COMP4075: Lecture 4

Pure Functional Programming: Exploiting Laziness

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Recap: Lazy Evaluation (1)

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- A redex is evaluated ***only if needed***.
- ***Sharing*** employed to avoid duplicating redexes.
- Once evaluated, a redex is ***updated*** with the result to avoid evaluating it more than once.

As a result, under lazy evaluation, any one redex is evaluated at most once.

Recap: Lazy Evaluation (2)

Recall:

```
sqr x = x * x
dbl x = x + x
main =
   sqr (dbl (2+3))
```

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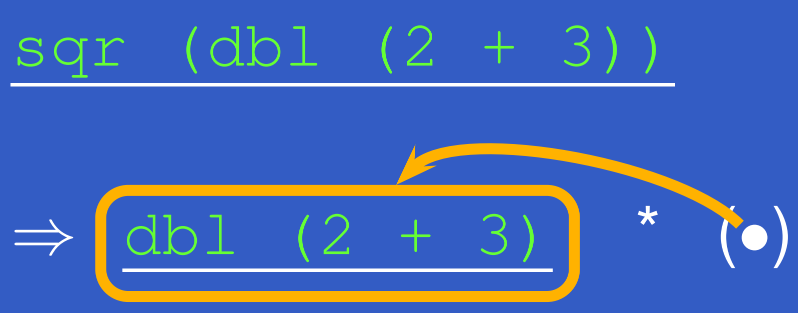

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⇒ dbl (2 + 3) * (•)
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```

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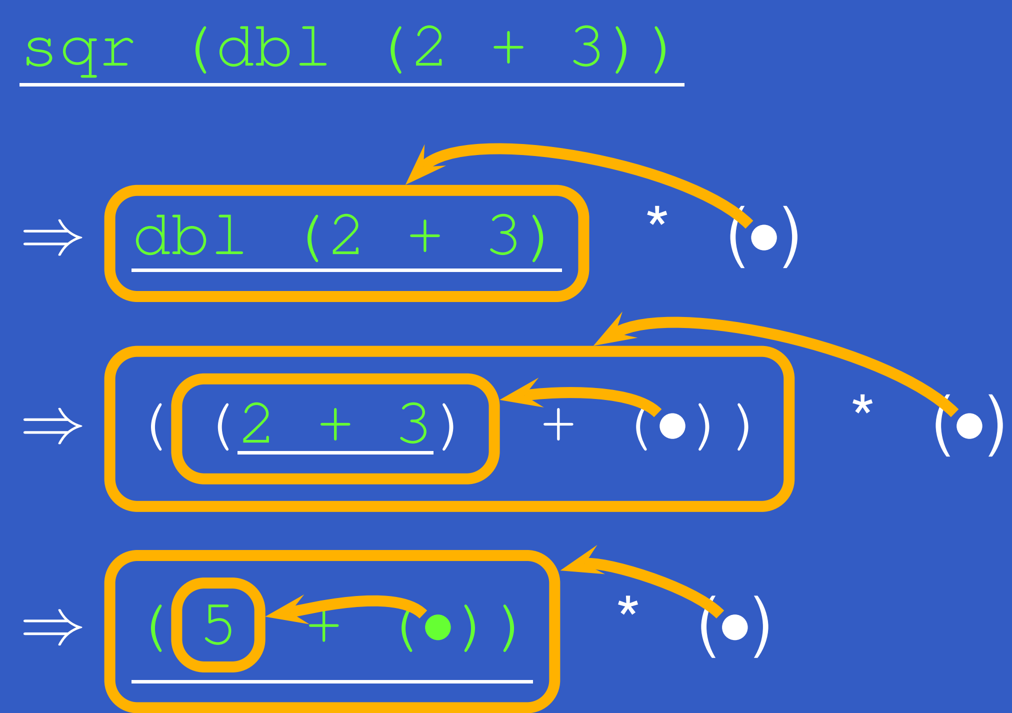

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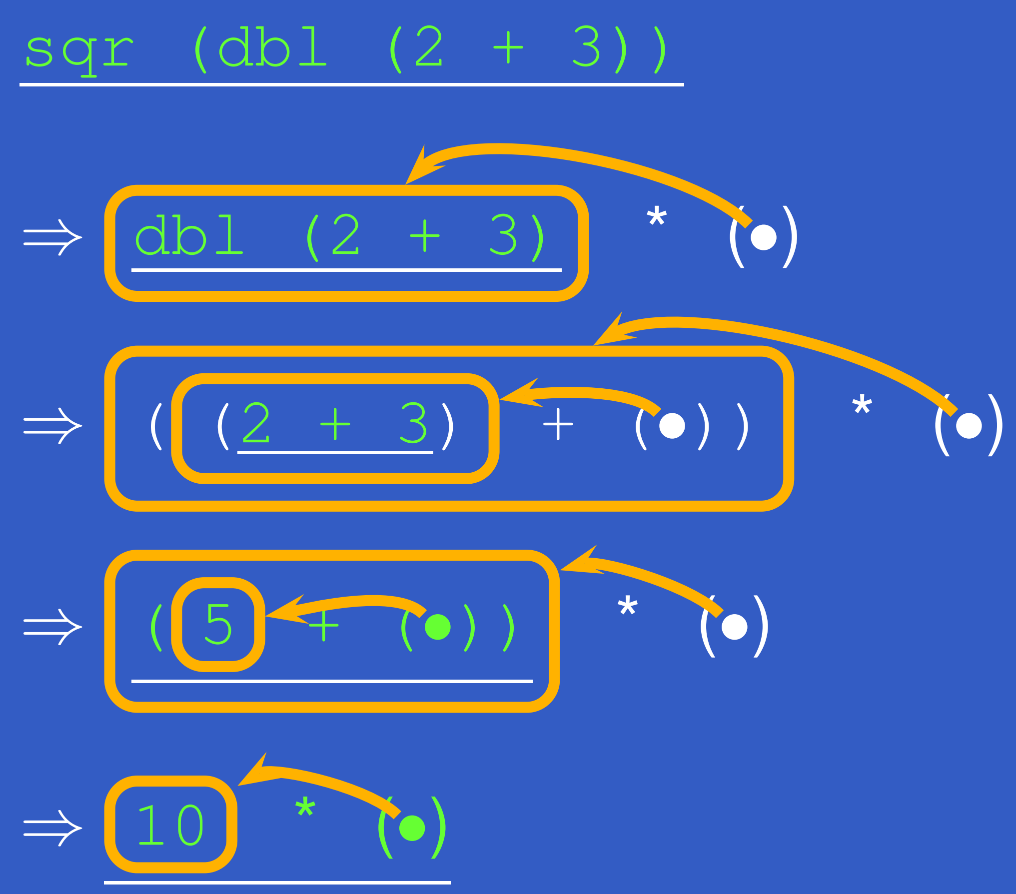

