \sqsubseteq y$$

" a finite element always approximates an element in a directed set if it approximates the lub."

<u>Notes</u>:

① The defn. <u>doesn't</u> mention finite or infinite sets!

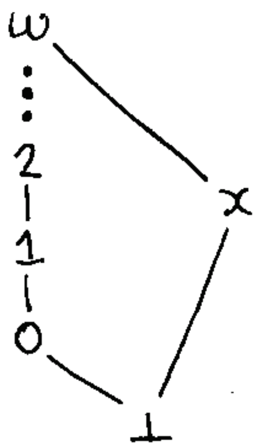
② The defn. is a generalisation to cpo's of the standard notion of a "finite element" from the theory of <u>algebraic lattices</u>.

③ We write $K(A)$ for the set $\{x \in A \mid x \text{ is compact}\}$ of compact elements in a cpo A.

Example: every element in a finite cpo (or more generally, a cpo of finite height) is compact.

(since finite directed sets contain their lub.)

Example



every element is compact, except ω and x; e.g. for x we have $x \sqsubseteq \bigsqcup \mathbb{N} = \omega$, but there is no element $y \in \mathbb{N}$ s.t. $x \sqsubseteq y$.

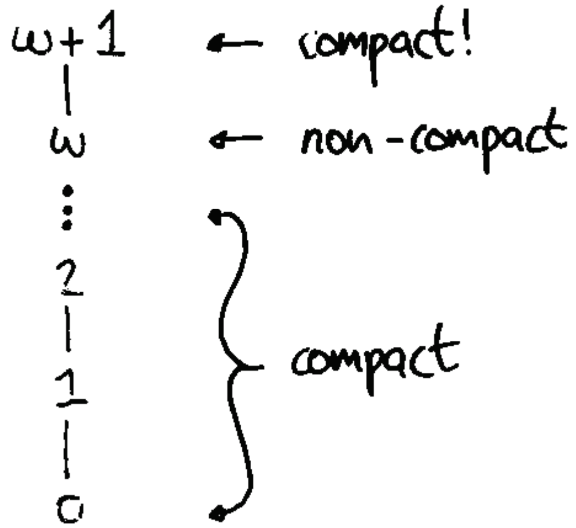
Example: consider the cpo $\mathbb{P}(X)$ of subsets of a set X, ordered by inclusion $\subseteq$. The compact elements of $\mathbb{P}(X)$ are precisely the (truly) finite sets.

Example: consider the cpo $\mathbb{N} \rightharpoonup \mathbb{N}$ of partial functions on $\mathbb{N}$, ordered by inclusion of graphs. The compact elements of $\mathbb{N} \rightharpoonup \mathbb{N}$ are the fns with finite domains.

However... our notion of "finite" is an abstract one: it does not always match our intuition.

<u>Example</u> (finite $\not\Rightarrow$ finite height)

Consider the cpo obtained by adding a new top $(\omega+1)$ onto the cpo $\mathbb{N}^\omega$. Now $\omega+1$ is <u>compact</u> (unlike ω, it does not arise as the lub of an interesting directed set: if $\omega+1 = \bigsqcup X$ then $\omega+1 \in X$) but has an <u>infinite</u> number of approximations!



<u>Asside</u>: adding the requirement that finite elements only have a (truly) finite number of approximations is the basis for the theory of "<u>Berry domains and stable functions</u>".

Example (infinite sets can be compact)

A submonoid of a monoid $(X, \oplus, e)$ is a subset $Y \subseteq X$ such that Y is closed under $\oplus$ and $e \in Y$.

Fact: the submonoids of X form a cpo under $\subseteq$.

Now... the infinite set E of even numbers is a finite element of the cpo of submonoids of $(\mathbb{N}, +, 0)$!

proof: Let $Y \subseteq \mathbb{N}$ be directed and $E \subseteq \bigcup Y$.
Then since $2 \in E$ there must exist $y \in Y$ s.t $2 \in y$.
Now since $(y, +, 0)$ must be a monoid, $E \subseteq y$.

But... E is "finitely generated": the (truly) finite set $\{2\} \subseteq \mathbb{N}$ is s.t E is the smallest submonoid of $(\mathbb{N}, +, 0)$ such that $\{2\} \subseteq E$. In fact, the compact submonoids of $(\mathbb{N}, +, 0)$ are precisely the finitely generated ones.

Conclusion: finite can mean "essentially finite".

ALGEBRAIC CPO'S

Defn: a cpo D is algebraic iff for all $x \in D$,

(1) $\downarrow(x) = \{ y \in K(D) \mid y \sqsubseteq x \}$ is directed;

(2) $x = \bigsqcup \downarrow(x)$.

"a cpo is algebraic iff every element is the lub of its compact approximations."

