G52MAL
Machines and Their Languages
Lecture 18 & 19
Turing Machines and Decidability

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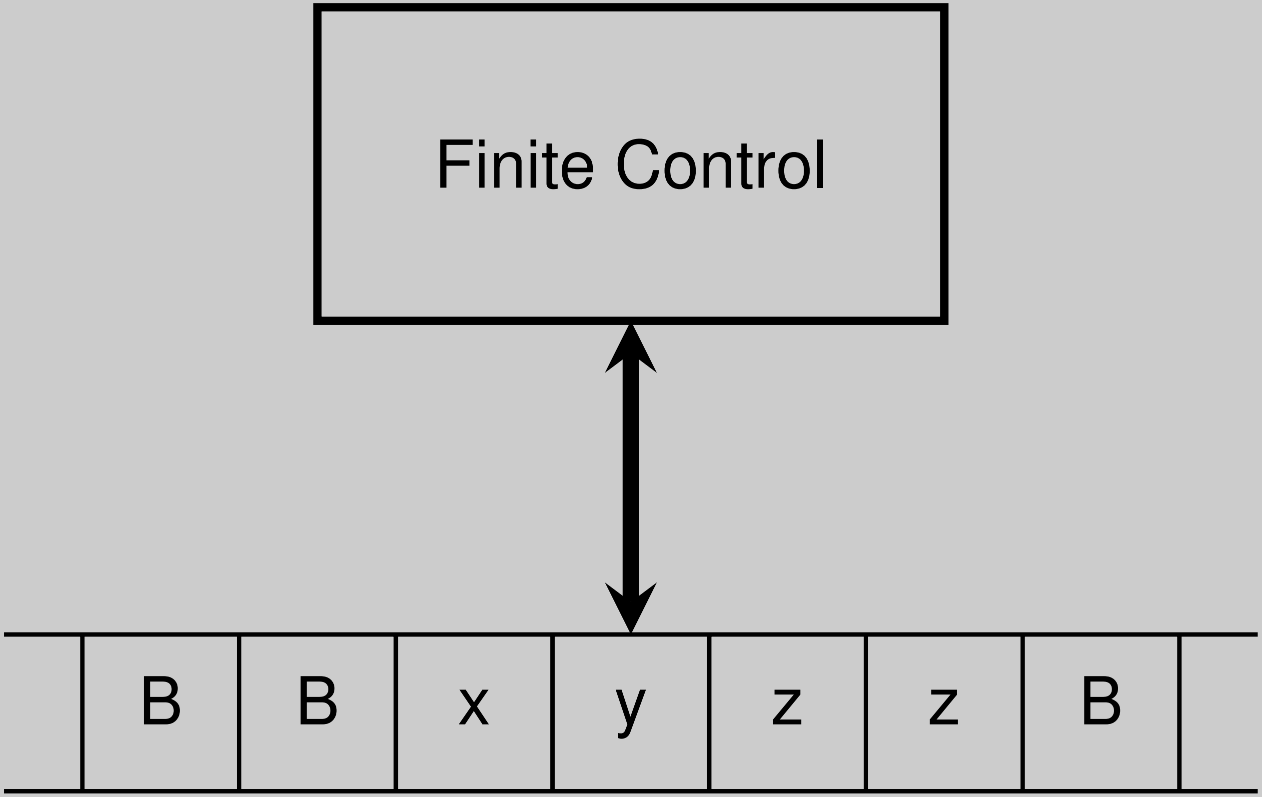
- Mainly used to study the ***notion of computation***: what (exactly!) can computers do (given sufficient time and memory) and what can they not do.
- There are other notions of computation, e.g. the ***λ-calculus*** introduced by Alonzo Church (G54FOP!).

