LiU-FP2016: Lecture 13

Arrows

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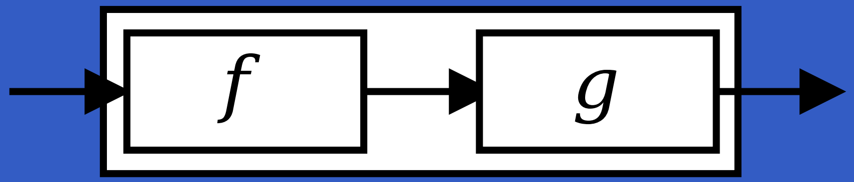
Arrows (1)

System descriptions in the form of block diagrams are very common. Blocks have inputs and outputs and can be combined into larger blocks. For example, serial composition:



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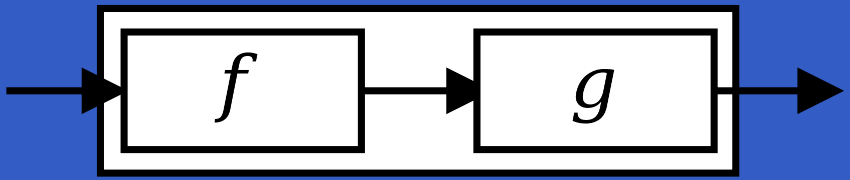
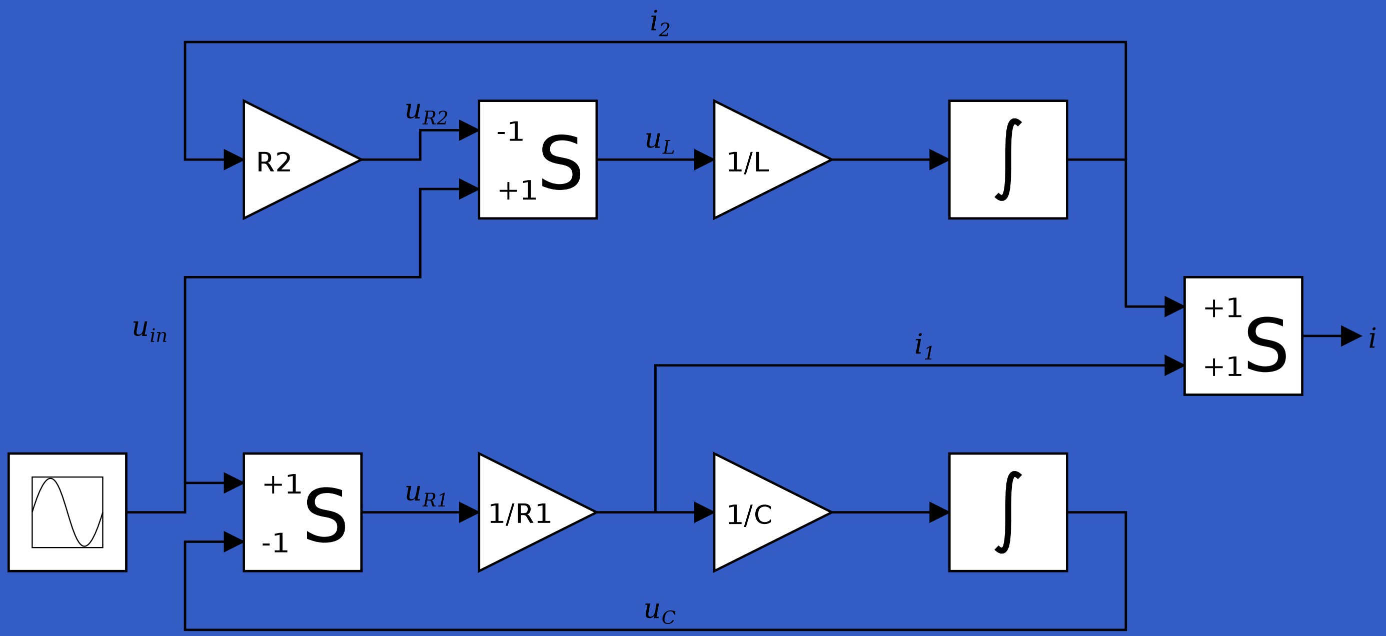


A *combinator* can be defined that captures this idea:

```
(>>>) :: B a b -> B b c -> B a c
```

Arrows (2)

But systems can be complex:



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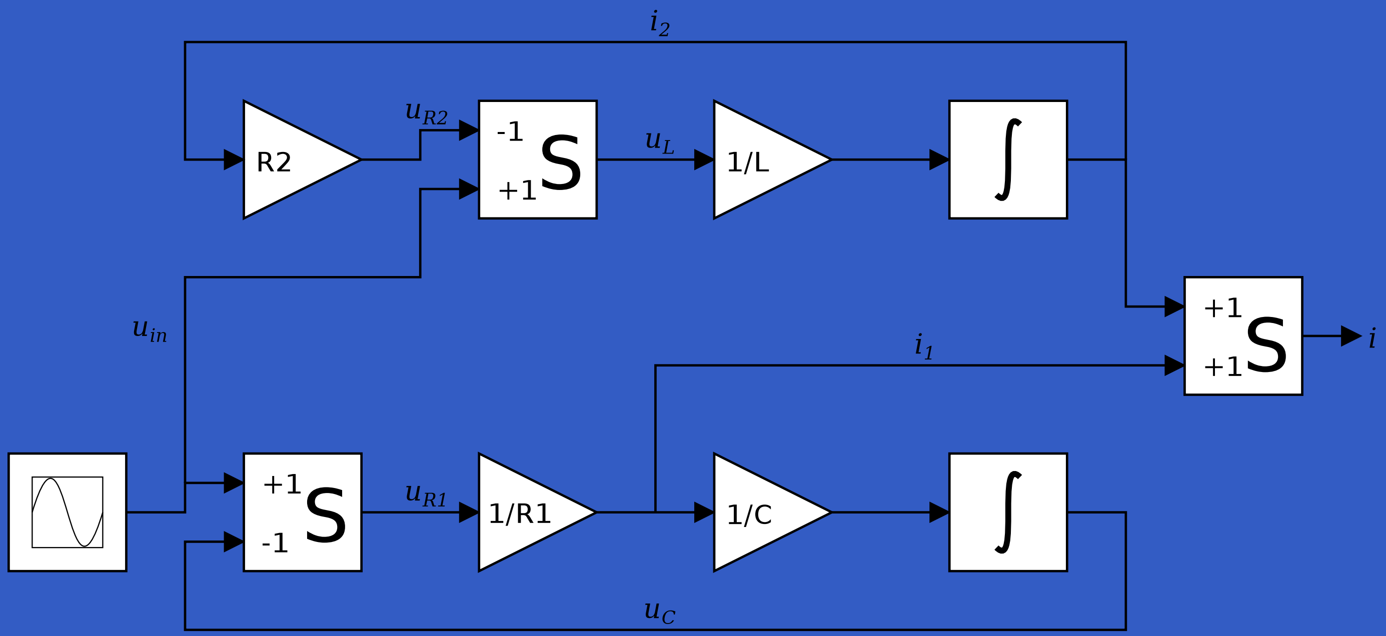

How many and what combinators do we need to be able to describe arbitrary systems?

Arrows (3)

John Hughes’ ***arrow*** framework:

- Abstract data type interface for function-like types (or “blocks”, if you prefer).

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- Particularly suitable for types representing process-like computations.
- Related to ***monads***, since arrows are computations, but more general.

Arrows (3)

John Hughes' ***arrow*** framework:

- Abstract data type interface for function-like types (or "blocks", if you prefer).
- Particularly suitable for types representing process-like computations.
- Related to ***monads***, since arrows are computations, but more general.
- Provides a minimal set of "wiring" combinators.

What is an arrow? (1)

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  - ***widening***:

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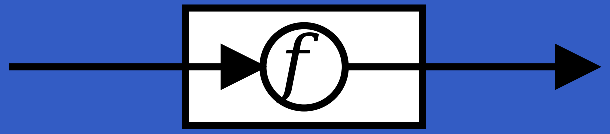
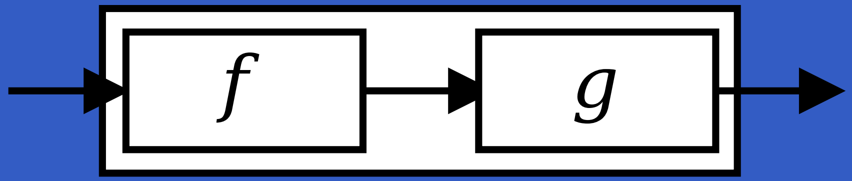
- A set of ***algebraic laws*** that must hold.

What is an arrow? (2)

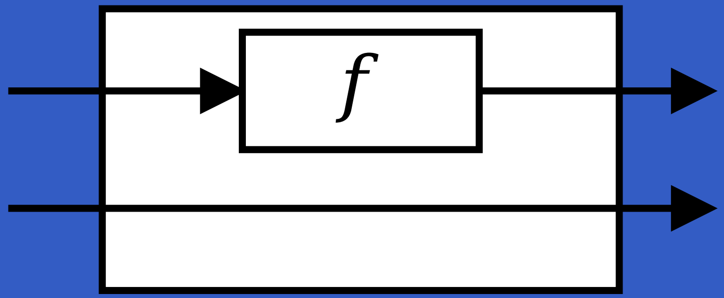
These diagrams convey the general idea:



`arr` f



f `>>>` g



`first` f

The `Arrow` class

In Haskell, a ***type class*** is used to capture these ideas (except for the laws):

```
class Arrow a where
    arr   :: (b -> c) -> a b c
    (>>>) :: a b c -> a c d -> a b d
    first :: a b c -> a (b,d) (c,d)
```

Functions are arrows (1)

Functions are a simple example of arrows, with `(->)` as the arrow type constructor.

