LiU-FP2016: Lecture 2

The Untyped λ-Calculus: Introduction

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The λ-Calculus: What is it? (1)

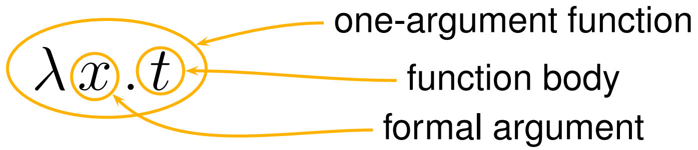
- Pure notion of effective computation procedure: ***all*** computation reduced to function definition and application.
- Invented in the 1920s by Alonzo Church.
- Cf. other formalisations of the notion of effective computation; e.g., the Turing machine.
- The λ-calculus and Turing Machines are equivalent in that they capture the exact same notion of what “computation” means.

The λ-Calculus: What is it? (2)

- The Church-Turing Hypothesis: The λ-calculus, Turing Machines, etc. coincides with our intuitive understanding of what “computation” means.
- The λ-calculus is important because it is at once:
  - very simple, yet in essence a practically useful programming language
  - mathematically precise, allowing for formal reasoning.

Key Idea

λ-abstraction (or anonymous function):



Multiple arguments handled by “returning” lambda abstractions that then are applied to further arguments: ***Currying.***

Syntax

t	$\rightarrow$		terms:
		x	variable
	$\mid$	$\lambda x.t$	abstraction
	$\mid$	$t\ t$	application

Note:

- x is the syntactic category of variables. We will use actual names like x, y, z, u, v, w, ...
- λ-abstractions often named for convenience. E.g. $I \equiv \lambda x.x$. ***Just an abbreviation!*** So e.g. $F \equiv \lambda x.(\ldots F \ldots)$ ***not*** valid def. Why?

Scope

- An ***occurrence*** of x is ***bound*** if it occurs in the body t of a λ-abstraction $\lambda x.t$.
- A non-bound occurrence is ***free***.
- A λ-term with ***no free*** variables is called ***closed***. Otherwise ***open***.
- A closed λ-term is called a ***combinator***.

Exercise

In the following:

- Which variables are free and which are bound?
- Which terms are open and which are closed?

(a)	x	(d)	$\lambda x.\lambda y.x\ y$
(b)	$\lambda x.x$	(e)	$(\lambda x.x)\ x$
(c)	$\lambda x.y$	(f)	$\lambda x.\lambda y.(\lambda x.x\ y)\ (\lambda z.x\ y)$

Operational Semantics (1)

Sole means of computation: ***β-reduction*** or ***function application***:

$$(\lambda x.t_1)\ t_2 \underset{\beta}{\rightarrow} [x \mapsto t_2]t_1$$

where

$$[x \mapsto t_2]t_1$$

means "term t_1 with all ***free*** occurrences of x (with respect to t_1) replaced by t_2."

Subtle problems concerning ***name clashes*** will be considered later.

Operational Semantics (2)

A term that can be β-reduced is called a ***(β-)redex***.

Exercise: Underline the redexes in

$$(\lambda x.x)\ ((\lambda x.x)\ (\lambda z.(\lambda x.x)\ z))$$

Programming In the λ-Calculus

How can such a simple language express arbitrary computations?

Nothing that looks like arithmetic, or conditionals, and seems not even recusrion allowed?

To make it plausible that the λ-calculus indeed is a general notion of computation, we will see how to express:

- Booleans
- Arithmetic
- Recursion

Church Booleans

True, false, and conditional:

$$\begin{aligned} T &\equiv \lambda t.\lambda f.t \\ F &\equiv \lambda t.\lambda f.f \\ \mathit{IF} &\equiv \lambda l.\lambda m.\lambda n.l\ m\ n \end{aligned}$$

Exercise: Evaluate $\mathit{IF}\ T\ v\ w$
Logical connectives:

$$\begin{aligned} \mathit{AND} &\equiv \lambda b.\lambda c.b\ c\ F \\ \mathit{OR} &\equiv \lambda b.\lambda c.b\ T\ c \\ \mathit{NOT} &\equiv \lambda b.b\ F\ T \end{aligned}$$

Pairs

If we can represent pairs, then we can represnt any kind of compound data:

$$\begin{aligned} \mathit{PAIR} &\equiv \lambda f.\lambda s.\lambda b.b\ f\ s \\ \mathit{FST} &\equiv \lambda p.p\ T \\ \mathit{SND} &\equiv \lambda p.p\ F \end{aligned}$$

Church Numerals (1)

Idea: The natural number n is represented by a function that applies its first argument n times to its second argument.

$$\begin{aligned} C_0 &\equiv \lambda s.\lambda z.z \\ C_1 &\equiv \lambda s.\lambda z.s\ z \\ C_2 &\equiv \lambda s.\lambda z.s\ (s\ z) \\ C_3 &\equiv \lambda s.\lambda z.s\ (s\ (s\ z)) \end{aligned}$$

Etc.

Church Numerals (2)

Operations:

$$\begin{aligned} \mathit{SUCC} &\equiv \lambda n.\lambda s.\lambda z.s\ (n\ s\ z) \\ \mathit{PLUS} &\equiv \lambda m.\lambda n.\lambda s.\lambda z.m\ s\ (n\ s\ z) \\ \mathit{TIMES} &\equiv \lambda m.\lambda n.\lambda s.m\ (n\ s) \\ \mathit{POWER} &\equiv \lambda m.\lambda n.m\ n \\ \mathit{ISZERO} &\equiv \lambda m.m\ (\lambda x.F)\ T \end{aligned}$$

Church Numerals (3)

Subtraction is more intricate. Let us consider the predecessor function:

$$\begin{aligned} \mathit{ZZ} &\equiv \mathit{PAIR}\ C_0\ C_0 \\ \mathit{SS} &\equiv \lambda p.\mathit{PAIR}\ (\mathit{SND}\ p)\ (\mathit{SUCC}\ (\mathit{SND}\ p)) \\ \mathit{PRED} &\equiv \lambda m.\mathit{FST}\ (m\ \mathit{SS}\ \mathit{ZZ}) \end{aligned}$$

Idea: SS maps (m, n) to $(n, n+1)$. Iterating SS n times on $(0, 0)$ means that the first component of the result is $n - 1$.