

G54FOP: Lecture 11

Untyped λ -calculus: Operational Semantics and Reduction Orders

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Capture-Avoiding Substitution (1)

$$\begin{aligned}
 [x \mapsto s]y &= \begin{cases} s, & \text{if } x \equiv y \\ y, & \text{if } x \not\equiv y \end{cases} \\
 [x \mapsto s](t_1 t_2) &= ([x \mapsto s]t_1) ([x \mapsto s]t_2) \\
 [x \mapsto s](\lambda y.t) &= \begin{cases} \lambda y.t, & \text{if } x \equiv y \\ \lambda y.[x \mapsto s]t, & \text{if } x \not\equiv y \wedge y \notin \text{FV}(s) \\ \lambda z.[x \mapsto s]([y \mapsto z]t), & \text{if } x \not\equiv y \wedge y \in \text{FV}(s), \\ & \text{where } z \text{ is fresh} \end{cases}
 \end{aligned}$$

where s, t and indexed variants denote lambda-terms; x, y , and z denote variables; $\text{FV}(t)$ denotes the free variables of term t ; and \equiv denotes syntactic equality.

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α - and η -conversion

- Renaming bound variables is known as **α -conversion**. E.g.

$$(\lambda x.x) \xrightarrow{\alpha} (\lambda y.y)$$

- Note that $(\lambda x.F x) G \xrightarrow{\beta} F G$ if x **not free** in F .

This justifies **η -conversion**:

$$\lambda x.F x \xrightarrow{\eta} F \quad \text{if } x \notin \text{FV}(F)$$

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Name Capture

Recall that

$$[x \mapsto t]F$$

means “substitute t for free occurrences of x in F .”

$$[x \mapsto y](\lambda x.x) =$$

$$[x \mapsto y](\lambda y.x)$$

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Capture-Avoiding Substitution (2)

The condition “ z is fresh” can be relaxed:

$$z \not\equiv x \wedge z \notin \text{FV}(s) \wedge z \notin \text{FV}(t)$$

is enough.

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Capture-Avoiding Substitution (4)

If we adopt the convention that terms that differ only in the names of bound variables are interchangeable in all contexts, then the following *partial* definition can be used as long as it is understood that an α -conversion has to be carried out if no case applies:

$$[x \mapsto s]y = \begin{cases} s, & \text{if } x \equiv y \\ y, & \text{if } x \not\equiv y \end{cases}$$

$$[x \mapsto s](t_1 t_2) = ([x \mapsto s]t_1) ([x \mapsto s]t_2)$$

$$[x \mapsto s](\lambda y.t) = \lambda y.[x \mapsto s]t, \quad \text{if } x \not\equiv y \wedge y \notin \text{FV}(s)$$

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Substitution Caveats

We have seen that there are some caveats with substitution:

- Must only substitute for **free** variables:

$$[x \mapsto t](\lambda x.x) \neq \lambda x.t$$

- Must avoid **name capture**:

$$[x \mapsto y](\lambda y.x) \neq \lambda y.y$$

“Substitution” almost always means **capture-avoiding substitution**.

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Capture-Avoiding Substitution (3)

A slight variation:

$$\begin{aligned}
 [x \mapsto s]y &= \begin{cases} s, & \text{if } x \equiv y \\ y, & \text{if } x \not\equiv y \end{cases} \\
 [x \mapsto s](t_1 t_2) &= ([x \mapsto s]t_1) ([x \mapsto s]t_2) \\
 [x \mapsto s](\lambda y.t) &= \begin{cases} \lambda y.t, & \text{if } x \equiv y \\ \lambda y.[x \mapsto s]t, & \text{if } x \not\equiv y \wedge y \notin \text{FV}(s) \\ [x \mapsto s](\lambda z.[y \mapsto z]t), & \text{if } x \not\equiv y \wedge y \in \text{FV}(s), \\ & \text{where } z \notin \text{FV}(s) \\ & \wedge z \notin \text{FV}(t) \end{cases}
 \end{aligned}$$

Homework: Why isn't $z \not\equiv x$ needed in this case?

