A Case Study to Illustrate the Use of Non-Convex Membership Functions for Linguistic Terms

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***Abstract*— Terms used in fuzzy systems are almost invariably normalised, convex and distinct. The shapes of these terms are generated by certain accepted membership functions: piecewise linear functions, Gaussians or Sigmoids are almost exclusively used. This paper extends previous work in which it was suggested that non-convex membership functions might be considered for use in the context of modelling human decision making utilising fuzzy expert systems. In particular, the merits of non-convex fuzzy sets are discussed and a case study is presented to investigate inferencing with non-convex fuzzy sets in a practical implementation. It is shown that it is indeed possible to build a fuzzy expert system featuring usual Mamdani style fuzzy inference in which a time-related non-convex fuzzy set is used together with 'traditional' fuzzy sets. An examination is made of the resultant output surface generated by four different sub-classes of non-convex membership functions.**

I. Introduction

This paper builds on ideas presented previously [1] in which the use of complex non-convex membership functions in the context of human decision making systems was suggested. In particular, in this paper, we attempt to address criticisms that were made as to whether the shapes being presented were really 'true', 'allowable' or in any way 'meaningful' membership functions.

Since Zadeh first introduced the concept of a fuzzy set [2] and subsequently went on to extend the notion via the concept of linguistic variables [3] the popularity and use of fuzzy logic has been extraordinary. Fuzzy principles have been applied to a huge and diverse range of problems such as aircraft flight control, robot control, car speed control, power systems, nuclear reactor control, fuzzy memory devices and the fuzzy computer, control of a cement kiln, focusing of a camcorder, climate control for buildings, shower control and mobile robots [4], [5]. The use of fuzzy logic is not limited to control. Other successful applications include, for example, stock tracking on the Nikkei stock exchange [5], information retrieval [6] and the scheduling of community transport [7].

We are particularly interested in the role of linguistic variables, and their associated terms as used in the fuzzy inferencing process. Within the general category of inferencing (rule-based) systems there are two broad aspects: *control systems* and *expert systems* (emulating human reasoning). Although human reasoning has been investigated since the inception of fuzzy logic (e.g. [3], [8]), by far the majority of published work has been concerned with fuzzy control. Indeed, both the two main methods of implementing fuzzy inferencing, namely the Mamdani method and the Takagi-Sugeno method, were introduced to solve control applications [9], [10].

This historical bias towards the control domain has, we believe, led to a relative neglect of aspects of inferencing in the context of human decision making. Thus, there has been a tendency to restrict membership functions (mfs) to well-known forms. Triangular, left-shoulder, right-shoulder and trapezoidal, or more generally piecewise linear, functions are common. Also used are standard Gaussian or Sigmoid type curves. We suggest that in many applications involving the modelling of human decision making (expert systems) the more traditional mfs do not provide a wide enough choice for the system developer. They are therefore missing an opportunity to, potentially, produce simpler or better systems. This paper highlights a number of membership functions outside the paradigm of fuzzy control. In particular, the merits of non-convex fuzzy sets are discussed and a case study is presented which investigates whether it is possible to build an expert system featuring usual Mamdani style fuzzy inference in which a time-related non-convex fuzzy set is used together with traditional fuzzy sets. It is shown that this is indeed possible and an examination is made of the resultant output surface generated by four different sub-classes of non-convex mfs.

The rest of the paper is structured as follows. Sections II and III are abbreviated from the previous paper and are included for completness. The new case study of using a time-related non-convex membership function is presented in Section IV. Finally, Section V presents a discussion of the issues raised.

II. Conventions of Fuzzy Terms

To enable a discussion of membership functions, we need to formally define the terminology used. In this Section we restate accepted definitions for completeness.

Definition 1 (Linguistic variable): A linguistic variable [11] is characterised by a quintuple $(X, T(X), U, G, M)$ in which X is the name of the variable, $T(X)$ is the term set, U is a universe of discourse, G is a syntactic rule for generating the elements of $T(X)$ and M is a semantic rule for associating meaning with the linguistic values of X.

Definition 2 (Normal): A fuzzy set, A, is *normal* if $\exists x'$ such that $\mu_A(x') = 1$.

Definition 3 (Sub-normal): A fuzzy set, A, is *sub-normal* if it is not normal i.e. $\exists$ no x' such that $\mu_A(x') = 1$.

