MGS 2009: FUN Lecture 1

Lazy Functional Programming

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What Is a Functional Language? (1)

- ***Imperative Languages***:
  - Implicit state.
  - Computation essentially a sequence of side-effecting actions.
- ***Declarative Languages*** (Lloyd 1994):
  - ***No*** implicit state.
  - A program can be regarded as a theory.
  - Computation can be seen as deduction from this theory.
  - Examples: Logic and Functional Languages.

What Is a Functional Language? (2)

Another perspective:

- ***Algorithm = Logic + Control***
- Declarative programming emphasises the logic (“what”) rather than the control (“how”).
- Examples:
  - Resolution (logic programming)
  - Lazy evaluation (found in some functional and logic languages)

What Is a Functional Language? (3)

Declarative languages for practical use tend to be only ***weakly declarative***; i.e., not totally free of control aspects. For example:

- Equations in functional languages are directed.
- Order of patterns often matters for pattern matching.
- Constructs for taking control over the order of evaluation.

What Is a Functional Language? (4)

Exactly what constitute a functional language is somewhat contentious.

Pragmatically, a functional language is one that encourages a mostly declarative, ***functional style*** of programming.

Typical features/characteristics:

- Functions are first-class entities.
- Computation expressed through function application.
- Recursive (and co-recursive) definitions.

What Is a Functional Language? (5)

This “definition” covers both:

- ***Pure*** functional languages: no side effects
  - (Weakly) declarative: equational reasoning valid (referentially transparent).
  - Example: Haskell
- ***Mostly*** functional languages: some side effects, e.g. for I/O.
  - Equational reasoning with care.
  - Examples: ML, OCaml, Scheme, Erlang

This and the Following Lectures

- In this and the following lectures we will explore ***Purely Functional Programming*** through the use of Haskell.
- Theme of today: ***Relinquishing control: exploiting lazy evaluation***

Will assume some familiarity with functional programming in a language like Haskell or ML. Will explain Haskell syntax and other points as needed: ***Just ask!***

Evaluation Orders (1)

Consider:

```
sqr x = x * x
dbl x = x + x
main = sqr (dbl (2 + 3))
```

Many possible reduction orders. Innermost, leftmost ***redex*** first is called ***Applicative Order Reduction*** (AOR):

```
main ⇒ sqr (dbl (2 + 3)) ⇒ sqr (dbl 5)
⇒ sqr (5 + 5) ⇒ sqr 10 ⇒ 10 * 10 ⇒ 100
```

This is just ***Call-By-Value***.

Evaluation Orders (2)

Outermost, leftmost redex first is called ***Normal Order Reduction*** (NOR):

```
main ⇒ sqr (dbl (2 + 3))
⇒ dbl (2 + 3) * dbl (2 + 3)
⇒ ((2 + 3) + (2 + 3)) * dbl (2 + 3)
⇒ (5 + (2 + 3)) * dbl (2 + 3)
⇒ (5 + 5) * dbl (2 + 3) ⇒ 10 * dbl (2 + 3)
⇒ ... ⇒ 10 * 10 ⇒ 100
```

(Applications of arithmetic operations only considered redexes once arguments are numbers.)
Demand-driven evaluation or ***Call-By-Need***

Why Normal Order Reduction? (1)

NOR seems rather inefficient. Any use?

- Best possible termination properties. Two important theorems from the λ-calculus:
  - Church-Rosser Theorem I:
    No term has more than one normal form.
  - Church-Rosser Theorem II:
    If a term has a normal form, then NOR will find it.

Why Normal Order Reduction? (2)

- More expressive power; e.g.:
  - "Infinite" data structures
  - Circular programming
- More declarative code as control aspects (order of evaluation) left implicit.

Strict vs. Non-strict Semantics (1)

- $\perp$, or "bottom", the ***undefined value***, representing ***errors*** and ***non-termination***.
- A function f is ***strict*** iff:

$$f\ \perp = \perp$$

For example, $+$ is strict in both its arguments:

$$\begin{aligned}(0/0) + 1 &= \perp + 1 &= \perp\\ 1 + (0/0) &= 1 + \perp &= \perp\end{aligned}$$

Strict vs. Non-strict Semantics (2)

Consider:

```
foo x = 1
```

What is the value of `foo (0/0)`?

