Combinatorial Games

Impartial, 2-person games with complete information.

- 2 players take turns to move
- valid moves are the same for both players
- the full state of the game is known to both
- play is guaranteed to terminate
- play ends when no move is possible, loser is person whose turn it is,
**Move**
Remove one or two adjacent petals

**Strategy**
Maintain symmetry
How to win.

Identify a property of positions (the strategy invariant) such that

- all end positions satisfy the property.
- every move from a position satisfying the property falsifies the property.
- for every position that does not satisfy the property there is a move that truthifies the property.
Matchstick Game

Move: remove at least 1 and at most M matches (where M is fixed in advance).

Strategy: \( m \mod (M+1) = 0 \)

where \( m \) is the number of matches.

<table>
<thead>
<tr>
<th>Moves</th>
<th>Losing Position</th>
<th>Strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 or 2 matches</td>
<td>$m \mod 3 = 0$</td>
<td>take $m \mod 3$ matches</td>
</tr>
<tr>
<td>1,2 or 3 matches</td>
<td>$m \mod 4 = 0$</td>
<td>$\text{---} m \mod 4 \text{---}$</td>
</tr>
<tr>
<td>1 thru $M$ matches</td>
<td>$m \mod (M+1) = 0$</td>
<td>$\text{---} m \mod (M+1) \text{---}$</td>
</tr>
</tbody>
</table>

Key:

$m$ no. of matches

$\mod$ remainder after dividing
A Game

Positions are the nodes of the graph.
Moves are directed edges.
Game Sum

A game with two piles of matches is the "sum" of two single-pile matchstick games.
Another "sum" of two games:

Move: choose either remove some matches or remove a petal
The *sum* of two games is a game defined as follows:

Call the two games the *left game* and the *right game*.

A position in the sum game is an ordered pair \((l, r)\) of positions, where \(l\) is a position in the left game and \(r\) is a position in the right game.

A move \((l, r) \mapsto (l', r')\) in the sum game satisfies either: \(l \mapsto l'\) is a move in the left game, and \(r = r'\), or : \(l = l'\) and \(r \mapsto r'\) is a move in the right game.
How to win.

Identify a property of positions (the *strategy invariant*) such that

- all end positions satisfy the property.
- every move from a position satisfying the property falsifies the property.
- for every position that does not satisfy the property there is a move that *truthifies* the property.
Conjecture: there is a function $\oplus$ such that

$$W.(l+r) = W.l \oplus W.r .$$

Sum game 1

Sum game 2
m matches

n matches

Move in left or right game: remove any +ve no. of matches.

Strategy: truthify $m = n$. 
Key:

l  position in left game
r  position in right game
v  "value" of position
Mex number

The mex (minimal excludant) number of a position is defined to be:

the minimum natural number that is not included in the mex numbers of the position's successors.

Winning strategy:

Truthify \( \text{mex}.l = \text{mex}.r \).