Definition 4 (Convex): A fuzzy set, A, is said to be *convex* if and only if all of its α-cuts are convex in the classical sense. That is, for each α-cut, A_α, for any $r, s \in A_\alpha$ and any $\lambda \in [0, 1]$ then $\lambda r + (1 - \lambda)s \in A_\alpha$.

Definition 5 (Non-convex): A fuzzy set, A, is said to be *non-convex* if it is not convex.

As well as being interested in sub-normal, non-convex fuzzy sets we also consider fuzzy sets that are *contained* in, or *included* in, another fuzzy set(s).

Definition 6 (Distinct): A fuzzy set, A, for a particular linguistic variable L, on the universe of discourse X is *distinct* from a fuzzy set, B (another term of L), on the universe of discourse X if and only if for all $x' \in X$ when $\mu_A(x') > 0$ then $\mu_B(x') = 0$ and when $\mu_B(x') > 0$ then $\mu_A(x') = 0$.

Definition 7 (Non-distinct): A fuzzy set, A, for a particular linguistic variable L, is *non-distinct* if $\exists$ a fuzzy set B (another term of L) such that A is not distinct from B.

Non-distinct fuzzy sets are also referred to as *overlapping* fuzzy sets. There are many types of non-distinct fuzzy sets. For clarity, we further define *partially overlapping* and *subsumed* fuzzy sets.

Definition 8 (Partially overlapping): A fuzzy set, A, on the universe of discourse X is *partially overlapping* another fuzzy set, B, on the universe of discourse X if and only if $\exists x'$ where $\mu_A(x') = max(\mu_A)$ but $\mu_B(x') \neq max(\mu_B)$, and $\exists x''$ where $\mu_B(x'') = max(\mu_B)$ but $\mu_A(x'') \neq max(\mu_A)$.

Definition 9 (Regular): Fuzzy terms that are normal, convex and distinct using the above definitions will be referred to as *regular* terms.

It is often implicitly accepted, and occasionally explicitly stated (e.g. [12], [13]), that the terms of a linguistic variable should be *justifiable in number* (5 ± 2), *distinct*, *normalised* and covering the entire universe of discourse.

III. Non-Convex Membership Functions

It would appear that the class of fuzzy sets which might have non-convex membership functions can be naturally split into three sub-classes:

- Those where the universe of discourse is not time-related. Such sets will be termed *elementary* non-convex membership functions.
- Those where the universe of discourse is time-related. Such sets will be termed *time-related* non-convex membership functions.
- Those which result from the inferencing process in the Mamdani method. Such sets are termed *consequent* non-convex membership functions.

1) Elementary Non-Convex Sets: Plausible discrete domain non-convex fuzzy sets which are not defined over a time-related universe of discourse are quite easy to imagine. There are three 'well-known' principles that govern the ideal number of people for forming a mountain rescue team:

1) there should be an odd number of people so that in any decision-making vote a simple majority is possible (i.e. voting does not result in ties);
2) three is not a good number to have, because there is a tendency to end up with a 2-1 split which causes the single person to feel resentful; and
3) too many people cause too many arguments.

Hence a discrete fuzzy set expressing the compatibility of various numbers of people with a suitable mountain-rescue team might look as in Fig. 1.

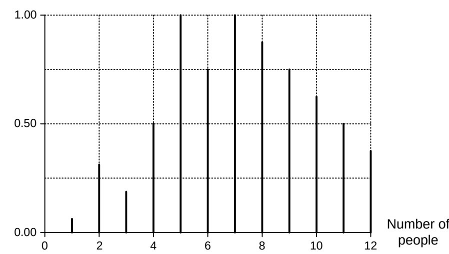

Fig. 1. A discrete non-convex set: suitability of number of people to comprise a mountain rescue team

Continuous domain non-convex fuzzy sets may be less common. Consider though, as an example, the desirability (drinkability) of a glass (cup) of milk according to the temperature of the milk. Most people (who like drinking milk) prefer it 'ice-cold' out of the fridge as opposed to room temperature (although actually 'ice-cold' refers to several degrees above freezing). Many people also agree that hot milk is also quite pleasant to drink. Hence a fuzzy set expressing the drinkability of milk by temperature might look like Fig. 2.