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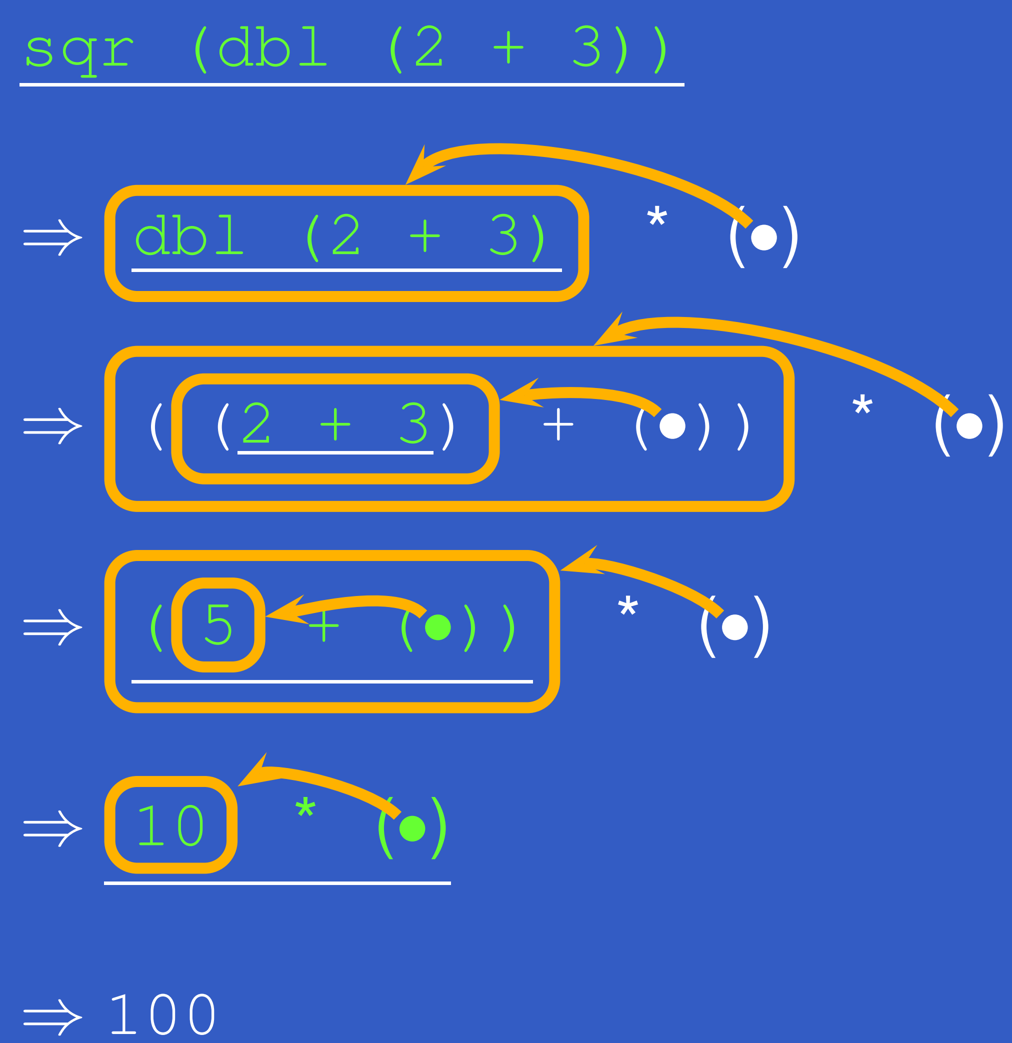


Circular Data Structures (1)

```
take 0 _      = []
take n []     = []
take n (x:xs) = x : take (n-1) xs

ones = 1 : ones

main = take 5 ones
```

Circular Data Structures (2)

main

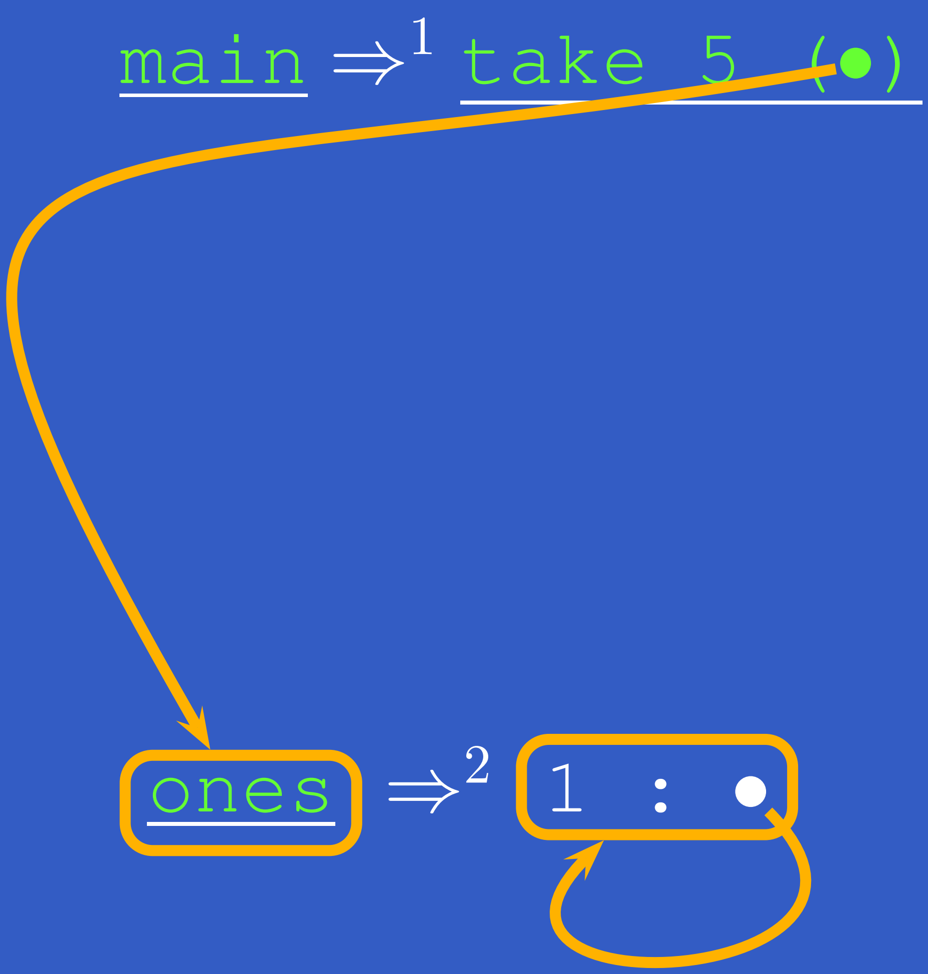
ones

Circular Data Structures (2)

`main` $\Rightarrow^1$ `take 5 (•)`

`ones`

Circular Data Structures (2)