Turing Machines (2)



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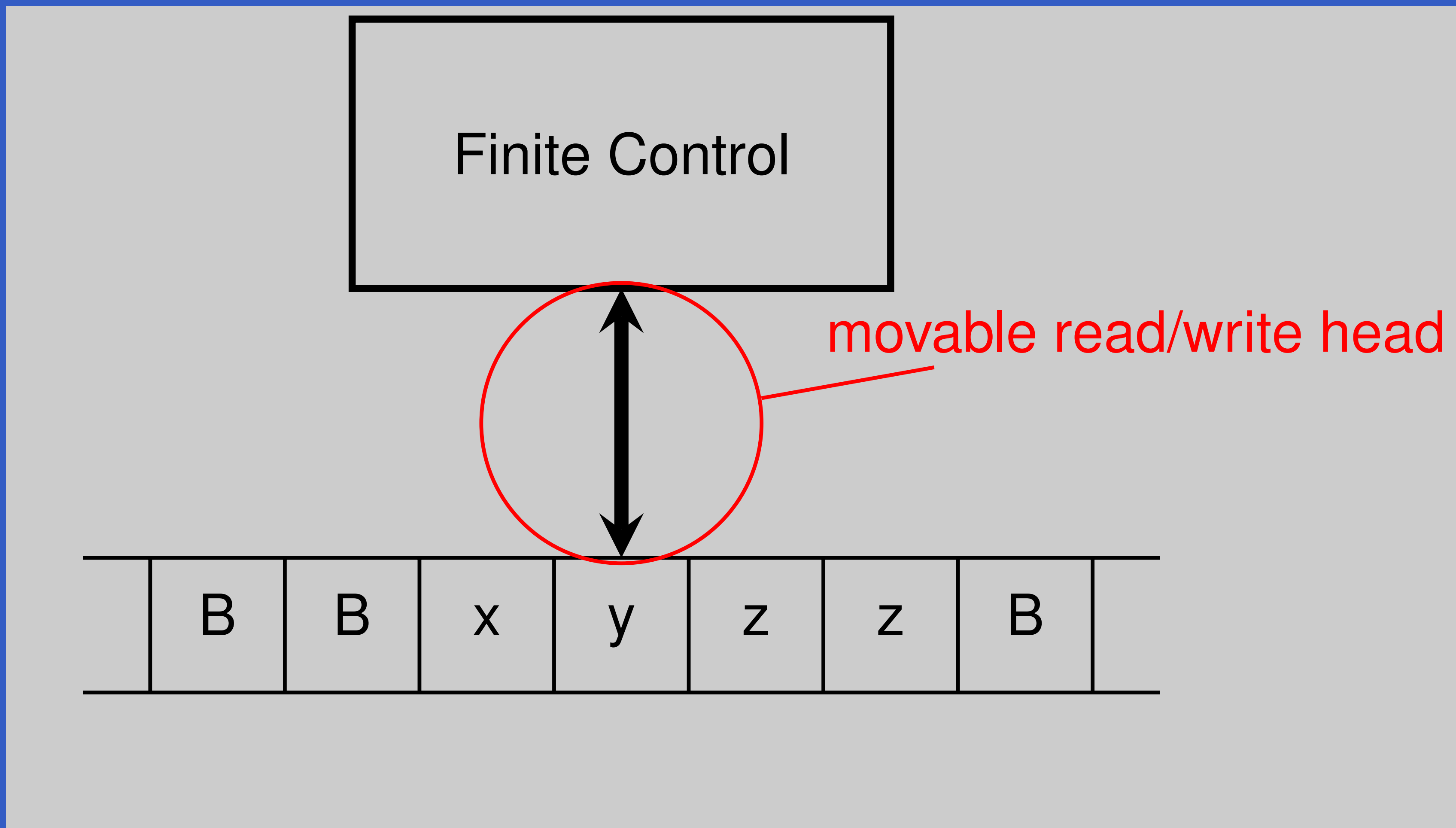