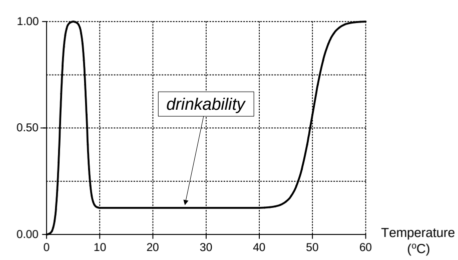

Fig. 2. Drinkability of milk by temperature

Alternatively, we could transform the representation of this concept by defining a variable *temperature* with perhaps four convex terms, *icy*, *cold*, *medium*, and *hot* (Fig. 3), an output set *drinkability* with two convex terms, *low* and *high* (Fig. 4), and an associated set of rules of the form:

IF *temp* is *cold* THEN *drinkability* is *high*
IF *temp* is *medium* THEN *drinkability* is *low*

IF *temp* is *high*   THEN *drinkability* is *high*

The drinkability for a given temperature could then be found by inputting the temperature into the above set of fuzzy rules, executing the rules and then defuzzifying the consequent set by, for example, the centroid method. A plot of the resultant drinkability obtained for each temperature is shown in Fig. 5. Note that the resultant set (Fig. 5) is also now sub-normal. Of course, it can be normalised to obtain a closer match to Fig. 2.

But how do we elicit the rules and membership functions to obtain the precise shape required? And why incur the additional time and effort of eliciting the 5-6 mfs required when it is simpler to elicit the set shown in Fig. 2 directly?

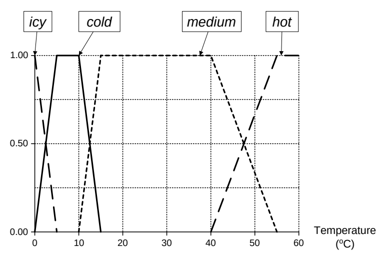

Fig. 3. Temperature as an input variable

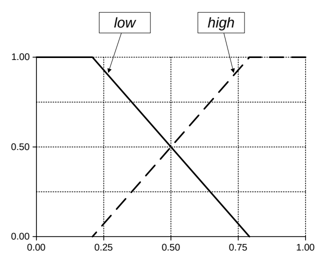

Fig. 4. Drinkability as an output variable

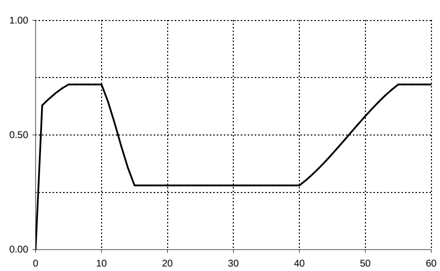

Fig. 5. Drinkability of milk elicited from the rule base: c.f. Fig. 2

2) Time-Related Non-Convex Sets: As a plausible time-related fuzzy set, suppose that an energy-supply company is creating an expert system to predict demand load. Amongst other factors that are considered may be the time of day and the prevailing temperature outside. We want to capture the concept that energy demand increases at mealtimes. Of course, mealtime is a fuzzy concept as breakfasts, lunches and dinners occur at variable times and indeed *may* occur at any time. Hence a non-convex fuzzy set for *mealtime* defined on time-of-day may be defined as shown in Fig. 6. Rules may then be created of the form:

IF *time-of-day* is *mealtime* AND *temp* is *low*
THEN *energy-demand* is *high*

Note that this fuzzy set is interesting as it is also sub-normal and never has a membership of zero.

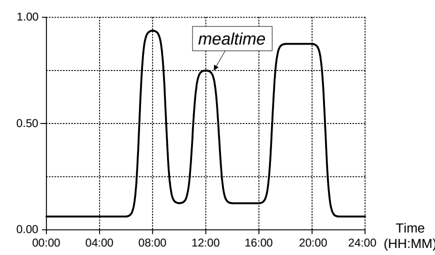

Fig. 6. Mealtime by time of day

Another example of a time-related fuzzy set is that of disposable income. By this we mean the amount of money (as a percentage of salary) somebody has available after paying out all their commitments (e.g. loan(s), electricity bill, etc.). It is well known that if you are young you have more disposable income than if you are middle aged (typically with a mortgage and children) and also as you get past middle age your disposable income increases. Depending on the application we could look at high disposable income in two ways. In the first case we might have a fuzzy set *high* for the linguistic variable *disposable income* as in Fig. 7 that has a domain which is the percentage of disposable income. However we may not know this information but have someone's age. In this case the domain would be age and we would have, for example, the non-convex fuzzy set in Fig. 8, in which the domain is time but the fuzzy set relates to income.