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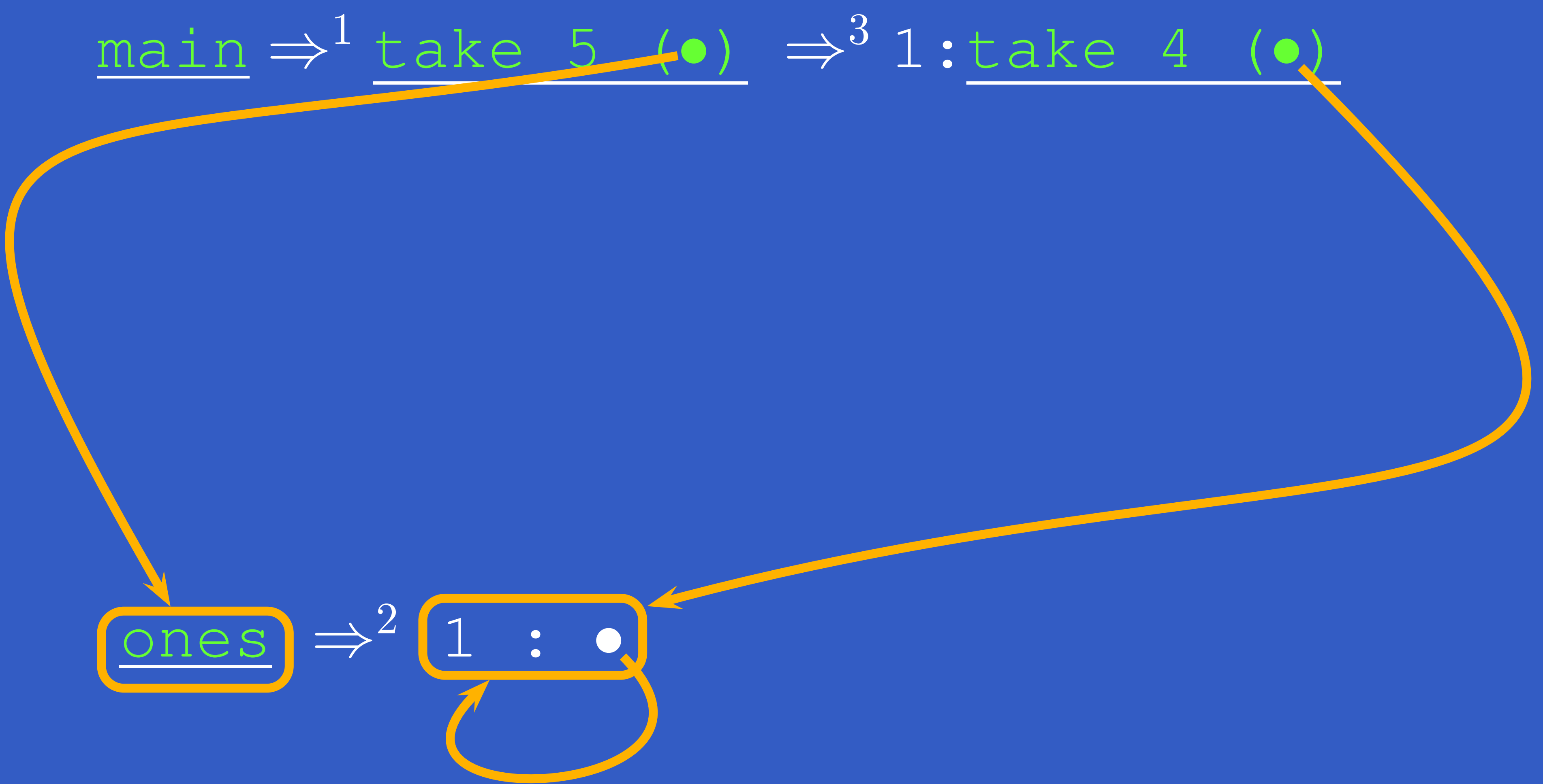

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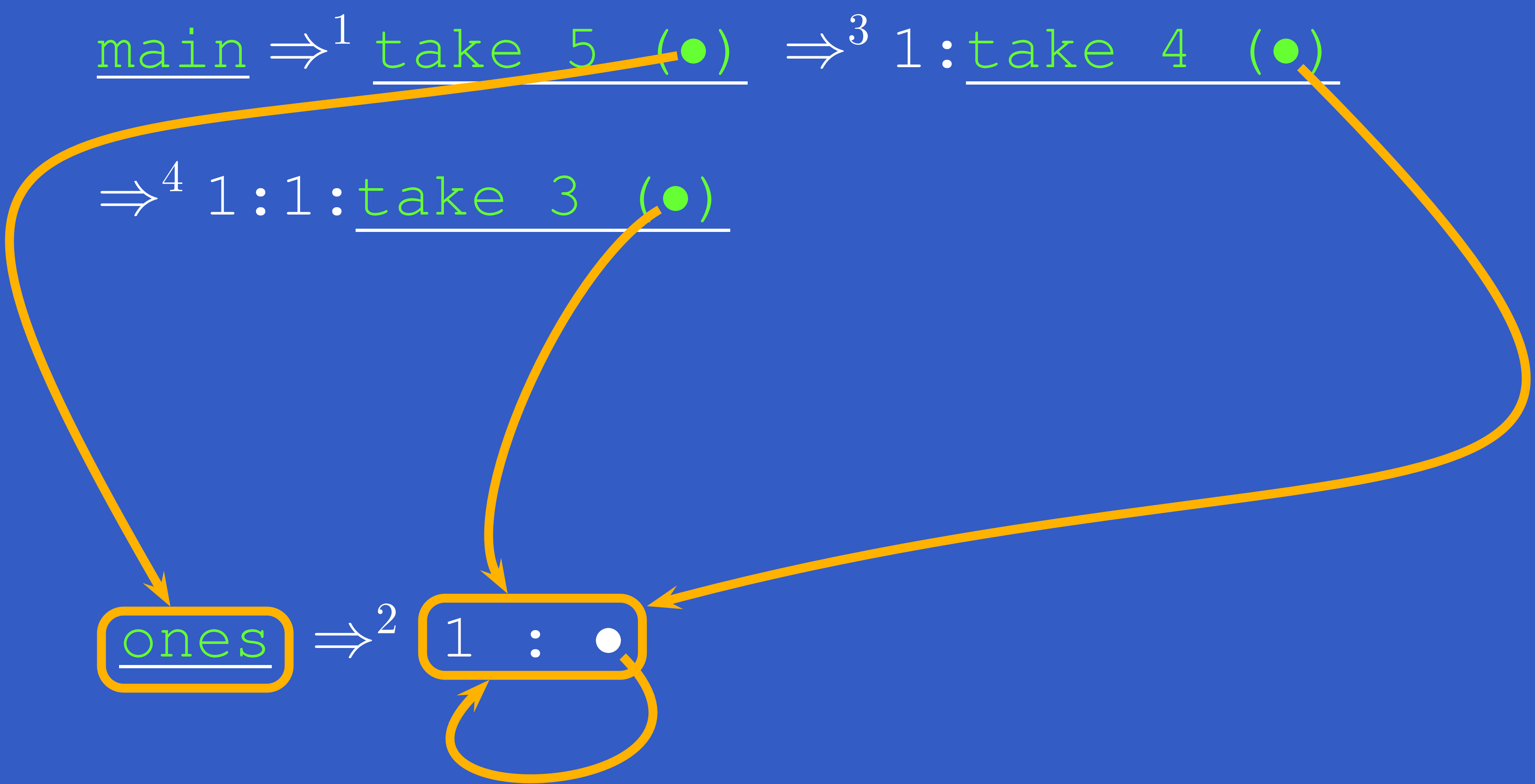

