Solutions to Exercises, Set 2

15 February 2015

1. (a) i. $\epsilon \notin L(N)$
   ii. $a \in L(N)$
   iii. $abc \notin L(N)$
   iv. $abd \in L(N)$
   v. $abca \in L(N)$
   vi. $abda \notin L(N)$
   vii. $aabcecbdb \in L(N)$
   viii. $aabbccbbd \notin L(N)$

[Marking: 1 mark each for a total of 8 marks]

(b) $\hat{\delta}_N(\{q_0, q_1\}, ab) = \hat{\delta}_N(\hat{\delta}_N(\{q_0, q_1\}, a), b)$
\[= \hat{\delta}_N(\delta_N(q_0, a) \cup \delta_N(q_1, a), b)\]
\[= \hat{\delta}_N(\{q_0, q_2\} \cup \{q_1\}, b)\]
\[= \hat{\delta}_N(\{q_0, q_1, q_2\}, b)\]
\[= \hat{\delta}_N(\delta_N(q_0, b) \cup \delta_N(q_1, b) \cup \delta_N(q_2, b), \epsilon)\]
\[= \delta_N(\{q_0\} \cup \{q_1, q_2\} \cup \emptyset, \epsilon)\]
\[= \hat{\delta}_N(\{q_0, q_1, q_2\}, \epsilon)\]
\[= \{q_0, q_1, q_2\}\]

[Marking: 7 marks]

2. (a) Starting from $S_A = \{q_0, q_1, q_3\}$, the start state of $D(A)$, we compute $\hat{\delta}_A(S_A, x)$ for each $x \in \Sigma_A$. Whenever we encounter a state $P \subseteq Q_A$ of $D(A)$ that has not been considered before, we add $P$ to the table and proceed to tabulate $\hat{\delta}_A(P, x)$ for each $x \in \Sigma_A$. We repeat the process until no new states are encountered. Finally, we identify the initial state ($\rightarrow$ to the left of the state) and all accepting states ($*$ to the left of the state). Note that a DFA state is accepting if it contains at least one accepting NFA state (as this means it is possible to reach at least one accepting state on a given word, which means that word is considered to be in the language of the NFA).

\[
\begin{array}{|c|c|c|c|}
\hline
\delta_{D(A)} & a & b & c \\
\hline
\rightarrow & \{q_0, q_1, q_3\} & \{q_1, q_3\} \cup \emptyset \cup \emptyset & \{q_0\} \cup \emptyset \cup \{q_4\} & \{q_0\} \cup \{q_2\} \cup \emptyset \\
\{q_0, q_4\} & \emptyset \cup \emptyset = \emptyset & \emptyset \cup \{q_4\} & \{q_0\} \cup \emptyset = \{q_2\} & \{q_0\} \cup \emptyset = \{q_2\} \\
\* & \{q_1, q_3\} & \{q_1, q_3\} \cup \{q_4\} & \{q_0\} \cup \emptyset \cup \{q_4\} & \{q_0\} \cup \emptyset = \{q_0\} \\
\* & \{q_0, q_2\} & \emptyset \cup \emptyset = \emptyset & \{q_0\} \cup \emptyset = \{q_0\} & \{q_0\} \cup \emptyset = \{q_0\} \\
\* & \{q_4\} & \{q_4\} & \{q_4\} & \emptyset \\
\* & \{q_2\} & \emptyset & \emptyset & \emptyset \\
\* & \{q_1, q_3, q_4\} & \emptyset \cup \emptyset \cup \{q_4\} & \emptyset \cup \{q_4\} \cup \{q_4\} & \{q_2\} \cup \emptyset \cup \emptyset \\
\{q_0\} & \{q_1, q_3\} & \{q_0\} & \{q_0\} & \{q_0\} \\
\hline
\end{array}
\]
(Note that we only needed to consider 9 states. That is a lot fewer than the \(2^5 = 32\) possible states in this case. \(32 - 9 = 23\) states are thus not reachable from the initial state, and we do not need to worry about those.)