3) Consequent Non-Convex Sets: In a rule-based fuzzy system the result of, for example, Mamdani fuzzy inferencing, is a fuzzy set. Fig. 9 provides an example of a typical result of Mamdani inferencing (prior to defuzzification) where the antecedent and consequent fuzzy sets are triangular and/or trapezoidal. In the context of fuzzy control, this is usually defuzzified to produce the precise value required for the output variable. However, when modelling human decision making in a rule-based fuzzy system we might want to use the output directly as part of a chained inference process or we might like the output to be defuzzified somehow to a linguistic term. If it is argued that sub-normal, non-convex sets have no meaning, then what should be done with consequent sets? If, on the other hand, it is accepted that such consequent sets are meaningful, so that they can be interpretted or chained in further processing, then why shouldn't the original inputs be similarly formed. Hence, we believe we need to improve our understanding of sub-normal, non-convex sets in order to lead us towards better 'Computing with Words' [14].

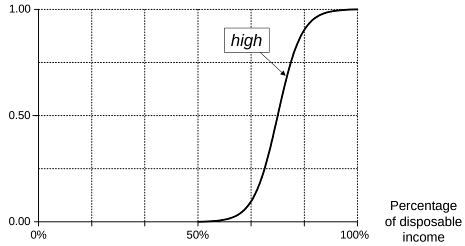

Fig. 7. A convex set *high* defined on the percentage of disposable income

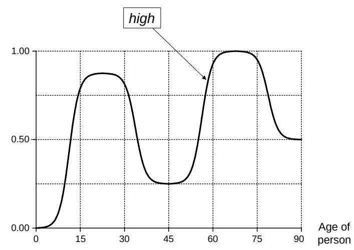

Fig. 8. A non-convex set *high-disposable-income* defined on the age

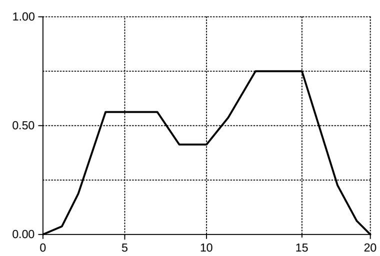

Fig. 9. An example of a typical non-convex, sub-normal consequent set

IV. Case Study

This section focuses on time related non-convex membership functions (mfs). It investigates whether the generated expert systems would work properly or not when time related non-convex mfs are used together with normal mfs. Suppose that an energy supply company is developing an expert system to predict demand load. Although there may be many factors that effect the demand load, *time* of the day and the prevailing *temperature* outside are chosen as the two system inputs. A simple model is more appropriate at this stage since it is only aimed to demonstrate the feasibility of the system.

A. Methodology

The illustrative system consists of two input variables, *Time* and *Temperature*, and one output variable, *Energy Demand* as shown in Fig. 10 - 15. In addition to other usual membership functions, the *Time* variable is associated with the term *MealTime* which is a time-related non-convex membership function. In order to observe the influence of *MealTime* on the performance, four systems are created by only changing the term *MealTime* in each system. The four different shapes of *MealTime* mfs that are used are as follows:

Case 1: mf of MealTime is in [0,1]
Case 2: mf of MealTime is in [0.2,1]
Case 3: mf of MealTime is in [0,0.9]
Case 4: mf of MealTime is in [0.2,0.9]

We note that membership functions which never reach zero are also unconventional but, once again, we see no reason why this convention cannot be violated.

