The University of Nottingham

SCHOOL OF COMPUTER SCIENCE and IT

A LEVEL 1 MODULE, AUTUMN 2006–2007

MATHEMATICS FOR COMPUTER SCIENCE

Time allowed 90 minutes

Candidates must NOT start writing their answers until told to do so.

Model Solutions
1  a)  
   i) true  
   ii) false  
   iii) false  
   iv) true  
   v) false  

   b)  
   i) false  
   ii) true  
   iii) true  
   iv) true  
   v) false  

   c)  
   i) true  
   ii) true  
   iii) true  
   iv) true  
   v) true  

   d)  
   i) false  
   ii) false  
   iii) true  
   iv) true  
   v) true  

   e)  
   i) true  
   ii) false  
   iii) true  
   iv) true  
   v) false  

2  a) i)
\[
\begin{array}{c|ccc}
 p & q & (p \Rightarrow q) & \land p \equiv p \land q \\
 \hline
 t & t & t & t \\
 t & f & f & t \\
 f & t & f & t \\
 f & f & t & f \\
\end{array}
\]

\[
\begin{array}{c|cc}
 p & q & \neg p \equiv p \iff q \\
 \hline
 t & t & f \\
 t & f & f \\
 f & t & f \\
 f & f & t \\
\end{array}
\]

\[
\begin{array}{c|cccc}
 p & q & p \Rightarrow q & \equiv & \neg p \iff \neg q \\
 \hline
 t & t & t & f & f \\
 t & f & f & t & f \\
 f & t & f & f & f \\
 f & f & t & f & f \\
\end{array}
\]

b) \[(p \iff q) \lor (p \Rightarrow q)\]
\[
= \{ \text{Definitions} \}\]
\[
(p \equiv p \lor q) \lor (q \equiv p \lor q)\]
\[
= \{ \text{distributivity of disjunction over equivalences} \}\]
\[
p \lor (q \equiv p \lor q) \equiv p \lor q \lor (q \equiv p \lor q)\]
\[
= \{ \text{distributivity of disjunction over equivalences} \}\]
\[
p \lor q \equiv p \lor p \lor q \equiv p \lor q \lor p \lor q \equiv p \lor q \lor p \lor q\]
\[
= \{ \text{idempotence of disjunction} \}\]
\[
p \lor q \equiv p \lor q \equiv p \lor q \equiv p \lor q\]
\[
= \{ \text{reflexivity of equivalences} \}\]
\[
\text{true}.
\]
(Note that associativity of equivalences is used implicitly from step 3 onwards.)

3 a) i) \[m < n+1 \leq m+1\]
\[
= \{ \text{meaning of continued inequality} \}\]
\[ m < n + 1 \land n + 1 \leq m + 1 \]
\[ = \{ \text{left operand: property of integers} \]
\[ \text{right operand: monotonicity of addition and multiplication} \} \]
\[ m \leq n \land n \leq m \]
\[ = \{ \text{antisymmetry} \} \]
\[ m = n \]

ii)
\[ 2 \times m < 2 \times n \leq 2 \times m + 1 \]
\[ = \{ \text{meaning of continued inequalities} \} \]
\[ 2 \times m < 2 \times n \land 2 \times n \leq 2 \times m + 1 \]
\[ = \{ \text{left conjunct: multiplication by 2 preserves inequalities,} \]
\[ \text{right conjunct: property of integers} \} \]
\[ m < n \land 2 \times n < 2 \times m + 2 \]
\[ = \{ \text{right conjunct: multiplication by 2 preserves inequalities} \} \]
\[ m < n \land n < m + 1 \]
\[ = \{ \text{there is no integer between } m \text{ and } m + 1 \} \]
\[ \text{false} \].

b)i) By instantiating \( k \) to \( m \div n \), the left side is truthified; hence the right side is truthified. Formally,

\[
\text{true} = \{ \text{definition, } k := m \div n \}
\]
\[
m \div n \leq m \div n \equiv (m \div n) \times n \leq m
\]
\[
= \{ \text{reflexivity of } \leq \}
\]
\[
\text{true} \equiv (m \div n) \times n \leq m
\]
\[
= \{ \text{reflexivity of } \equiv \}
\]
\[
(m \div n) \times n \leq m
\]

ii) We have, for all \( k \),
\[ k \leq m \div 1 \]
\[ = \{ \text{definition, } m \}
\]
\[ k \times 1 \leq m \]
\[ = \{ \text{arithmetic} \}
\]
\[ k \leq m \]

