Type-Based Structural Analysis for Modular Systems of Equations

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The Problem (1)

- A core aspect of equation-based modelling: modular description of models through composition of equation system fragments.
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- Central questions:
  - *Does a system of equations have a (unique) solution?*
  - *Does an individual fragment “makes sense”?*
The Problem (1)

- A core aspect of equation-based modelling: modular description of models through composition of equation system fragments.

- Central questions:
  - *Does a system of equations have a (unique) solution?*
  - *Does an individual fragment “makes sense”?*

- Desirable to detect problematic fragments and compositions as early as possible.
The Problem (2)

- Consider:

\[ x + y + z = 0 \] (1)
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\[ x + y + z = 0 \]  \hspace{1cm} (1)

- Does not have a (unique) solution.
The Problem (2)

- Consider:

\[ x + y + z = 0 \quad (1) \]

- Does not have a (unique) solution.
- Could be part of a system that does have a (unique) solution.
Consider:

\[ x + y + z = 0 \]  

- Does not have a (unique) solution.
- Could be part of a system that does have a (unique) solution.

The same holds for:

\[ x - y + z = 1 \]
\[ z = 2 \]
The Problem (3)

- Composing (1) and (2):

\[
\begin{align*}
  x + y + z &= 0 \\
  x - y + z &= 1 \\
  z &= 2
\end{align*}
\]

Does have a solution.
The Problem (3)

- Composing (1) and (2):

\[
\begin{align*}
    x + y + z &= 0 \\
    x - y + z &= 1 \\
    z &= 2
\end{align*}
\]  

Does have a solution.

- However, the following fragment is over-constrained:

\[
\begin{align*}
    x &= 1 \\
    x &= 2
\end{align*}
\]  

(4)
The Problem (4)

- Cannot answer questions regarding solvability comprehensively before we have a complete system. *Not very modular!*
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  - *Variable-equation balance*
• Cannot answer questions regarding solvability comprehensively before we have a \textit{complete} system. \textit{Not very modular!}

• However, maybe violations of certain \textit{necessary} conditions for solvability can be checked modularly?
  - \textit{Variable-equation balance}
  - \textit{Structural singularity}
The Problem (4)

- Cannot answer questions regarding solvability comprehensively before we have a complete system. *Not very modular!*

- However, maybe violations of certain *necessary* conditions for solvability can be checked modularly?
  - *Variable-equation balance*
  - *Structural singularity*

- This talk: preliminary investigation into modular checking of structural singularity. (Paper at EOOLT’08)
Modular Systems of Equations (1)

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- a system of equations specifies a *relation* among a set of variables
- specifically, our interest is relations on time-varying values or *signals*
- an equation system fragment needs an *interface* to distinguish between local variables and variables used for composition with other fragments.
These ideas can be captured through a notion of **typed signal relations**:

\[
foo :: SR (\text{Real}, \text{Real}, \text{Real})
\]

\[
foo = \text{sigrel} (x_1, x_2, x_3) \text{ where } \begin{align*}
  f_1 & \quad x_1 \quad x_2 \quad x_3 = 0 \\
  f_2 & \quad x_2 \quad x_3 = 0
\end{align*}
\]

A signal relation is an **encapsulated equation system fragment**.
Modular Systems of Equations (2)

These ideas can be captured through a notion of **typed signal relations**:

\[
\text{foo} :: \text{SR} \left( \text{Real}, \text{Real}, \text{Real} \right) \\
\text{foo} = \text{sigrel} \left( x_1, x_2, x_3 \right) \text{ where} \\
\begin{align*}
    f_1 \ x_1 \ x_2 \ x_3 &= 0 \\
    f_2 \ x_2 \ x_3 &= 0
\end{align*}
\]

A signal relation is an **encapsulated equation system fragment**.
Of course, the ideas are general and not limited to equations over signals.
Modular Systems of Equations (3)