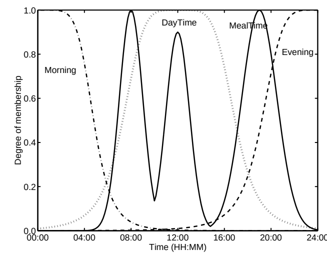

Fig. 10. Case 1: Time with MealTime mf in [0,1]

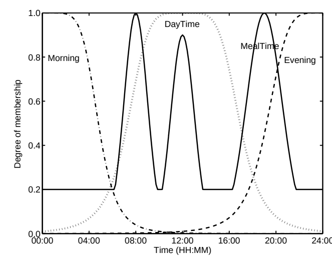

Fig. 11. Case 2: Time with MealTime mf in [0.2,1]

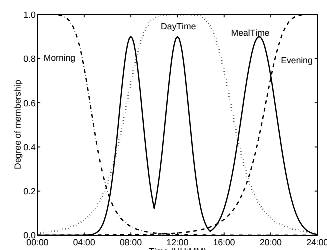

Fig. 12. Case 3: Time with MealTime mf in [0,0.9]

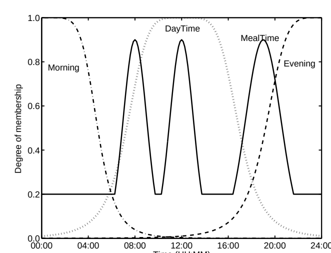

Fig. 13. Case 4: Time with MealTime mf in [0.2,0.9]

As seen in Fig. 10 - 13, *Time* is an input variable which consists of four membership functions; *Morning*, *DayTime*, *Evening*, and *MealTime* where the membership values of MealTime vary in the range of [0,1], [0.2,1], [0,0.9], and

[0.2,0.9], for each generated system respectively. The values of *time* is between 0 and 24.

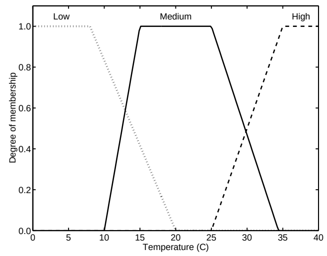

Fig. 14. Temperature as an input variable with normal mfs

Fig. 14 shows the input variable *Temperature* which consists of three membership functions; *Low*, *Medium*, and *High*. The same linguistic variable, *Temperature*, is used in all four systems. The values of temperature is between 0°C and 40°C.

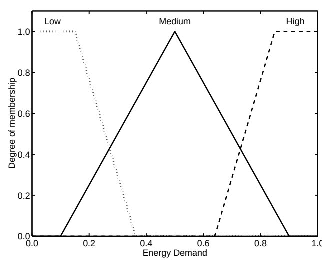

Fig. 15. Energy Demand as an output variable with normal mfs

Fig. 15 shows the output variable *Energy Demand* which consists of three membership functions; *Low*, *Medium*, and *High*. The same linguistic variable, *Energy Demand*, is used in all four generated systems. The values of Energy Demand varies between 0 and 1.

The following 12 rules are used within the four expert systems. These rules are purely illustrative at the moment and do not correspond to a real application. We are currently pursuing potential collaborations with energy generating companies in order to derive a fuzzy expert system to attempt to map real data. The rules are:

1. IF *Time* is *Mealtime* AND *Temperature* is *Low* THEN *Energy-Demand* is *High*
2. IF *Time* is *Mealtime* AND *Temperature* is *Medium* THEN *Energy-Demand* is *High*
3. IF *Time* is *Mealtime* AND *Temperature* is *High* THEN *Energy-Demand* is *Medium*
4. IF *Time* is *Evening* AND *Temperature* is *Low* THEN *Energy-Demand* is *High*
5. IF *Time* is *Evening* AND *Temperature* is *Medium* THEN *Energy-Demand* is *Medium*
6. IF *Time* is *Evening* AND *Temperature* is *High* THEN *Energy-Demand* is *Medium*
7. IF *Time* is *DayTime* AND *Temperature* is *Low* THEN *Energy-Demand* is *High*
8. IF *Time* is *DayTime* AND *Temperature* is *Medium* THEN *Energy-Demand* is *Medium*
9. IF *Time* is *DayTime* AND *Temperature* is *High* THEN *Energy-Demand* is *Low*
10. IF *Time* is *Morning* AND *Temperature* is *Low* THEN *Energy-Demand* is *Medium*
11. IF *Time* is *Morning* AND *Temperature* is *Medium* THEN *Energy-Demand* is *Medium*
12. IF *Time* is *Morning* AND *Temperature* is *High* THEN *Energy-Demand* is *Low*

B. Results

After time related non-convex membership function (*MealTime*) is added into linguistic variable *Time*, it is observed that all four systems work perfectly well (as expected). The prediction results of *energy demand* are shown as three dimensional plots in Fig. 16 - 19. Table I shows the summary of the results obtained from each system. The difference in the predicted results of each system is due to the different time-related non-convex *MealTime* term added to the variable *time*. It is clearly observed that the *energy demand* predictions have incorporated the information introduced by addition of the *MealTime* term. The surface plots, particularly in Fig. 17 and 19 exhibit a marked increase in output at the peak meal times.