- AOR: `foo (0/0)` $\Rightarrow \perp$
  Conceptually, `foo` $\perp = \perp$; i.e., `foo` is strict.
- NOR: `foo (0/0)` $\Rightarrow$ `1`
  Conceptually, `foo` $\perp = 1$; i.e., `foo` is non-strict.

Thus, NOR results in non-strict semantics.
Note: NOR gave well-defined result, AOR did not.

Lazy Evaluation (1)

Lazy evaluation is an ***technique for implementing NOR*** more efficiently:

- An expression is evaluated ***only if needed***.
- ***Sharing*** employed to ensure any one expression evaluated at most once.

Lazy Evaluation (2)

```
sqr (dbl (2 + 3))
⇒ dbl (2 + 3) * (•)
⇒ ((2 + 3) + (•)) * (•)
⇒ (5 + (•)) * (•)
⇒ 10 * (•)
⇒ 100
```


Infinite Data Structures (1)

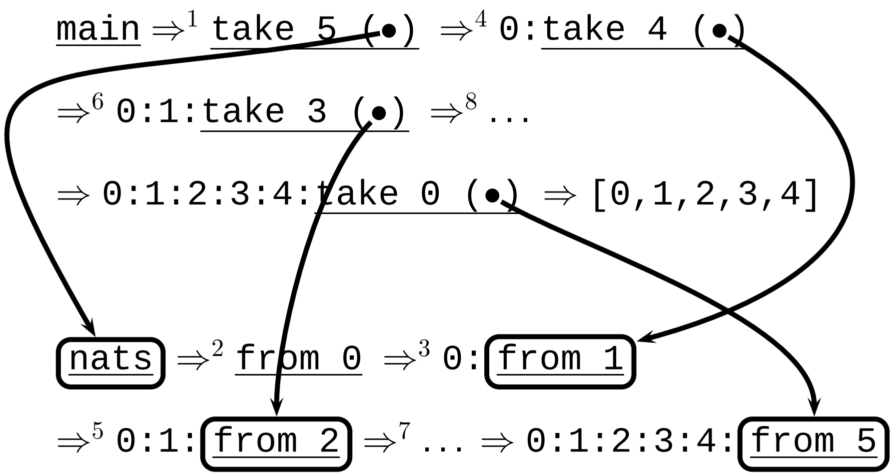
```
take 0 xs     = []
take n []     = []
take n (x:xs) = x : take (n-1) xs

from n = n : from (n+1)

nats = from 0

main = take 5 nats
```

Infinite Data Structures (2)



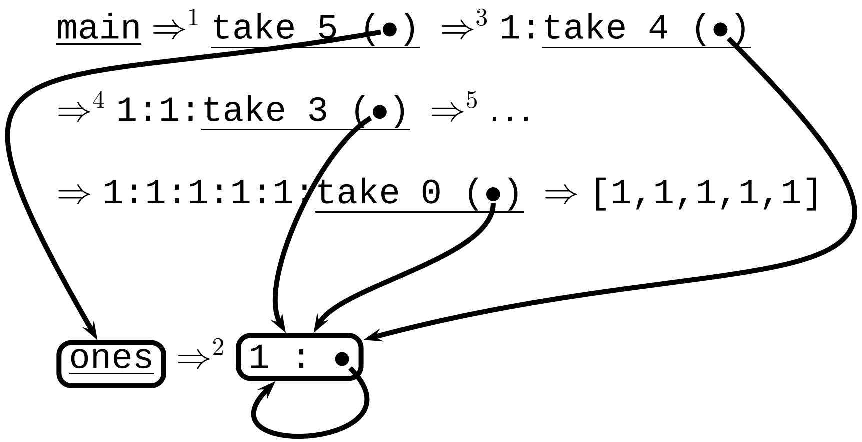
Circular Data Structures (2)

```
take 0 xs     = []
take n []     = []
take n (x:xs) = x : take (n-1) xs

ones = 1 : ones

main = take 5 ones
```


Circular Data Structures (2)



Circular Programming (1)

A non-empty tree type:

```
data Tree = Leaf Int | Node Tree Tree
```

Suppose we would like to write a function that replaces each leaf integer in a given tree with the ***smallest*** integer in that tree.