Composition can by expressed through *signal relation application*:

\[
\text{foo} \odot (u, v, w) \\
\text{foo} \odot (w, u + x, v + y)
\]

yields

\[
\begin{align*}
f_1 \quad &u \quad v \quad w &= 0 \\
f_2 \quad &v \quad w &= 0 \\
f_1 \quad &w \quad (u + x) \quad (v + y) &= 0 \\
f_2 \quad &u + x \quad (v + y) &= 0
\end{align*}
\]
Signal relations are *first class entities* at the functional layer. Offers way to parametrise the relations:

$$
\text{foo2} :: \text{Int} \rightarrow \text{Real} \rightarrow \text{SR} (\text{Real, Real, Real})
$$

$$
\text{foo2} \ n \ k = \text{sigrel} \ (x_1, x_2, x_3) \ \text{where}
\begin{align*}
    f_1 \ n \ x_1 \ x_2 \ x_3 &= 0 \\
    f_2 \ x_2 \ x_3 &= k
\end{align*}
$$
Example: Resistor Model

\[\text{twoPin} :: SR (Pin, Pin, Voltage)\]
\[\text{twoPin} = \text{sigrel} (p, n, u) \text{ where}\]
\[u = p.v - n.v\]
\[p.i + n.i = 0\]

\[\text{resistor} :: \text{Resistance} \rightarrow SR (Pin, Pin)\]
\[\text{resistor } r = \text{sigrel} (p, n) \text{ where}\]
\[\text{twoPin } \diamond (p, n, u)\]
\[r \ast p.i = u\]
Equal number of equations and variables is a necessary condition for solvability. For a modular analysis, one might keep track of the balance in the signal relation type:

\[ SR (\ldots) n \]
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\[ SR(\ldots) n \]

But very weak assurances:

\[
\begin{align*}
  f(x, y, z) &= 0 \\
  g(z) &= 0 \\
  h(z) &= 0
\end{align*}
\]
A system of equations is \textit{structurally singular} iff it is not possible to put variables and equations in a one-to-one correspondence such that each variable occurs in the equation it is related to.
A Possible Refinement (1)

A system of equations is *structurally singular* iff not possible to put variables and equations in a one-to-one correspondence such that each variable occurs in the equation it is related to.

Structural singularities are typically indicative of problems.
Structural singularities can be discovered by studying the \textit{incidence matrix}:

\begin{align*}
  f_1(x, y, z) &= 0 \\
  f_2(z) &= 0 \\
  f_3(z) &= 0
\end{align*}

\begin{equation}
  \begin{pmatrix}
    1 & 1 & 1 \\
    0 & 0 & 1 \\
    0 & 0 & 1
  \end{pmatrix}
\end{equation}
A Possible Refinement (3)

So maybe we can index signal relations with incidence matrices?

\[
\textit{foo} :: \text{SR (Real, Real, Real)} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}
\]

\[
\textit{foo} = \text{sigrel (x}_1, \ x_2, \ x_3) \quad \text{where}
\]

\[
f_1 \ x_1 \ x_2 \ x_3 = 0
\]

\[
f_2 \ x_2 \ x_3 \quad = 0
\]
Structural Type (1)

- The **Structural Type** represents information about which variables occur in which equations.
- Denoted by an incidence matrix.
- Two interrelated instances:
  - Structural type of a *system of equations*
  - Structural type of a *signal relation*
Structural Type (2)

- Structural type for composition of signal relations: *Straightforward*.
- The structural type of signal *relation* obtained by *abstraction* over the structural type of a system of equations: *Less straightforward*. 
Recall

\[ \text{foo} :: SR (\text{Real}, \text{Real}, \text{Real}) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \]

Consider

\[ \text{foo} \diamond (u, v, w) \]
\[ \text{foo} \diamond (w, u + x, v + y) \]

in a context with five variables \( u, v, w, x, y \).
The structural type for the equations obtained by instantiating \(foo\) is simply obtained by Boolean matrix multiplication. For \(foo \diamond (u, v, w)\):