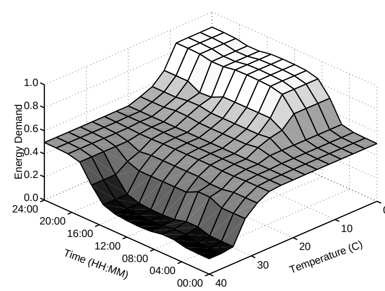

Fig. 16. Three dimensional plot of the result from Case 1

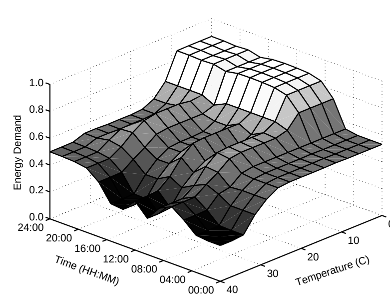

Fig. 17. Three dimensional plot of the result from Case 2

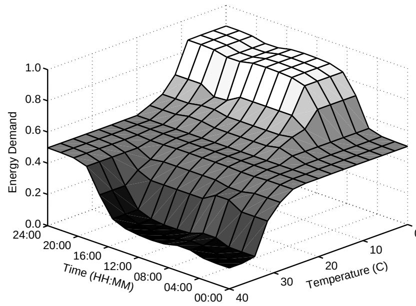

Fig. 18. Three dimensional plot of the result from Case 3

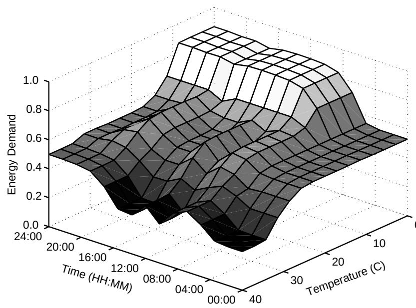

Fig. 19. Three dimensional plot of the result from Case 4

TABLE I
SUMMARY RESULTS FROM THE OUTPUTS

System	Mean	STD	Max	Min
Case 1	0.5696	0.1690	0.8679	0.1334
Case 2	0.5822	0.1554	0.8679	0.2659
Case 3	0.5679	0.1688	0.8679	0.1334
Case 4	0.5808	0.1552	0.8679	0.2659

The maximum predicted result is the same for each system as seen in Table I. This is 0.8679 because it is produced when the consequents of the rules are *High* = 1, *Low* = 0, and *Medium* = 0 and the COG method is used for defuzzification results in 0.8679.

V. DISCUSSION

This paper has described the well-known properties *normal*, *convex* and *distinct* used in the vast majority of terms implemented in fuzzy systems in the literature. It has been argued that while these properties are undoubtedly useful in the context of fuzzy control, they restrict the more general shapes of terms that might be used within linguistic variables in fuzzy systems. Examples have been given in which potentially useful terms do not adhere to each of these three properties, and a case study is presented to demonstrate the use of such non-regular membership functions in a fuzzy expert system.

We repeat our assertion in our previous paper that we are not cognitive scientists and are *not* arguing that the unusual membership shapes described in this paper are how such concepts are internally represented at a cognitive level. Whether concepts can be non-convex at a cognitive level has been discussed by, for example, Gärdenfors [15], in which he asserts that:

> "*most* properties expressed by simple words in natural language *can* be analysed as convex regions of a domain in a conceptual space" (our italics)

However, while he supports this (rather hedged) assertion with some examples, it remains far from proven. Whatever the reality at the cognitive level, we merely assert that non-regular fuzzy sets may be useful to consider when *modelling* human reasoning in a fuzzy system.

The results of the case study presented in this paper has shown that non-regular terms can be used in a fuzzy logic system and they can perform together with regular mfs. From these illustrations, we firmly believe that non-convex mfs such as *MealTime* featured in the *Time* (of day) variable are plausible, reasonable membership functions in the sense originally intended by Zadeh. Therefore, in the next stage of this study, we are planning to re-generate the systems and work with time-related real world data.

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