Exercise 1: Suggest suitable definitions of

- `arr`
- `(>>>)`
- `first`

for this case!

(We have not looked at what the laws are yet, but they are “natural”.)

Functions are arrows (2)

Solution:

- `arr = id`

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  To see this, recall

```
id :: t -> t
arr :: (b->c) -> a b c
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- `arr = id`

  To see this, recall

  ```
  id :: t -> t
  arr :: (b->c) -> a b c
  ```

  Instantiate with

  ```
  a = (->)
  t = b->c = (->) b c
  ```

Functions are arrows (3)

- `f >>> g = \a -> g (f a)`

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- `f >>> g = g . f` ***or even***
- `(>>>) = flip (.)`
- `first f = \(b,d) -> (f b,d)`

Functions are arrows (4)

`Arrow` instance declaration for functions:

```
instance Arrow (->) where
    arr     = id
    (>>>)   = flip (.)
    first f = \(b,d) -> (f b,d)
```

Some arrow laws

```
(f >>> g) >>> h = f >>> (g >>> h)
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(f >>> g) >>> h = f >>> (g >>> h)
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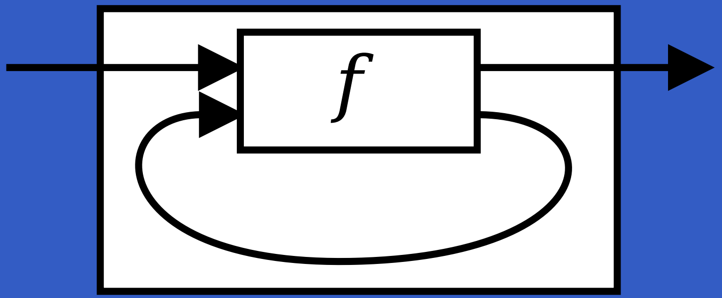
   arr id >>> f = f
              f = f >>> arr id
  first (arr f) = arr (first f)
first (f >>> g) = first f >>> first g
```

The `loop` combinator (1)

Another important operator is `loop`: a fixed-point operator used to express recursive arrows or ***feedback***:



`loop` f

The `loop` combinator (2)

Not all arrow instances support `loop`. It is thus a method of a separate class:

```
class Arrow a => ArrowLoop a where
    loop :: a (b, d) (c, d) -> a b c
```

Remarkably, the four combinators `arr`, `>>>`, `first`, and `loop` are sufficient to express any conceivable wiring!

Some more arrow combinators (1)

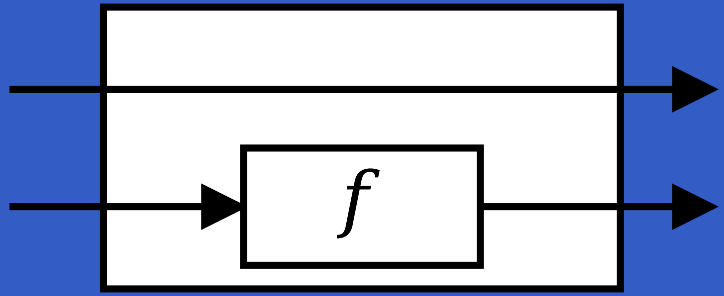
```
second :: Arrow a =>
    a b c -> a (d,b) (d,c)

(***) :: Arrow a =>
    a b c -> a d e -> a (b,d) (c,e)

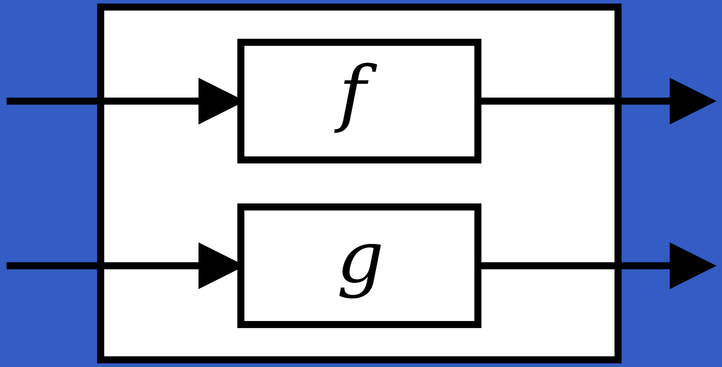
(&&&) :: Arrow a =>
    a b c -> a b d -> a b (c,d)
```

Some more arrow combinators (2)

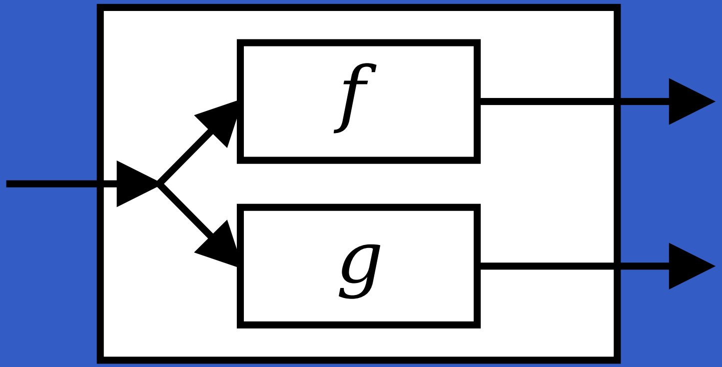
As diagrams:



second f



f *** g



f &&& g

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```
second :: Arrow a => a b c -> a (d,b) (d,c)
second f = arr swap >>> first f >>> arr swap
swap (x,y) = (y,x)
```