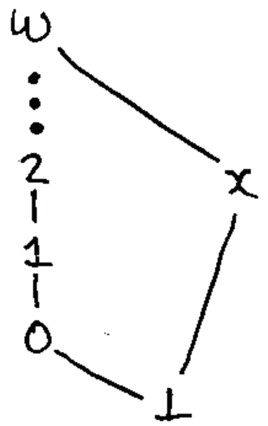
Note: in semantics it is common to only consider algebraic cpo's with a countable set of compact elements. Such cpo's are called ω-algebraic.

Examples (ω-algebraic cpo's)

finite cpo's,	[all elements are compact]
subsets $\mathbb{P}(X)$ of a set X,	[compact $\equiv$ finite subset]
partial fns $\mathbb{N} \rightharpoonup \mathbb{N}$,	[compact $\equiv$ finite domain]
submonoids of a monoid.	[compact $\equiv$ finitely generated]

<u>Note</u>: in general, any cpo of subalgebras (e.g. subgroups of a group, subrings of a ring, etc) is ω-algebraic. In fact, these examples are the origin of the term "algebraic".

<u>Example</u> (a non ω-algebraic cpo)



this element is not the lub of its compact approx.:

$\downarrow(x) = \{\perp\}$, but $x \neq \bigsqcup \{\perp\}$.

<u>Asside</u>: there is an equivalent defn of ω-algebraic cpo's in which directed sets are replaced throughout by ω-chains; see Plotkin p 60.

But ... the ω-chain defn. is not so appealing: one speaks of <u>an</u> ω-chain of compact approximations, rather than <u>the</u> directed set of such.

CLOSURE PROPERTIES

Fact: if D and E are algebraic cpo's, so is $D \times E$.

The following result is useful for the proof.

> Defn: $X \subseteq \mathbf{K}(A)$ is a basis for a cpo A iff for all $x \in A$, $x = \bigsqcup \{a \in X \mid a \sqsubseteq x\}$.
>
> Fact: if X is a basis for A then A is algebraic and $\mathbf{K}(A) = X$, i.e. $\mathbf{K}(A)$ is the unique basis for an algebraic cpo A.

proof: we will show that $K(D) \times K(E)$ is a basis for $D \times E$, which implies $D \times E$ is algebraic with $K(D \times E) = K(D) \times K(E)$.

part 1: $K(D) \times K(E)$ is a compact subset of $D \times E$.

Let $(x,y) \in K(D) \times K(E)$ and suppose $(x,y) \sqsubseteq \bigsqcup X$, where $X \subseteq D \times E$ is directed. Then (defn of $\bigsqcup$ on products), $x \sqsubseteq \bigsqcup \pi_0 X$ and $y \sqsubseteq \bigsqcup \pi_1 X$. Now since x and y are compact, there exists $x' \in \pi_0 X$ and $y' \in \pi_1 X$ such that $x \sqsubseteq x'$ and $y \sqsubseteq y'$...

It does not follow that $(x', y') \in X$, but since X is directed, there exists $(a,b) \in X$ s.t $x' \sqsubseteq a$ and $y' \sqsubseteq b$. Hence, $(a,b) \in X$ satisfies $(x,y) \sqsubseteq (a,b)$. $\square$

<u>part 2. $\downarrow(x,y)$ is directed for all $(x,y) \in D \times E$.</u>

$$\begin{aligned} \downarrow(x,y) &= \{(a,b) \in K(D) \times K(E) \mid (a,b) \sqsubseteq (x,y)\} \\ &= \{a \in K(D) \mid a \sqsubseteq x\} \times \{b \in K(E) \mid b \sqsubseteq y\} \\ &= \downarrow(x) \times \downarrow(y). \end{aligned}$$

Now because D and E are algebraic, $\downarrow(x)$ and $\downarrow(y)$ are directed, and hence $\downarrow(x,y)$ is too ($\times$ preserves directed sets). $\square$

<u>part 3. each (x,y) is the lub of its finite approx.</u>

$$\begin{aligned} \bigsqcup \downarrow(x,y) &= \bigsqcup(\downarrow(x) \times \downarrow(y)) && [\text{part 2}] \\ &= \left(\bigsqcup \downarrow(x), \bigsqcup \downarrow(y)\right) && [\text{defn. } \textstyle\bigsqcup \text{ on products}] \\ &= (x,y). \ \square && [D,E \text{ are algebraic}] \end{aligned}$$

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<u>Fact</u>: ω-algebraic cpo's are closed under <u>all</u> the cpo constructions of lecture 3, ... <u>except</u> $\rightarrow$ and $\multimap$.

SOME FACTS ABOUT ALGEBRAIC CPO'S

Let D and E be algebraic cpo's. Then,

① A function $f : D \to E$ is continuous iff $\forall x \in D$,

$$f(x) = \bigsqcup \{ fa \mid a \in \downarrow(x) \}.$$

"in an algebraic cpo, continuous functions are completely determined by their behaviour on finite arguments."

<u>Note</u>: this makes precise our earlier slogan (slide 2.12) about continuity: "nothing is suddenly invented at infinity."