Substituting \( m \) for \( k \), we get \( m \leq m \div 1 \), and substituting \( m \div 1 \) for \( k \), we get \( m \div 1 \leq m \). Thus, by anti-symmetry of \( \leq \), \( m \div 1 = m \). Also,
\[ k \leq m \div m \]
\( \{ \text{definition, } n := m \} \)
\( k \times m \leq m \)
\( = \{ \text{ } 1 \times m = m, \text{ multiplication preserves inequalities} \}{\ (m \text{ is strictly positive)} \} \)
\( k \leq 1 \).
Substituting 1 for \( k \), we get \( 1 \leq m \div m \), and substituting \( m \div m \) for \( k \), we get \( m \div m \leq 1 \). Thus, by anti-symmetry of \( \leq \), \( m \div m = 1 \).

iii) Choose \( k \) equal to 0, \( n \) equal to 2 and \( m \) equal to 1.
From the definition, we have \( 0 \leq (−1) \div 2 \equiv 0 \times 2 \leq −1 \). That is, \( 0 \leq (−1) \div 2 \equiv \text{false} \). Also,
\[ k \leq 1 \div 2 \]
\[ = \{ \ \text{(3.1)} \ \} \]
\[ 2 \times k \leq 1 \]
\[ = \{ \text{integer inequalities} \} \]
\[ 2 \times k < 2 \]
\[ = \{ \text{multiplication preserves } < \} \]
\[ k < 1 \]
\[ = \{ \text{integer inequalities} \} \]
\[ k \leq 0 \).
It follows that \( 1 \div 2 = 0 \). But then, \( 0 \leq (−1) \div 2 \equiv \text{true} \). So \( (−1) \div 2 \) and \( −(1 \div 2) \) cannot be equal.

4 a) The three inscriptions are formalised as:
\[ (g \equiv G) \land (s \equiv \neg S) \land (l \equiv \neg s) \land (G \lor S \lor L). \]
In addition, we have that the portrait is placed inside at least one of the caskets. That is,
\[ G \lor S \lor L. \]
b) We calculate:
\[ (g \equiv G) \land (s \equiv \neg S) \land (l \equiv \neg s) \land (G \lor S \lor L) \]
\[ = \{ \text{reflexivity of } \equiv \} \]
\[ (g \equiv G) \land (s \equiv \neg S) \land \neg s \land (G \lor S \lor L) \]
\[ = \{ \text{substitution of equals for equals} \} \]
\[ (G \equiv g) \land S \land \neg s \land (G \lor S \lor L) \]
We conclude that the portrait could be in any of the caskets, but it is certainly in the silver casket. The inscription on the silver casket is false. Nothing can be concluded about the inscription on the lead casket. All that can be concluded about the gold casket is \( G \equiv g \).
a) This is \(0^2+1^2+2^2+3^2+4^2\), which equals 30.

b) This is \(2\times25 \leq 60 \land 2\times27 \leq 60 \land 2\times29 \leq 60\), which equals true.

c) \[
\langle \exists j : 0 \leq j \leq 250 : (\exists k : k = 1 \lor k = 3 : 2\times k = 3\times j) \rangle
\]
\[
= \{ \text{rearrangement and one-point} \}
\]
\[
\langle \exists j : 0 \leq j \leq 250 : 2\times 1 = 3\times j \lor 2\times 3 = 3\times j \rangle
\]
\[
= \{ \text{rearrangement, arithmetic} \}
\]
\[
\langle \exists j : 0 \leq j \leq 250 : 2 = 3\times j \lor \langle \exists j : 0 \leq j \leq 250 : 2 = j \rangle \rangle
\]
\[
= \{ \text{arithmetic, one-point rule} \}
\]
true.

d) \[
even. \langle \prod_{k} : 1 \leq k \leq 500 \land odd.k : k^2 \rangle
\]
\[
= \{ \text{distributivity and idempotence of disjunction} \}
\]
\[
\langle \exists k : 1 \leq k \leq 500 \land odd.k : even.k \rangle
\]
\[
= \{ \text{trading} \}
\]
\[
\langle \exists k : 1 \leq k \leq 500 \land odd.k \land even.k : true \rangle
\]
\[
= \{ odd.k \land even.k \equiv false \}
\]
\[
\langle \exists k : false : true \rangle
\]
\[
= \{ \text{empty range} \}
\]
false.

e) This is true (empty range rule and unit of conjunction is true).