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\begin{array}{cccccc}
u & v & w & x & y \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{array}
\end{pmatrix}
= 
\begin{pmatrix}
\begin{array}{cccccc}
u & v & w & x & y \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0
\end{array}
\end{pmatrix}
\]
For $foo \diamond (w, u + x, v + y)$:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
u & v & w & x & y \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
u & v & w & x & y \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1
\end{pmatrix}
\]
Composition of Structural Types (4)

Complete incidence matrix and corresponding equations:

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
\end{pmatrix}
\]

\[
\begin{align*}
& f_1 \ u \ v \ w = 0 \\
& f_2 \ v \ w = 0 \\
& f_1 \ w \ (u + x) \ (v + y) = 0 \\
& f_2 \ (u + x) \ (v + y) = 0 \\
\end{align*}
\]
Abstraction over Structural Types (1)

Now consider encapsulating the equations:

\[
\text{bar} = \text{sigrel} (u, y) \quad \text{where}
\]
\[
\text{foo} \diamond (u, v, w)
\]
\[
\text{foo} \diamond (w, u + x, v + y)
\]

The equations of the body of \text{bar} needs to be partitioned into

- **Local Equations**: equations used to solve for the local variables
- **Interface Equations**: equations contributed to the outside
Abstraction over Structural Types (2)

How to partition?
Abstraction over Structural Types (2)

How to partition?

- *A priori local equations*: equations over local variables only.
How to partition?

- **A priori local equations**: equations over local variables only.
- **A priori interface equations**: equations over interface variables only.
Abstraction over Structural Types (2)

How to partition?

• **A priori local equations**: equations over local variables only.

• **A priori interface equations**: equations over interface variables only.

• **Mixed equations**: equations over local and interface variables.
Abstraction over Structural Types (2)

How to partition?

- **A priori local equations**: equations over local variables only.

- **A priori interface equations**: equations over interface variables only.

- **Mixed equations**: equations over local and interface variables.

Note: too few or too many local equations gives an opportunity to catch *locally underdetermined* or *overdetermined* systems of equations.
Abstraction over Structural Types (3)

In our case:
Abstraction over Structural Types (3)

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- We have 1 a priori local equation, 3 mixed equations
Abstraction over Structural Types (3)

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- We need to choose 3 local equations and 1 interface equation
Abstraction over Structural Types (3)

In our case:

- We have 1 a priori local equation, 3 mixed equations
- We need to choose 3 local equations and 1 interface equation
- Consequently, 3 possibilities, yielding the following possible structural types for $\bar{a}r$:

\[
\begin{pmatrix}
  u & y \\
  1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
  u & y \\
  1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
  u & y \\
  1 & 1 \\
\end{pmatrix}
\]
Abstraction over Structural Types (4)

The two last possibilities are equivalent. But still leaves two distinct possibilities. How to choose?
Abstraction over Structural Types (4)

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- Note that a type with more variable occurrences is “better” as it gives more freedom when pairing equations and variables. Thus discard choices that are subsumed by better choices.
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- Assume the choice is free
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- As a last resort, approximate.
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- As a last resort, approximate.

Details in the EOOLT’08 paper.
Also in the Paper

- A more realistic modelling example:

- Structural types for components of this model

- Examples of errors caught by the proposed method, but that would not have been found by just counting equations and variables.
Problems

• Structural types not at all intuitive.
• The matrix notation is potentially cumbersome.
• User would likely often have to provide declarations of structural type explicitly.
• Type-checking is (currently) expensive.
• Sensible meta theory?
Questions

• How much do structural types buy over variable-equation balance in practice? Worth the complexity?

• Is there some sensible middle ground between structural types and variable-equation balance that provides most of the benefits of structural types, but in a simpler way?