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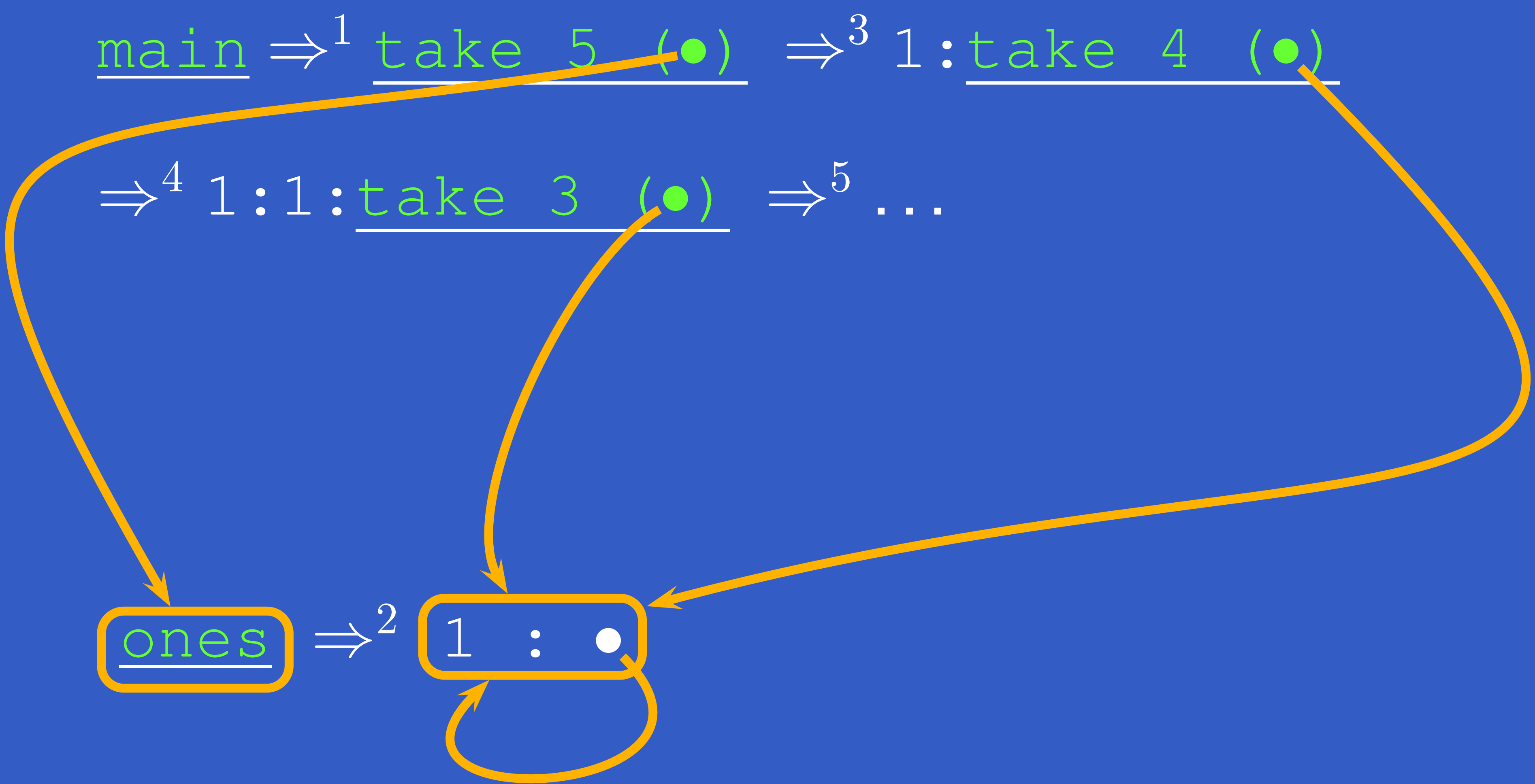

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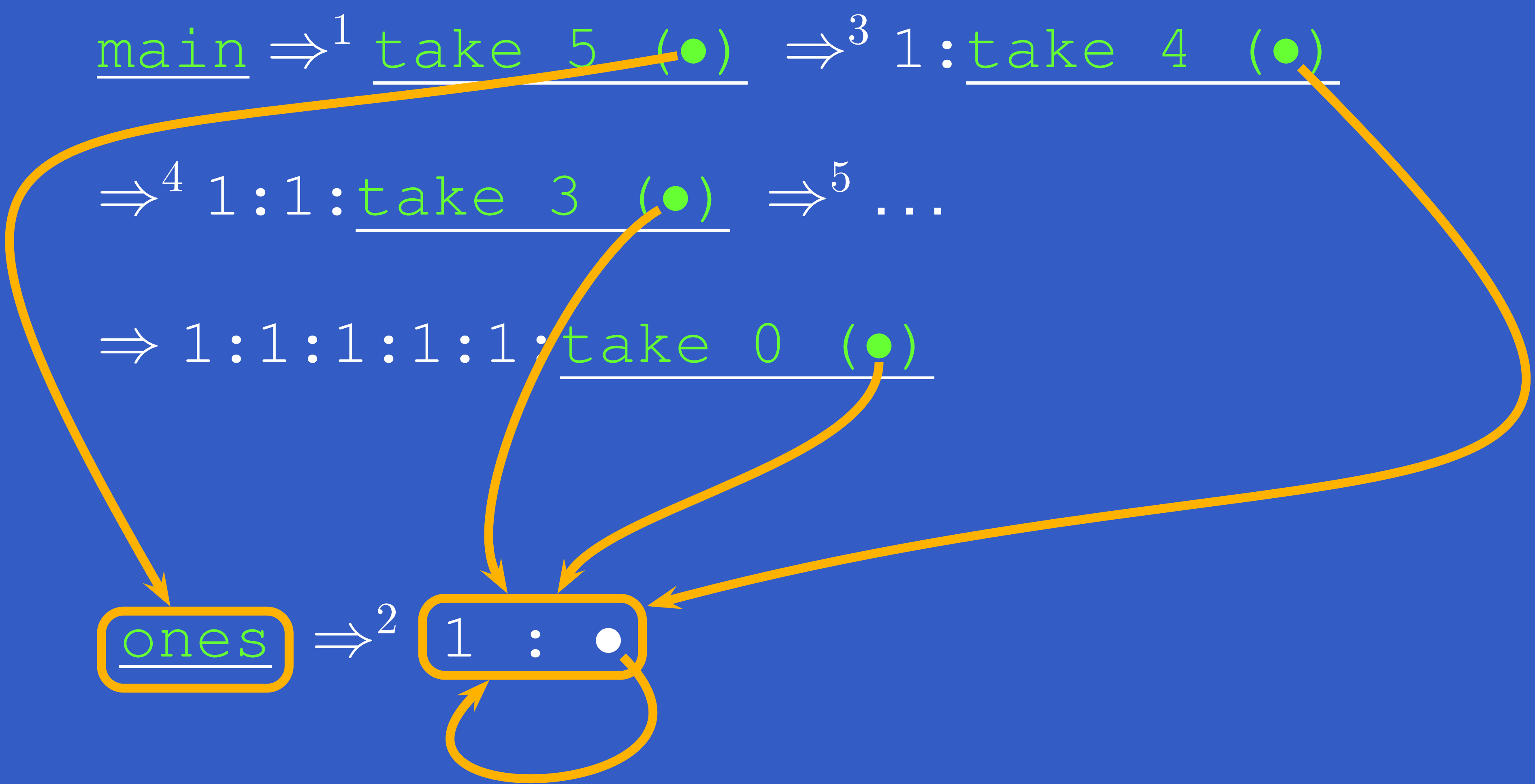