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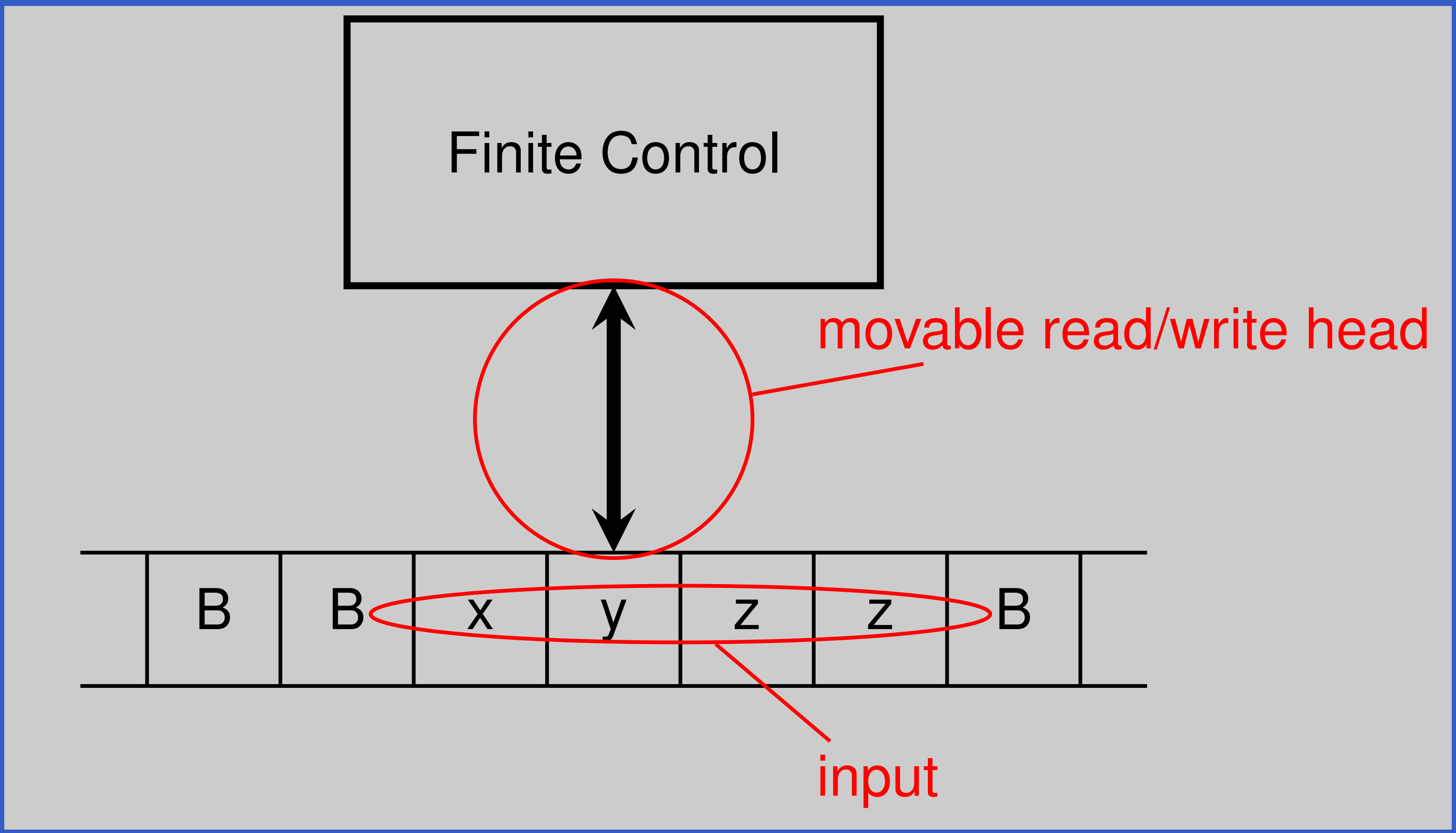

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- All suggested notions of computation have so far proved to be equivalent.
- ***The Church-Turing Thesis***: “Every function which would naturally be regarded as ‘computable’ can be computed by a TM”.
- At first, given how simple TMs are, it may seem surprising they can do much at all. E.g. how can they even add or multiply?
- We will see that a TM at least is more expressive than a PDA.

Definition of a Turing Machine

A TM $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ where

- Q is a finite set of states
- Σ is the input alphabet
- Γ is the tape alphabet, $\Sigma \subset \Gamma$ (finite)
- $\delta \in Q \times \Gamma \to \{\mathrm{stop}\} \cup Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ is the transition function
- $q_0 \in Q$ is the initial state
- B is the blank symbol, $B \in \Gamma$, $B \notin \Sigma$
- $F \subseteq Q$ are the accepting (final) states

Instantaneous Description (ID)

Instantaneous Descriptions (ID) describe the ***state*** of a TM computation:

$$ID = \Gamma^* \times Q \times \Gamma^*$$

$(\gamma_L, q, \gamma_R) \in ID$ means:

- TM is in state q
- γ_L is the non-blank part of the tape to the ***left*** of the head.
- γ_R is the non-blank part of the tape to the ***right*** of the head, ***including*** the current position.

The Next State Relation (1)

The next state relation on ID:

$$\underset{M}{\vdash} \subseteq ID \times ID$$

Read

$$id_1 \underset{M}{\vdash} id_2$$

"TM M moves in one step from id_1 to id_2."