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Op. Semantics: Call-By-Value (1)

Abstract syntax:

$$\begin{array}{l}
 t \rightarrow \begin{array}{l} \text{terms:} \\ x \quad \text{variable} \\ | \lambda x.t \quad \text{abstraction} \\ | t t \quad \text{application} \end{array}
 \end{array}$$

Values:

$$\begin{array}{l}
 v \rightarrow \begin{array}{l} \text{values:} \\ \lambda x.t \quad \text{abstraction value} \end{array}
 \end{array}$$

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Op. Semantics: Call-By-Value (2)

Call-by-value operational semantics:

$$\frac{t_1 \longrightarrow t'_1}{t_1 t_2 \longrightarrow t'_1 t_2} \quad (\text{E-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{v_1 t_2 \longrightarrow v_1 t'_2} \quad (\text{E-APP2})$$

$$(\lambda x.t) v \longrightarrow [x \mapsto v]t \quad (\text{E-APPABS})$$

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Op. Semantics: Call-By-Name

Call-by-name like normal order, but no evaluation under λ :

$$\frac{t_1 \longrightarrow t'_1}{t_1 t_2 \longrightarrow t'_1 t_2} \quad (\text{E-APP1})$$

$$(\lambda x.t_1) t_2 \longrightarrow [x \mapsto t_2]t_1 \quad (\text{E-APPABS})$$

Note: Argument not evaluated “prior to call”!

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The Church-Rosser Theorems (1)

Church-Rosser Theorem I:

For all λ -calculus terms t , t_1 , and t_2 such that $t \xrightarrow{*}_{\beta} t_1$ and $t \xrightarrow{*}_{\beta} t_2$, there exists a term

t_3 such that $t_1 \xrightarrow{*}_{\beta} t_3$ and $t_2 \xrightarrow{*}_{\beta} t_3$.

That is, β -reduction is **confluent**.

This is also known as the “diamond property”.

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Op. Semantics: Full β -reduction

Operational semantics for full β -reduction (non-deterministic). Syntax as before, but the syntactic category of values not used:

$$\frac{t_1 \longrightarrow t'_1}{t_1 t_2 \longrightarrow t'_1 t_2} \quad (\text{E-APP1})$$

$$\frac{t_2 \longrightarrow t'_2}{t_1 t_2 \longrightarrow t_1 t'_2} \quad (\text{E-APP2})$$

$$(\lambda x.t_1) t_2 \longrightarrow [x \mapsto t_2]t_1 \quad (\text{E-APPABS})$$

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Call-By-Value vs. Call-By-Name (1)

Exercises:

1. Evaluate the following term both by call-by-name and call-by-value:

$$(\lambda x.\lambda y.y) ((\lambda z.z z) (\lambda z.z z))$$

2. For some term t and some value v , suppose $t \xrightarrow{*}_{\beta} v$ in, say 100 steps. Consider $(\lambda x.x x) t$ under both call-by-value and call-by-name. How many steps of evaluation in the two cases? (Roughly)

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The Church-Rosser Theorems (2)

Church-Rosser Theorem II:

If $t_1 \xrightarrow{*}_{\beta} t_2$ and t_2 is a normal form (no redexes), then t_1 will reduce to t_2 under normal-order reduction.

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Op. Semantics: Normal-Order

Normal-order operational semantics is somewhat awkward to specify. Like full β -reduction, except left-most, outermost redex first.

Call-By-Value vs. Call-By-Name (2)

Questions:

- Do we get the same result (modulo termination issues) regardless of evaluation order?
- Which order is “better”?

Which Reduction Order? (1)

So, which reduction order is “best”?

- Depends on the application. Sometimes reduction under λ needed, sometimes not.
- Normal-order reduction has the best possible termination properties: if a term has a normal form, normal-order reduction will find it.

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Which Reduction Order? (2)

- In terms of reduction steps (fewer is more efficient), none is strictly better than the other. E.g.:
 - Call-by-value may run forever on a term where normal-order would terminate.
 - Normal-order often duplicates redexes (by substitution of reducible expressions for variables), thereby possibly duplicating work, something that call-by-value avoids.

Lazy Evaluation (1)

Lazy evaluation is an *implementation technique* that seeks to combine the advantages of the various orders by:

- evaluate on demand only, but
- evaluate any one redex at most once (avoiding duplication of work)

Idea: *Graph Reduction* to avoid duplication by explicit sharing of redexes.

Lazy Evaluation (2)

Result: normal-order/call-by-need semantics, but efficiency closer to call-by-value (when call-by-value doesn't do unnecessary work). However, there are inherent implementation overheads of lazy evaluation.

Lazy evaluation is used in languages like Haskell.