How many passes over the tree are needed?

One!

Circular Programming (2)

Write a function that replaces all leaf integers by a given integer, and returns the smallest integer of the given tree:

```
fmr :: Int -> Tree -> (Tree, Int)
fmr m (Leaf i) = (Leaf m, i)
fmr m (Node tl tr) =
    (Node tl' tr', min ml mr)
    where
        (tl', ml) = fmr m tl
        (tr', mr) = fmr m tr
```


Circular Programming (3)

For a given tree `t`, the desired tree is obtained as the ***solution*** to the equation:

```
(t', m) = fmr m t
```

Thus:

```
findMinReplace t = t'
    where
        (t', m) = fmr m t
```

Intuitively, this works because `fmr` can compute its result without needing to know the ***value*** of `m`.

A Simple Spreadsheet Evaluator

	a	b	c
1	c3 + c2		
2	a3 * b2	2	a2 + b2
3	7		a2 + a3

s

$\Rightarrow$

	a	b	c
1	37		
2	14	2	16
3	7		21

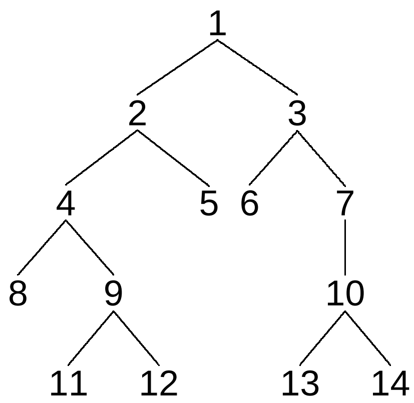
r

```
r = array (bounds s)
          [ ((i,j), eval r (s!(i,j)))
          | (i,j) <- indices s ]
```

The evaluated sheet is simply the solution to the stated equation. No need to worry about evaluation order. Any caveats?

Breadth-first Numbering (1)

Consider the problem of numbering a possibly infinitely deep tree in breadth-first order:



Breadth-first Numbering (2)

The following algorithm is due to G. Jones and J. Gibbons (1992), but the presentation differs.

Consider the following tree type:

```
data Tree a = Empty
            | Node (Tree a) a (Tree a)
```

Define:

$\mathrm{width}\ t\ i$	The width of a tree t at level i (0 origin).
$\mathrm{label}\ t\ i\ j$	The jth label at level i of a tree t (0 origin).

Breadth-first Numbering (3)

The following system of equations defines breadth-first numbering:

$$
\begin{aligned}
\mathrm{label}\ t\ 0\ 0 &= 1 && (1)\\
\mathrm{label}\ t\ (i+1)\ 0 &= \mathrm{label}\ t\ i\ 0 + \mathrm{width}\ t\ i && (2)\\
\mathrm{label}\ t\ i\ (j+1) &= \mathrm{label}\ t\ i\ j + 1 && (3)
\end{aligned}
$$

Note that $\mathrm{label}\ t\ i\ 0$ is defined for ***all*** levels i (as long as the widths of all tree levels are finite).

Breadth-first Numbering (4)

The code on the next slide sets up the defining system of equations.

- Streams (infinite lists) of labels are used as a mediating data structure to allow equations to be set up between adjacent nodes within levels and between the last node at one level and the first node at the next.
- As there manifestly are no cyclic dependences among the equations, we can entrust the details of solving them to the lazy evaluation machinery, in the safe knowledge that a solution will be found.

Breadth-first Numbering (5)

```
bfn :: Tree a -> Tree Integer
bfn t = t'
    where
        (ns, t') = bfnAux (1 : ns) t

bfnAux :: [Integer] -> Tree a
          -> ([Integer], Tree Integer)
bfnAux ns       Empty          = (ns, Empty)
bfnAux (n : ns) (Node tl _ tr) = ((n + 1) : ns'',
                                  Node tl' n tr')
    where
        (ns',  tl') = bfnAux ns tl
        (ns'', tr') = bfnAux ns' tr
```

Dynamic Programming

Dynamic Programming:

- Create a ***table*** of all subproblems that ever will have to be solved.
- Fill in table without regard to whether the solution to that particular subproblem will be needed.
- Combine solutions to form overall solution.