The Next State Relation (2)

Let $q, q' \in Q$, $x, y, z \in \Gamma$, $\gamma_L, \gamma_R \in \Gamma^*$

1. $(\gamma_L, q, x\gamma_R) \underset{M}{\vdash} (\gamma_L y, q', \gamma_R)$ if $\delta(q, x) = (q', y, R)$
2. $(\gamma_L z, q, x\gamma_R) \underset{M}{\vdash} (\gamma_L, q', zy\gamma_R)$ if $\delta(q, x) = (q', y, L)$
3. $(\epsilon, q, x\gamma_R) \underset{M}{\vdash} (\epsilon, q', By\gamma_R)$ if $\delta(q, x) = (q', y, L)$
4. $(\gamma_L, q, \epsilon) \underset{M}{\vdash} (\gamma_L y, q', \epsilon)$ if $\delta(q, B) = (q', y, R)$
5. $(\gamma_L z, q, \epsilon) \underset{M}{\vdash} (\gamma_L, q', zy)$ if $\delta(q, B) = (q', y, L)$
6. $(\epsilon, q, \epsilon) \underset{M}{\vdash} (\epsilon, q', By)$ if $\delta(q, B) = (q', y, L)$

The Language of a TM (1)

$$L(M) = \{w \in \Sigma^* \mid (\epsilon, q_0, w) \overset{*}{\underset{M}{\vdash}} (\gamma_L, q, \gamma_R) \ \wedge \ q \in F\}$$

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A TM stops if it reaches an accepting state.

A TM stops in a non-accepting state if the transition function returns stop for that state and current tape input.

However, it may also ***never*** stop!

This is unlike the machines we have encountered before.

The Language of a TM (2)

If a particular TM M ***always*** stops, either in an accepting or a non-accepting state, then M ***decides*** $L(M)$.

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What may come as a surprise is that there are languages for which a TM ***necessarily*** cannot decide membership; i.e., will loop on some inputs.

Example

Construct a TM that accepts the language $\{a^n b^n c^n \mid n \in \mathbb{N}\}$.

This is a language that cannot be defined by a CFG or recognized by a PDA.

On the whiteboard.

There are many TM similators on-line. Try this (or some other) example with one of those. E.g.:

`http://ironphoenix.org/tm`

Recursive Language

L is ***recursive*** if $L = L(M)$ for a TM M such that

1. if $w \in L$, then M accepts w (and thus halts)
2. if $w \notin L$, then M eventually halts without ever entering an accepting state.

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We also say that M ***decides*** L.

Recursively Enumerable (RE) Language

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Why “recursively enumerable”?

Because it is possible to construct a TM that enumerates all strings in such a language in some order. (But it will necessarily keep trying to enumerate strings forever.)

Decidable and Undecidable

There are even languages that have no TM! The non-RE languages.

- Decidable: a language or problem (encoded as a language) that is recursive.
- Undecidable: a language or problem that is RE but not recursive, or non-RE.

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(In fact, a simple cardinality argument shows that most languages are non-RE: there are “many more” languages than there are TMs.)

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Famous example of a RE language that is not recursive; i.e. an undecidable language.

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Proof sketch on whiteboard.

Other Undecidable Problems

- Whether two programs (computable functions) are equal
- Whether a CFG is ambiguous
- Whether two CFGs are equivalent
- Rice's Theorem: Whether the language of a given TM has some particular ***non-trivial*** property. (Non-trivial: holds for some but not all languages.)

Rice's Theorem (1)

(After Henry Gordon Rice; also known as the Rice-Myhill-Shapiro theorem.)

Let C be a set of languages. Define

$$L_C = \{ \ M \mid L(M) \in C \ \}$$

where M ranges over all TMs. Then either L_C is empty, or it contains all TMs, or it is undecidable.

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For example, C might be the set of regular languages. As there are some TMs that recognise regular languages, but not all do, L_C is undecidable in this case.

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Caveat: Rice's theorem is concerned with properties of the ***language*** accepted by a TM, not about properties of the TM (code) itself. E.g., it is certainly decidable if a TM has at most 10 states, if it terminates in less than 100 steps, etc.

`http://www.eecs.berkeley.edu/~luca/cs172/noterice.pdf`