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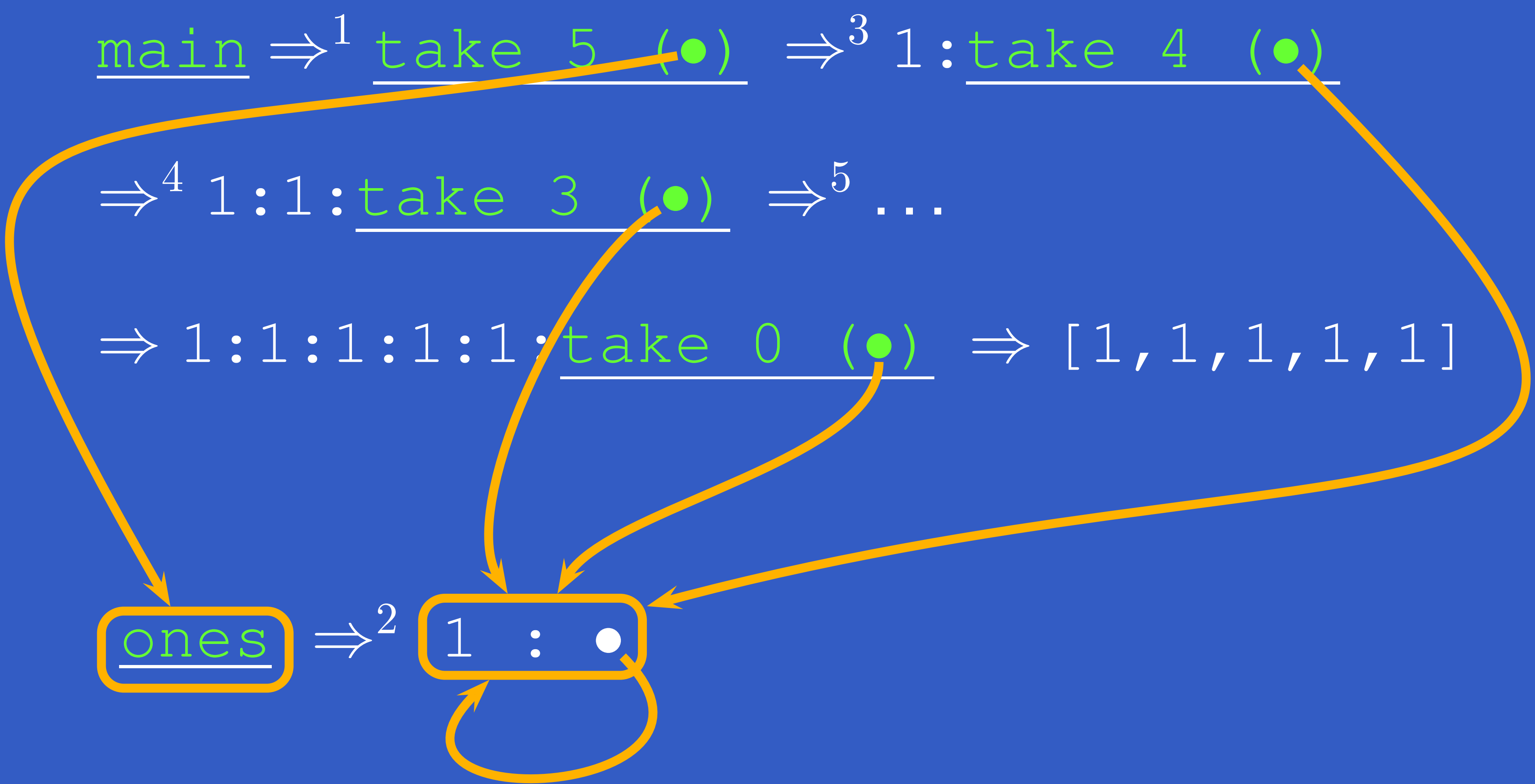

Exercise

Given the following tree type

```
data Tree = Empty
          | Node Tree Int Tree
```

define:

- An infinite tree where every node is labelled by `1`.
- An infinite tree where every node is labelled by its depth from the root node.

Exercise: Solution

```
treeOnes = Node treeOnes 1 treeOnes

treeFrom n = Node (treeFrom (n + 1))
                  n
                  (treeFrom (n + 1))

treeDepths = treeFrom 0
```

Circular Programming (1)

A non-empty tree type:

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A non-empty tree type:

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data Tree = Leaf Int | Node Tree Tree
```

Suppose we would like to write a function that replaces each leaf integer in a given tree with the ***smallest*** integer in that tree.

How many passes over the tree are needed?

One!

Circular Programming (2)

Write a function that replaces all leaf integers by a given integer, and returns the new tree along with the smallest integer of the given tree:

```
fmr :: Int -> Tree -> (Tree, Int)
fmr m (Leaf i) = (Leaf m, i)
fmr m (Node tl tr) =
    (Node tl' tr', min ml mr)
    where
        (tl', ml) = fmr m tl
        (tr', mr) = fmr m tr
```

Circular Programming (3)

For a given tree `t`, the desired tree is now obtained as the ***solution*** to the equation:

```
(t', m) = fmr m t
```

Thus:

```
findMinReplace t = t'
    where
        (t', m) = fmr m t
```

Intuitively, this works because `fmr` can compute its result without needing to know the ***value*** of `m`.

A Simple Spreadsheet Evaluator (1)

s

	a	b	c
1	c3 + c2		
2	a3 * b2	2	a2 + b2
3	7		a2 + a3

$\Rightarrow$

s′

	a	b	c
1	37		
2	14	2	16
3	7		21

```
s' = array (bounds s)
           [ (r, evalCell s' (s ! r))
           | r <- indices s ]
```

The evaluated sheet is again simply the ***solution*** to the stated equation. No need to worry about evaluation order. ***Any caveats?***

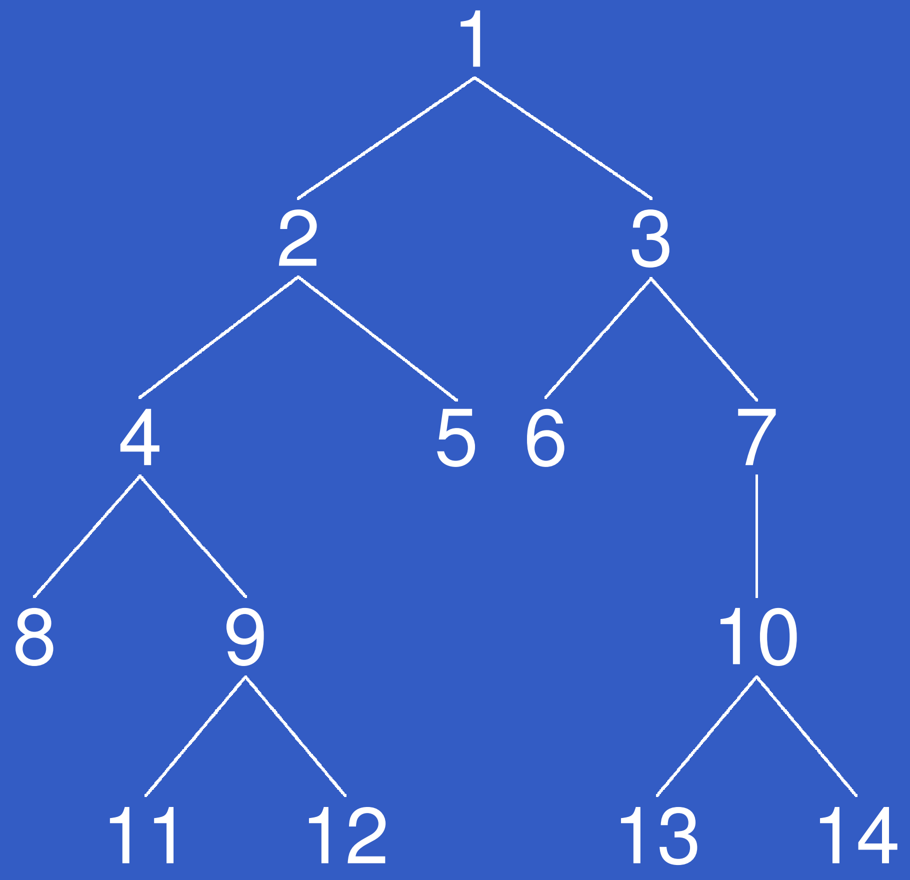
A Simple Spreadsheet Evaluator (2)

As it is quite instructive, let us develop this evaluator together. Some definitions to get us started:

```
type CellRef = (Char, Int)

type Sheet a = Array CellRef a

data BinOp = Add | Sub | Mul | Div

data Exp = Lit Double
         | Ref CellRef
         | App BinOp Exp Exp
```

Breadth-first Numbering (1)

Consider the problem of numbering a possibly infinitely deep tree in breadth-first order:



Breadth-first Numbering (2)

The following algorithm is due to G. Jones and J. Gibbons (1992), but the presentation differs.

Consider the following tree type:

```
data Tree a = Empty
            | Node (Tree a) a (Tree a)
```

Define:

$\text{width}\ t\ i$	The width of a tree t at level i (0 origin).
$\text{label}\ t\ i\ j$	The jth label at level i of a tree t (0 origin).

Breadth-first Numbering (3)

The following system of equations defines breadth-first numbering:

$$\begin{aligned} \text{label}\; t\; 0\; 0 &= 1 && (1)\\ \text{label}\; t\; (i+1)\; 0 &= \text{label}\; t\; i\; 0 + \text{width}\; t\; i && (2)\\ \text{label}\; t\; i\; (j+1) &= \text{label}\; t\; i\; j + 1 && (3) \end{aligned}$$

Note that $\text{label}\; t\; i\; 0$ is defined for ***all*** levels i (as long as the widths of all tree levels are finite).

Breadth-first Numbering (4)

The code that follows sets up the defining system of equations:

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- ***Streams*** (infinite lists) of labels are used as a ***mediating data structure*** to allow equations to be set up between adjacent nodes within levels and between the last node at one level and the first node at the next.

Breadth-first Numbering (4)

The code that follows sets up the defining system of equations:

- ***Streams*** (infinite lists) of labels are used as a ***mediating data structure*** to allow equations to be set up between adjacent nodes within levels and between the last node at one level and the first node at the next.
- Idea: the tree numbering function for a subtree takes a stream of labels for the ***first node*** at each level, and returns a stream of labels for the ***node after the last node*** at each level.

Breadth-first Numbering (5)

- As there manifestly are ***no cyclic dependences*** among the equations, we can entrust the details of solving them to the lazy evaluation machinery in the safe knowledge that a solution will be found.

Breadth-first Numbering (6)

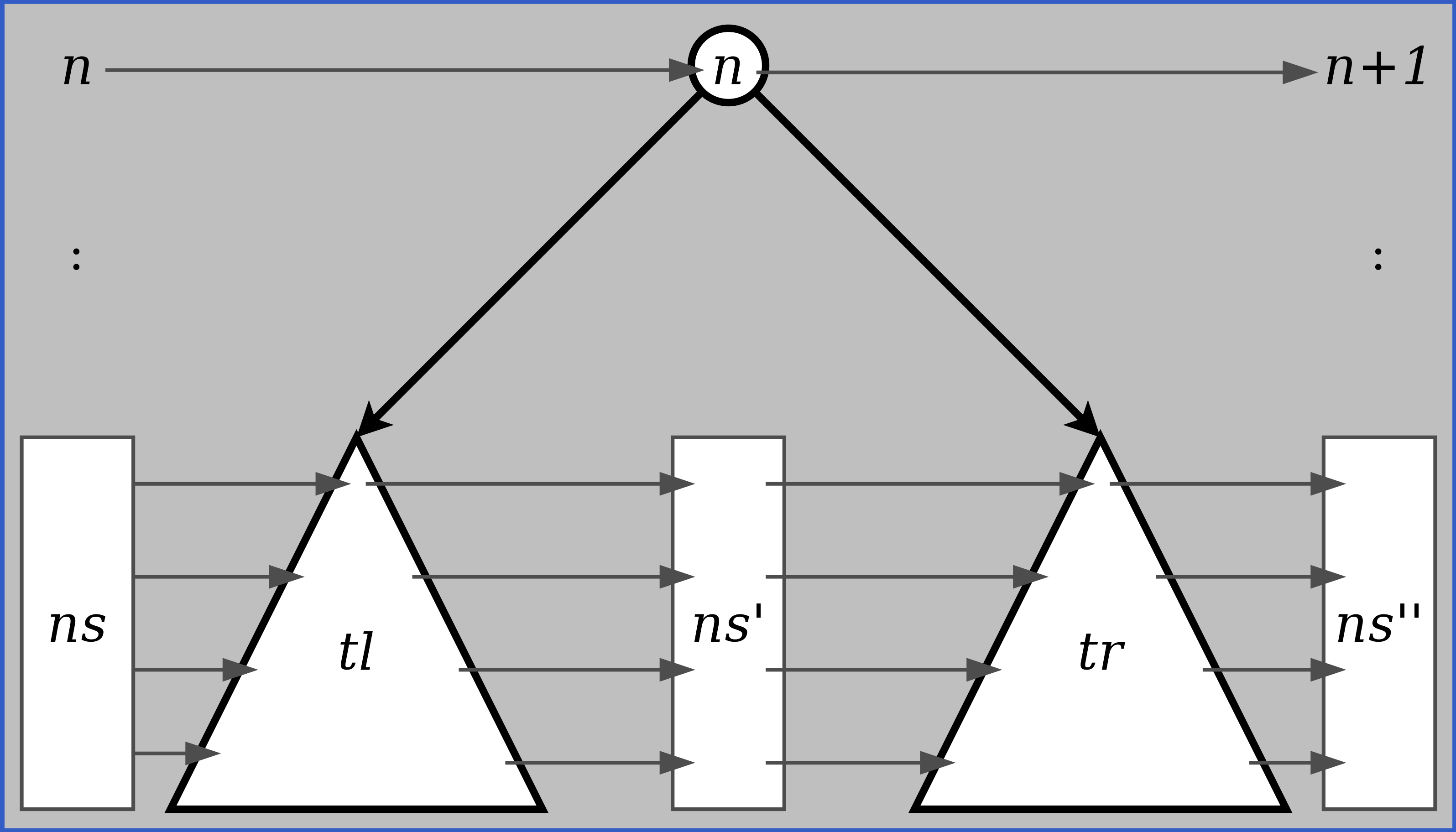
```
bfn :: Tree a -> Tree Integer
bfn t = t'
    where
        (ns, t') = bfnAux (1 : ns) t