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f *** g = first f >>> second g

(&&&) :: Arrow a => a b c -> a b d -> a b (c,d)
f &&& g = arr (\x->(x,x)) >>> (f *** g)
```
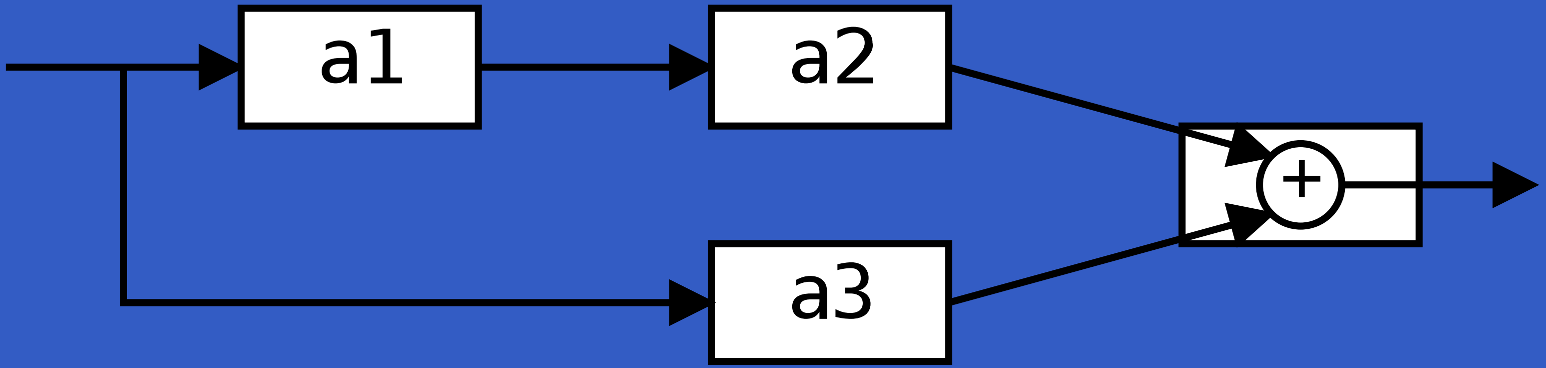

Exercise 2

Describe the following circuit using arrow combinators:



`a1, a2, a3 :: A Double Double`

Exercise 2: One solution

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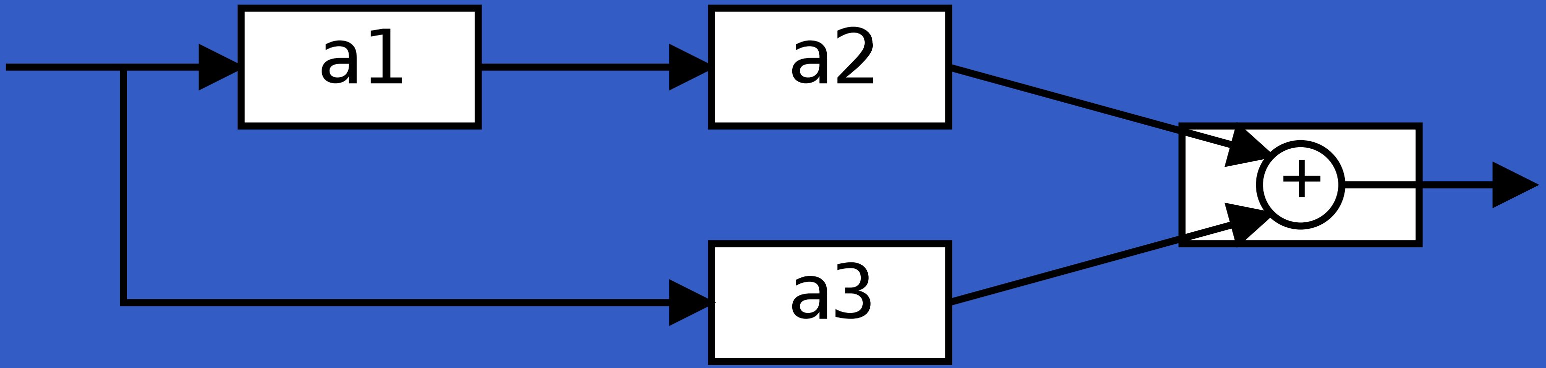

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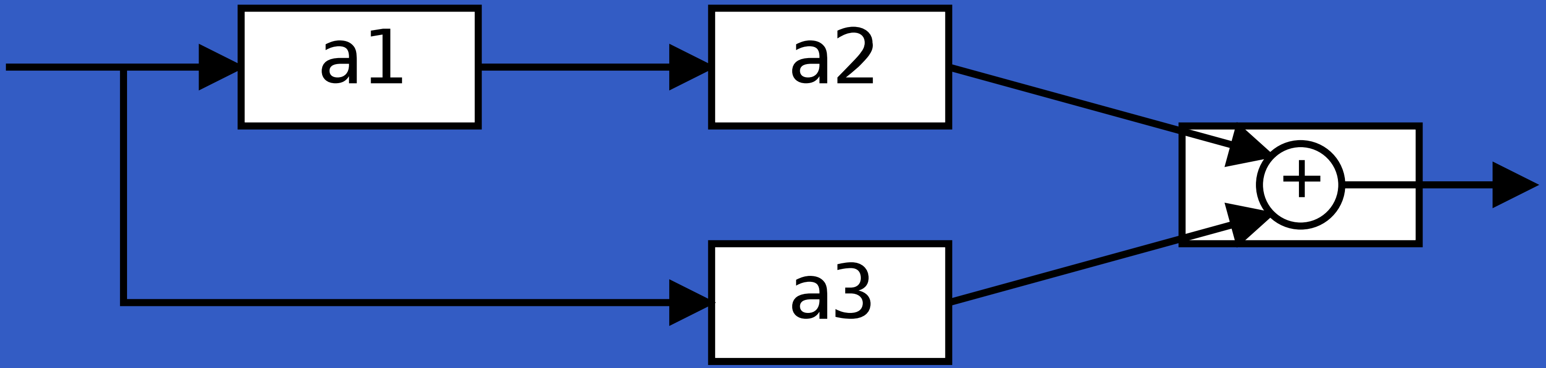