② Let $f : D \to E$ be continuous, and define

$$G_f = \{ (a,b) \in K(D) \times K(E) \mid b \sqsubseteq fa \}.$$

Then for all $x \in D$, we have

$$f(x) = \bigsqcup \{ b \mid (a,b) \in G_f \text{ and } a \sqsubseteq x \}.$$

This is very powerful; e.g. a continuous function $f : \mathbb{P}(\mathbb{N}) \to \mathbb{P}(\mathbb{N})$ on an <u>uncountable</u> cpo $\mathbb{P}(\mathbb{N})$ is completely determined by the <u>countable</u> relation G_f.

proof:

$$
\begin{aligned}
f(x) &= f(\bigsqcup \downarrow(x)) && [D \text{ is algebraic}] \\
&= \bigsqcup \{ fa \mid a \in \downarrow(x) \} && [f \text{ is continuous}] \\
&= \bigsqcup \{ \bigsqcup \downarrow(fa) \mid a \in \downarrow(x) \} && [E \text{ is algebraic}] \\
&= \bigsqcup \{ b \in K(E) \mid b \sqsubseteq fa \text{ and } a \in \downarrow(x) \} && [\text{lubs}] \\
&= \bigsqcup \{ b \mid (a,b) \in G_f \text{ and } a \sqsubseteq x \} \; \square && [\text{defn. } G_f]
\end{aligned}
$$

③ A representation theorem for algebraic cpo's.

Theorem: every algebraic cpo D is isomorphic to the "ideal completion" of its compact elements, i.e

$\overline{K(D)} = \{ X \subseteq K(D) \mid X \text{ is an ideal} \}$ is an algebraic cpo such that $D \cong \overline{K(D)}$.

Note: an ideal is a downwards-closed directed set; the compact elements of $\overline{K(D)}$ are $\downarrow(a)$ for $a \in K(D)$.

Asside: categorically, the category of algebraic cpo's and continuous functions is equivalent to the category of pre-orders and "approximable" relations.

SERIOUS PROBLEM

ω-algebraic cpos are not closed under $\rightarrow$.

This is a non-trivial problem, with many solutions.

Original solution [Scott 1970]

Using complete lattices (posets with lubs of all subsets) rather than cpo's solves the problem with $\rightarrow$, but brings some problems of its own...

1. Complete lattices must have a top $\top$; being required to add a fictitious top (representing "inconsistent info") to cpo's like $\mathbb{B}_\perp$ is strange...

2. Extending primitive functions on cpo's to complete lattices can spoil their algebraic properties.

   e.g. consider the two possible extensions to "if":

   (a) $\text{if}\,(\top, x, y) = x \sqcup y$,

   (b) $\text{if}\,(\top, x, y) = \top$.

... either extension results in the failure of some familiar (and useful) law for "if"; e.g. for (a),

$$\mathrm{if}\,(b,\ \mathrm{if}\,(b,x,y),\ z) \;=\; \mathrm{if}\,(b,x,z)$$

no longer holds unconditionally (it fails if $b = \top$).

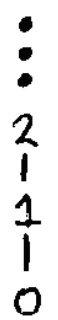
③ It is not possible to generalise the powerset operator $\mathbb{P}$ to semantic domains (see lecture 6) if complete lattices are used.

A SIMPLE SOLUTION: CONSISTENT COMPLETENESS

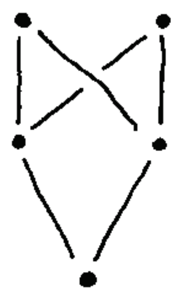
Defn: a poset A is consistent complete (or bounded complete) iff $\bigsqcup X$ exists for all consistent $X \subseteq A$.

($X \subseteq A$ is consistent iff X has an upper bound in A.)

Examples



consistent complete, but not directed complete



directed complete, but not consistent complete.

SCOTT DOMAINS

Defn: a cpo D is a Scott domain iff

1. D is an ω-algebraic cpo;
2. D is consistent complete.

Fact: in the context of 1 there are a number of equivalent ways to express condition 2, e.g.

2a $x \sqcup y$ exists for all consistent $x, y \in D$.

This condition is useful when one is required to show that a given cpo is a Scott domain.

2b $\sqcap X$ exists for all non-empty $X \subseteq D$.

Warning: don't read 2b as "D is a complete lattice without a top". In fact, an ω-algebraic cpo is a complete lattice iff $x \sqcup y$ exists for all x, y, not just the consistent ones; c.f 2a

Memory aid:

Scott domain = "a c^3po" (STAR WARS!)

ω-algebraic consistent-complete complete-partial-order

Fact: Scott domains are closed under all the cpo constructions of lecture 3, including $\rightarrow$ and $\multimap$.

$\boxed{\text{thesis: semantic domains are Scott domains.}}$

Asside: is there a best (largest) class of ω-algebraic cpo's which is closed under $\rightarrow$?

YES! These are the bifinite domains or SFP objects, and are characterised by a simple and beautiful property: a cpo is bifinite iff it is the limit of (truly) finite posets in the category cpo^{ep} of cpo's and "embedding-projection" pairs.