Lazy Evaluation is a perfect match as saves us from having to worry about finding a suitable evaluation order.

The Triangulation Problem (1)

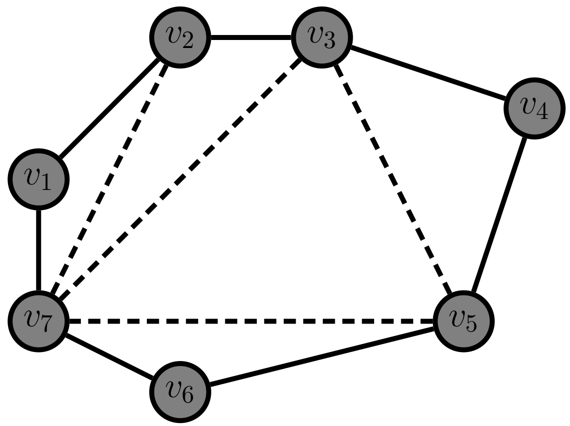
Select a set of ***chords*** that divides a convex polygon into triangles such that:

- no two chords cross each other
- the sum of their length is minimal.

We will only consider computing the minimal length.

See Aho, Hopcroft, Ullman (1983) for details.

The Triangulation Problem (2)



The Triangulation Problem (3)

- Let S_{is} denote the subproblem of size s starting at vertex v_i of finding the minimum triangulation of the polygon $v_i, v_{i+1}, \ldots, v_{i+s-1}$ (counting modulo the number of vertices).
- Subproblems of size less than 4 are trivial.
- Solving S_{is} is done by solving $S_{i,k+1}$ and $S_{i+k,s-k}$ for all k, $1 \leq k \leq s-2$
- The obvious recursive formulation results in 3^{s-4} recursive calls.
- But for n vertices there are only $n(n-4)$ non-trivial subproblems!

The Triangulation Problem (4)

- Let C_{is} denote the minimal triangulation cost of S_{is}.
- Let $D(v_p, v_q)$ denote the length of a chord between v_p and v_q (length is 0 for non-chords; i.e. adjacent v_p and v_q).
- For $s \geq 4$:

$$C_{is} = \min_{k \in [1, s-2]} \left\{ \begin{array}{l} C_{i,k+1} + C_{i+k,s-k} \\ +D(v_i, v_{i+k}) + D(v_{i+k}, v_{i+s-1}) \end{array} \right\}$$

- For $s < 4$, $S_{is} = 0$.

The Triangulation Problem (5)

These equations can be transliterated straight into Haskell:

```
triCost :: Polygon -> Double
triCost p = cost!(0,n) where
    cost = array ((0,0), (n-1,n))
                 ([ ((i,s),
                     minimum [ cost!(i, k+1)
                               + cost!((i+k) ‘mod‘ n, s-k)
                               + dist p i ((i+k) ‘mod‘ n)
                               + dist p ((i+k) ‘mod‘ n)
                                        ((i+s-1) ‘mod‘ n)
                             | k <- [1..s-2] ])
                  | i <- [0..n-1], s <- [4..n] ] ++
                  [ ((i,s), 0.0)
                  | i <- [0..n-1], s <- [0..3] ])
    n = snd (bounds b) + 1
```


Reading

- John W. Lloyd. Practical advantages of declarative programming. In *Joint Conference on Declarative Programming, GULP-PRODE’94*, 1994.
- John Hughes. Why Functional Programming Matters. *The Computer Journal*, 32(2):98–197, April 1989.

Reading

- Geraint Jones and Jeremy Gibbons. *Linear-time breadth-first tree algorithms: An exercise in the arithmetic of folds and zips.* Technical Report TR-31-92, Oxford University Computing Laboratory, 1992.
- Alfred Aho, John Hopcroft, Jeffrey Ullman. *Data Structures and Algorithms*. Addison-Wesley, 1983.