```
a1, a2, a3 :: A Double Double

circuit_v1 :: A Double Double
circuit_v1 = (a1 &&& arr id)
             >>> (a2 *** a3)
             >>> arr (uncurry (+))
```

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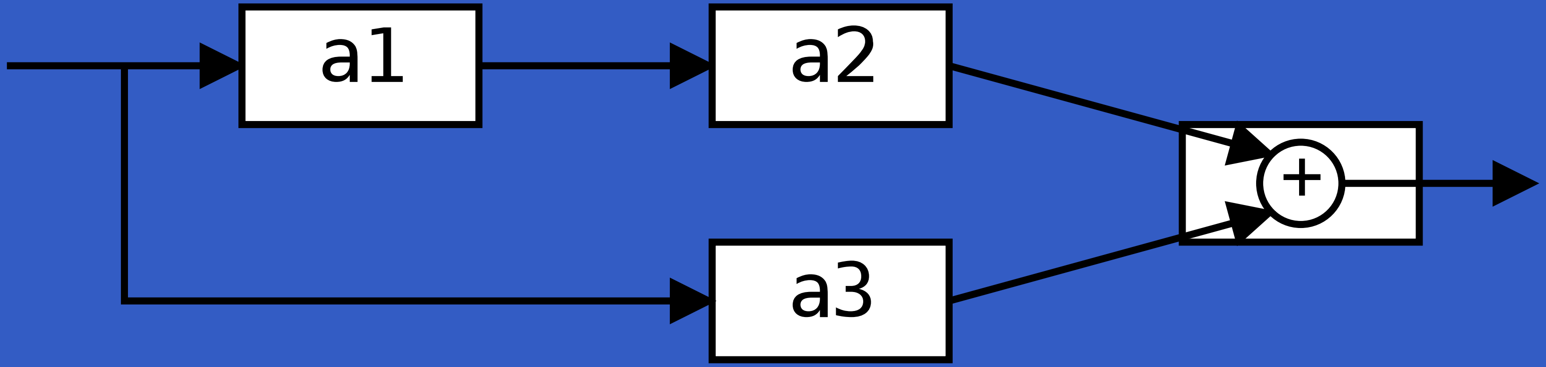

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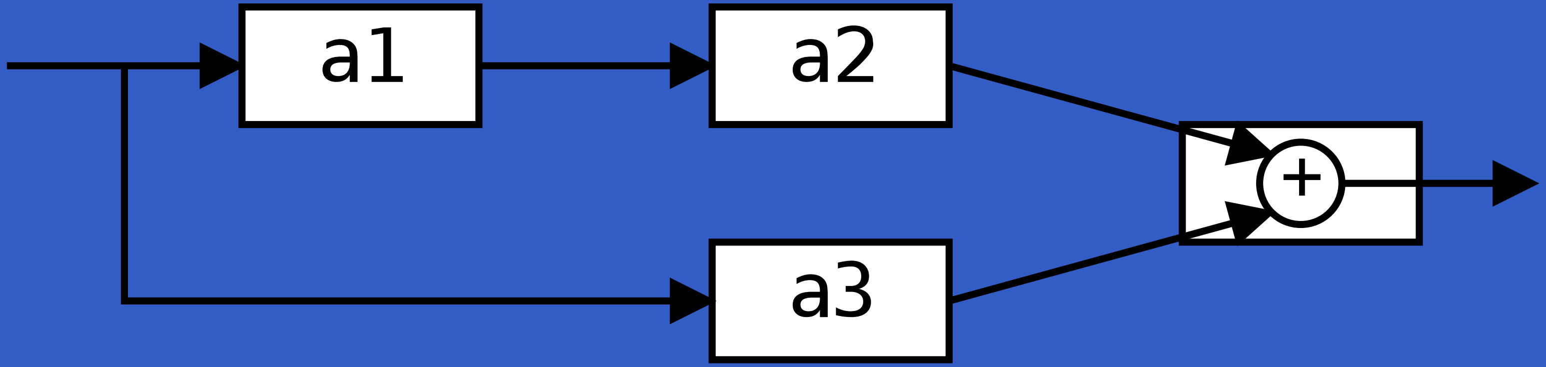