bfnAux :: [Integer] -> Tree a
          -> ([Integer], Tree Integer)
bfnAux ns      Empty          = (ns, Empty)
bfnAux (n : ns) (Node tl _ tr) = ((n + 1) : ns'',
                                  Node tl' n tr')
    where
        (ns',  tl') = bfnAux ns tl
        (ns'', tr') = bfnAux ns' tr
```

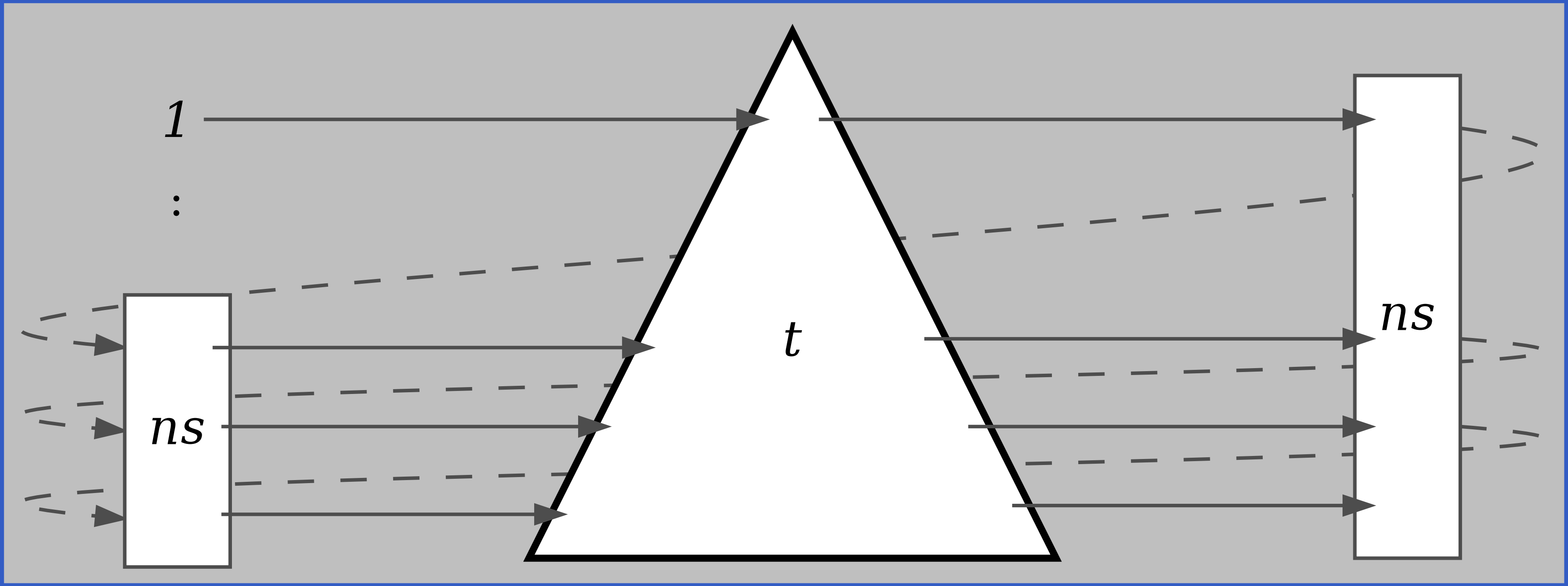
Eqns (1) & (2)

Eqn (3)

Breadth-first Numbering (7)



Breadth-first Numbering (8)



Dynamic Programming

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- Create a ***table*** of all subproblems that ever will have to be solved.
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In effect, using laziness to implement limited form of ***memoization***.

The Triangulation Problem (1)

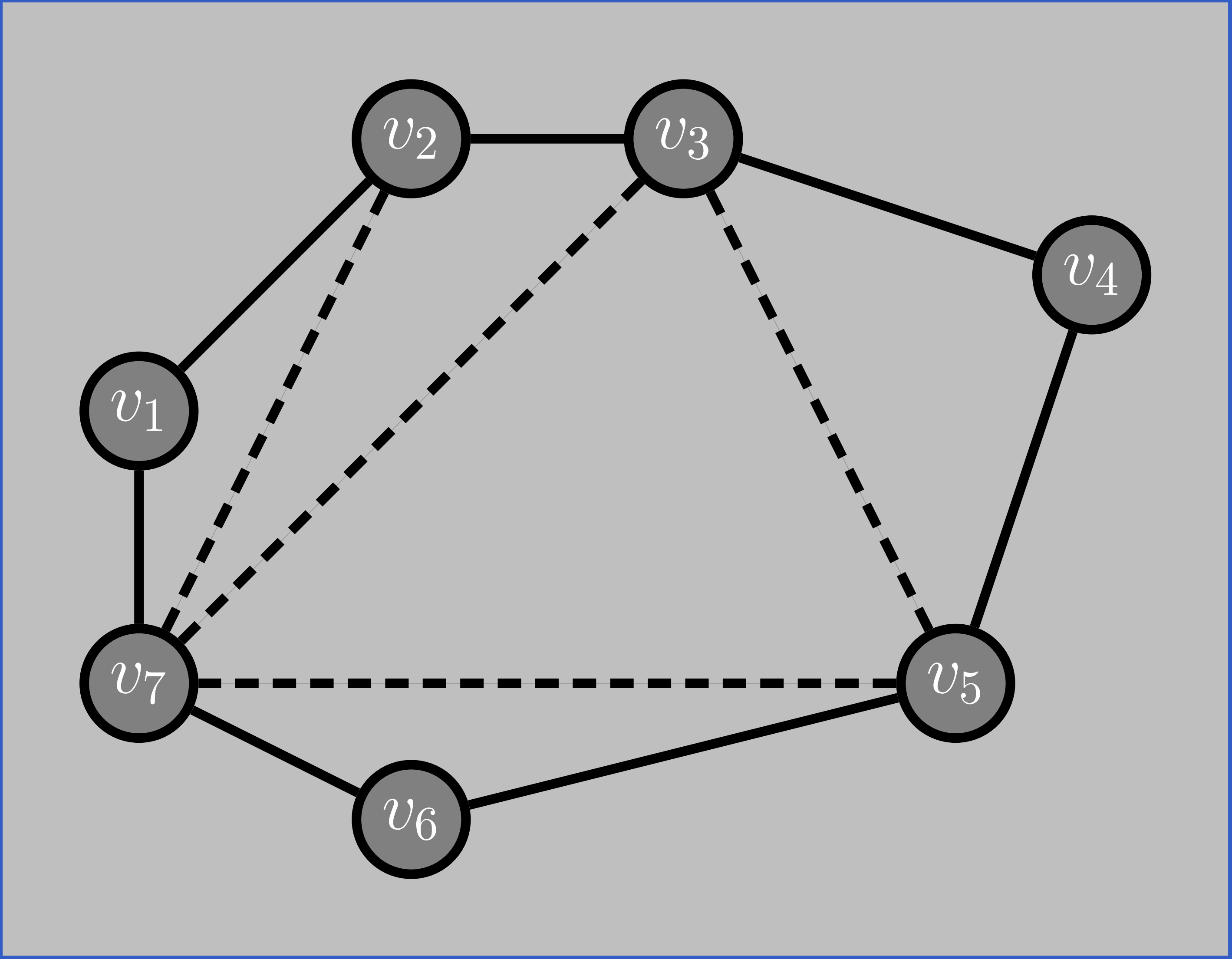
Select a set of ***chords*** that divides a convex polygon into triangles such that:

- no two chords cross each other
- the sum of their length is minimal.

We will only consider computing the minimal length.

See Aho, Hopcroft, Ullman (1983) for details.

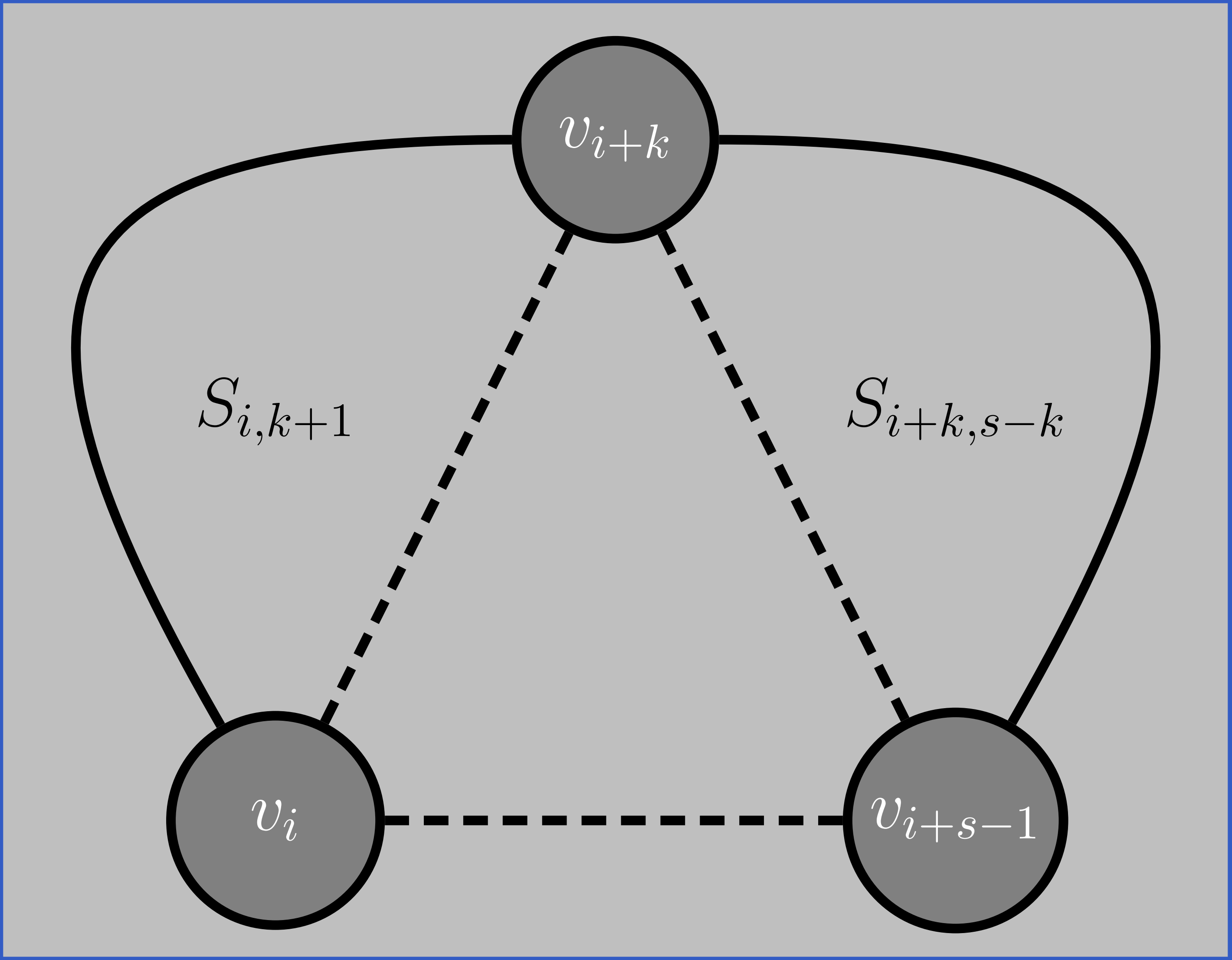
The Triangulation Problem (2)



The Triangulation Problem (3)

- Let S_{is} denote the subproblem of size s starting at vertex v_i of finding the minimum triangulation of the polygon $v_i, v_{i+1}, \ldots, v_{i+s-1}$ (counting modulo the number of vertices).
- Subproblems of size less than 4 are trivial.
- Solving S_{is} is done by solving $S_{i,k+1}$ and $S_{i+k,s-k}$ for all k, $1 \leq k \leq s-2$
- The obvious recursive formulation results in 3^{s-4} (non-trivial) calls.
- But for $n \geq 4$ vertices there are only $n(n-3)$ non-trivial subproblems!

The Triangulation Problem (4)



The Triangulation Problem (5)

- Let C_{is} denote the minimal triangulation cost of S_{is}.
- Let $D(v_p, v_q)$ denote the length of a chord between v_p and v_q (length is 0 for non-chords; i.e. adjacent v_p and v_q).
- For $s \geq 4$:

$$C_{is} = \min_{k \in [1, s-2]} \left\{ \begin{array}{l} C_{i,k+1} + C_{i+k,s-k} \\ +D(v_i, v_{i+k}) + D(v_{i+k}, v_{i+s-1}) \end{array} \right\}$$

- For $s < 4$, $C_{is} = 0$.

The Triangulation Problem (6)

These equations can be transliterated straight into Haskell:

```
triCost :: Polygon -> Double
triCost p = cost!(0,n) where
    cost = array ((0,0), (n-1,n))
                 ([ ((i,s),
                     minimum [ cost!(i, k+1)
                               + cost!((i+k) `mod` n, s-k)
                               + dist p i ((i+k) `mod` n)
                               + dist p ((i+k) `mod` n)
                                        ((i+s-1) `mod` n)
                             | k <- [1..s-2] ])
                  | i <- [0..n-1], s <- [4..n] ] ++
                  [ ((i,s), 0.0)
                  | i <- [0..n-1], s <- [0..3] ])
    n = snd (bounds b) + 1
```

Attribute Grammars (1)

Lazy evaluation is also very useful for evaluation of ***Attribute Grammars***:

- The attribution function is defined recursively over the tree:
  - takes inherited attributes as extra arguments;
  - returns a tuple of all synthesised attributes.
- As long as there exists ***some*** possible attribution order, lazy evaluation will take care of the attribute evaluation.

Attribute Grammars (2)

- The earlier examples on Circular Programming and Breadth-first Numbering can be seen as instances of this idea.

Reading

- John W. Lloyd. Practical advantages of declarative programming. In *Joint Conference on Declarative Programming, GULP-PRODE'94*, 1994.
- John Hughes. Why Functional Programming Matters. *The Computer Journal*, 32(2):98–197, April 1989.
- Thomas Johnsson. Attribute Grammars as a Functional Programming Paradigm. In *Functional Programming Languages and Computer Architecture, FPCA'87*, 1987

Reading

- Geraint Jones and Jeremy Gibbons.
  Linear-time breadth-first tree algorithms: An exercise in the arithmetic of folds and zips.
  Technical Report TR-31-92, Oxford University Computing Laboratory, 1992.
- Alfred Aho, John Hopcroft, Jeffrey Ullman.
  Data Structures and Algorithms.
  Addison-Wesley, 1983.