[Marking: 20 marks]

(b) We can now draw the transition diagram for \(D(A)\):

Accepting states have been marked by outgoing arrows in this case. That is an alternative to the double circle. [Marking: 10 marks]

3. (a) Note: these are not necessarily the only possibilities, nor necessarily the “simplest” ones in any formal sense. But they are all fairly simple, and your answers should not be much more complicated.

i. \((a + c)^*b(a + b + c)^*\)

ii. \((a + b)^*(b + c)^*\)

[Marking: 5 marks each for a total of 10 marks]

(b) i.

\[
L((ab + c + \epsilon)dd) = L(ab + c + \epsilon)L(d)L(d) = (L(ab) \cup L(c) \cup L(\epsilon))L(d)L(d) = (L(a)L(b) \cup L(c) \cup L(\epsilon)L(d)L(d) = \{\{a\} \{b\} \cup \{c\} \cup \{\epsilon\}\{d\}\{d\} = \{\{ab\} \cup \{c\} \cup \{\epsilon\}\{dd\} = \{ab, c, \epsilon\}\{dd\} = \{abby, cdd, dd\}
\]

\[
L(EF) = L(E)L(F) = \{L(E + F) = L(E) \cup L(F)\}
\]

\[
L(x) = \{x\}
\]

\[
\{\text{Concatenation of Languages}\}
\]

\[
\{\text{Set union}\}
\]

\[
\{\text{Concatenation of Languages}\}
\]
ii.

\[\begin{align*}
L((a^*0 + cb)c) &= \{L(EF) = L(E)L(F)\} \\
L(a^*0 + cb)L(c) &= \{L(E + F) = L(E) \cup L(F)\} \\
(L(a^*0) \cup L(cb))L(c) &= \{L(\emptyset) = \emptyset, L(\epsilon) = \{\epsilon\}\} \\
(L(a^*)L(\emptyset) \cup L(\epsilon)L(b))L(c) &= \{L(E^*) = (L(E))^*\} \\
(L(a^*)\emptyset \cup \{\epsilon\}L(b))L(c) &= \{L(x) = \{x\}\} \\
((a)^*\emptyset \cup \{\epsilon\}\{b\}\{c\}) &= \{L(\emptyset) = \emptyset, \{\epsilon\}L = L\} \\
(\emptyset \{b\}\{c\}) &= \{\emptyset \cup L = L\} \\
\{b\}\{c\} &= \{\text{Concatenation of Languages}\}
\end{align*}\]

[Marking: 10 marks each for a total of 20 marks]

4. Construct an NFA \(A\) for \((a(b + c))^*\) according to the lecture notes. Start with the innermost subexpressions and then join the NFAs together step by step. (I have named the states according to how they will be named in the final NFA to make it easier to follow the derivation. It is OK to leave states unnamed to the end.)

NFA for \(a\):

\[\begin{align*}
0 \xrightarrow{a} 6
\end{align*}\]

NFA for \(b + c\):

\[\begin{align*}
1 \xrightarrow{b} 2 \\
3 \xrightarrow{c} 4
\end{align*}\]

Join the above two NFAs to obtain an NFA for \(a(b + c)\):

\[\begin{align*}
0 \xrightarrow{a} 6 \quad 1 \xrightarrow{b} 2 \\
0 \xrightarrow{a} 3 \quad 1 \xrightarrow{a} 6 \\
3 \xrightarrow{c} 4
\end{align*}\]

The last step is to carry out the construction corresponding to the \(^*\) operator. States 1 and 3 both immediately precede a final state, and we should thus add corresponding transition edges from those back to all start states. But there is only one start state, 0, so only one edge from each. Additionally, we must not
forget to add an extra start state which is also final (here state 5) to ensure the NFA accepts $\epsilon$. Finally, state 6 is manifestly now a "dead end" and can thus be eliminated:

Note that the isolated state 5 also is part of the same automaton.

[Marking: 25 marks]