```
a1, a2, a3 :: A Double Double

circuit_v2 :: A Double Double
circuit_v2 = arr (\x -> (x,x))
             >>> first a1
             >>> (a2 *** a3)
             >>> arr (uncurry (+))
```

The arrow do notation (1)

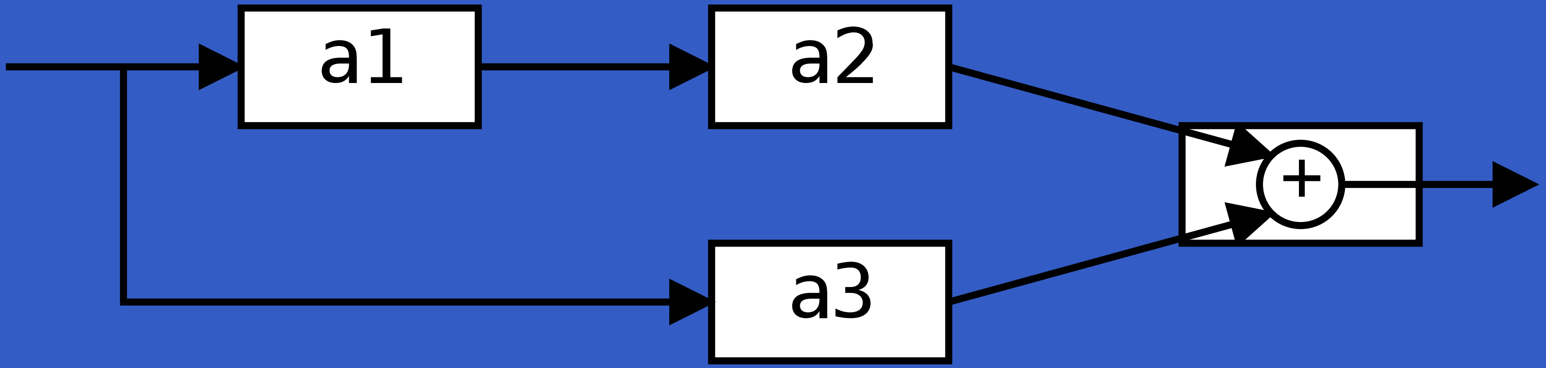
Ross Paterson's `do`-notation for arrows supports ***pointed*** arrow programming. Only ***syntactic sugar***.

$$\begin{array}{l}
\texttt{proc}\ \mathit{pat}\ \texttt{->}\ \texttt{do}\ [\ \texttt{rec}\ ] \\
\qquad \mathit{pat}_1\ \texttt{<-}\ \mathit{sfexp}_1\ \texttt{-<}\ \mathit{exp}_1 \\
\qquad \mathit{pat}_2\ \texttt{<-}\ \mathit{sfexp}_2\ \texttt{-<}\ \mathit{exp}_2 \\
\qquad \ldots \\
\qquad \mathit{pat}_n\ \texttt{<-}\ \mathit{sfexp}_n\ \texttt{-<}\ \mathit{exp}_n \\
\qquad \texttt{returnA}\ \texttt{-<}\ \mathit{exp}
\end{array}$$

Also: $\texttt{let}\ \mathit{pat} = \mathit{exp} \quad \equiv \quad \mathit{pat}\ \texttt{<-}\ \texttt{arr id}\ \texttt{-<}\ \mathit{exp}$

The arrow do notation (2)

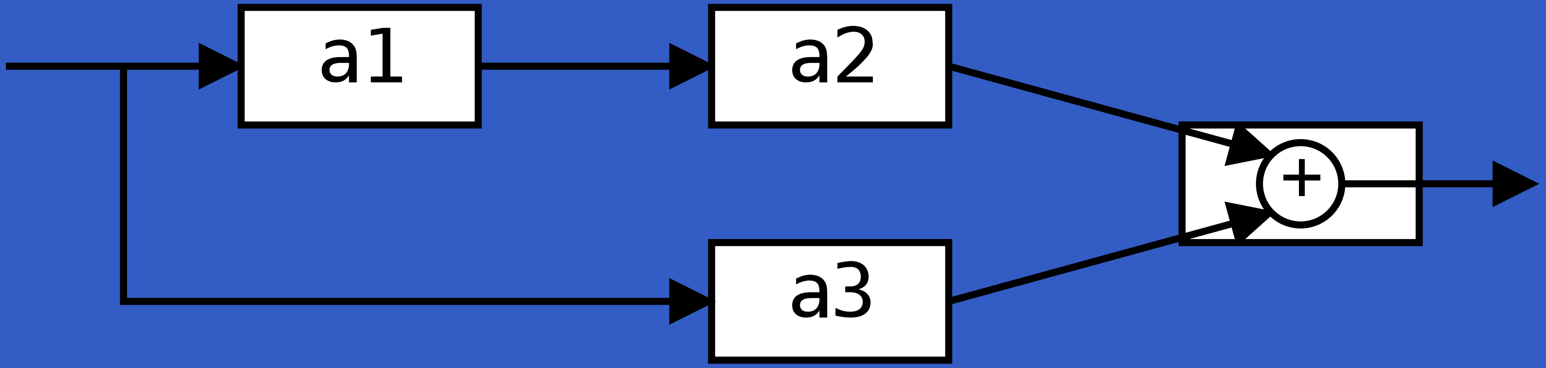
Let us redo exercise 2 using this notation:



```
circuit_v4 :: A Double Double
circuit_v4 = proc x -> do
    y1 <- a1 -< x
    y2 <- a2 -< y1
    y3 <- a3 -< x
    returnA -< y2 + y3
```

The arrow do notation (3)

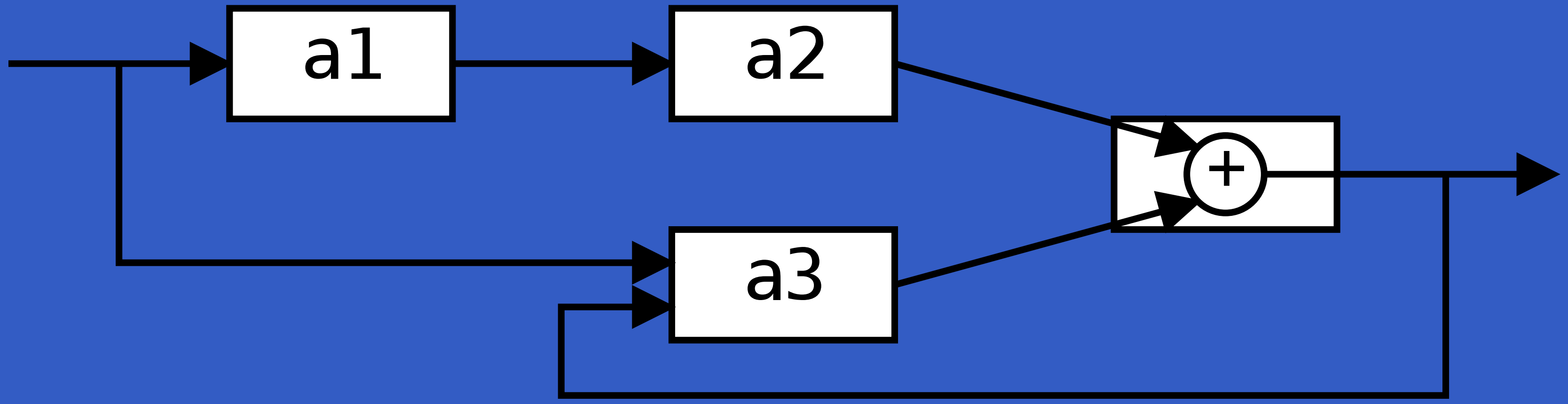
We can also mix and match:



```
circuit_v5 :: A Double Double
circuit_v5 = proc x -> do
    y2 <- a2 <<< a1 -< x
    y3 <- a3        -< x
    returnA -< y2 + y3
```

The arrow do notation (4)

Recursive networks: `do`-notation:



```
a1, a2 :: A Double Double
a3 :: A (Double,Double) Double
```

The arrow do notation (4)

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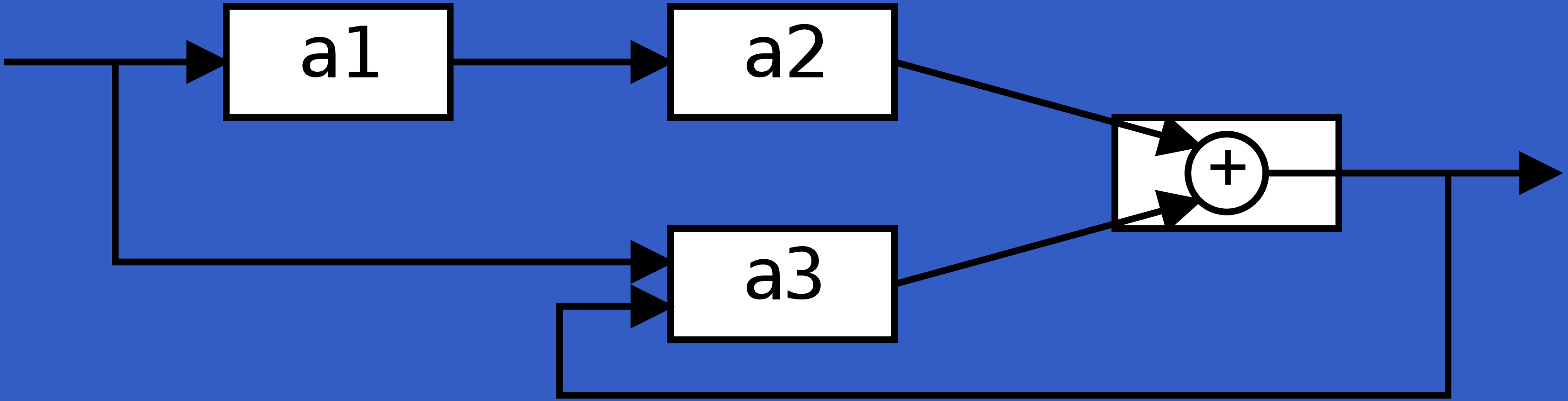
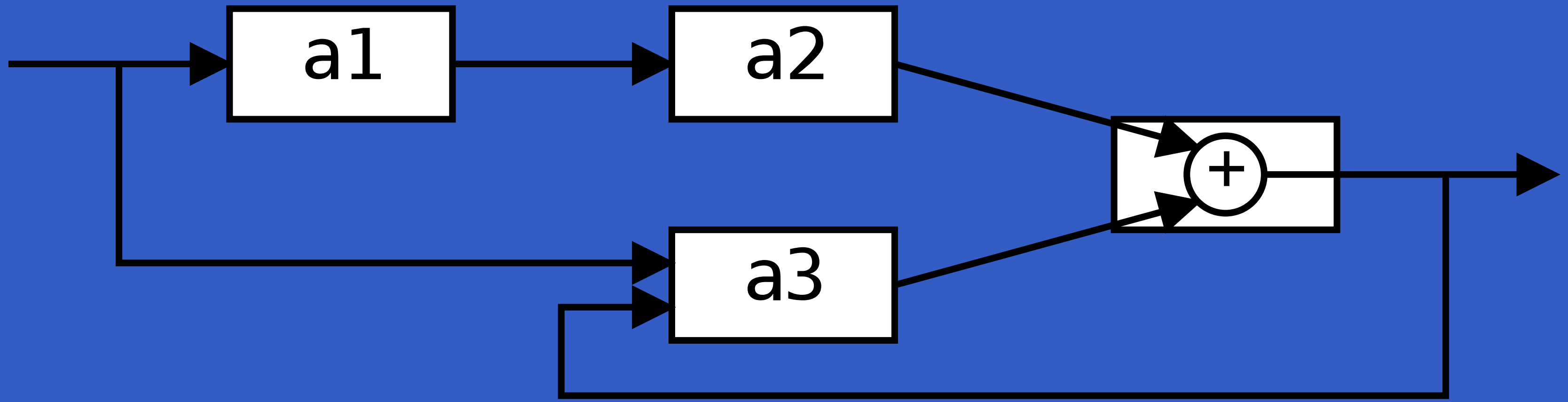


```
a1, a2 :: A Double Double
a3 :: A (Double,Double) Double
```

Exercise 3: Describe this using only the arrow combinators.

The arrow do notation (5)



```
circuit = proc x -> do
    rec
        y1 <- a1 -< x
        y2 <- a2 -< y1
        y3 <- a3 -< (x, y)
        let y = y2 + y3
    returnA -< y
```

Arrows and Monads (1)

Arrows generalize monads: for every monad type there is an arrow, the ***Kleisli category*** for the monad:

```
newtype Kleisli m a b = K (a -> m b)

instance Monad m => Arrow (Kleisli m) where
    arr f       = K (\b -> return (f b))
    K f >>> K g = K (\b -> f b >>= g)
```

Arrows and Monads (2)

But not every arrow is a monad. However, arrows that support an additional `apply` operation ***are*** effectively monads:

```
apply :: Arrow a => a (a b c, b) c
```

Exercise 4: Verify that

```
newtype M b = M (A () b)
```

is a monad if `A` is an arrow supporting `apply`; i.e., define `return` and `bind` in terms of the arrow operations (and verify that the monad laws hold).

Reading

- John Hughes. Generalising monads to arrows. *Science of Computer Programming*, 37:67–111, May 2000
- John Hughes. Programming with arrows. In *Advanced Functional Programming*, 2004. To be published by Springer Verlag.