G53KRR 2016-17
Answers to the exercise on syntax and semantics of FOL
Set in lecture 4, 13 October 2016.

In the exercises below, P is a unary predicate symbol, R is a binary predicate symbol, and f is a unary function symbol.

1. Consider an interpretation $J = (D, I)$ where $D = \{1, 2, 3\}$, $I(P) = \{1, 2\}$ and $I(R) = \{\langle 2, 2\rangle, \langle 2, 3\rangle\}$. Which of the following formulas are true in J under the variable assignment μ which assigns 1 to x and 2 to y:

   (a) $P(x)$
   Answer: yes, $J, \mu \models P(x)$ because $\mu(x) = 1$ and 1 is in $I(P)$.

   (b) $R(x, y)$
   Answer: no, $J, \mu \not\models R(x, y)$, because $\mu(x) = 1$, $\mu(y) = 2$, and $\langle 1, 2\rangle$ is not in $I(R)$.

   (c) $\exists x R(x, x)$:
   Answer: yes, $J, \mu \models \exists x R(x, x)$, because there is a value (2) which we can assign to x instead of the value that μ assigns to x, and for the resulting assignment $\mu\{2; x\}$ it holds that $J, \mu\{2; x\} \models R(x, x)$.

   (d) $\exists x \exists y(\neg(x = y) \wedge R(x, y))$
   Answer: yes, $J, \mu \models \exists x \exists y(\neg(x = y) \wedge R(x, y))$ because $J, \mu\{2; x, 3; y\} \models \neg(x = y) \wedge R(x, y)$.

   (e) $\exists x \forall y \neg R(x, y)$
   Answer: yes, $J, \mu \models \exists x \forall y \neg R(x, y)$: if x is given value 1, then for all possible values of y, $R(x, y)$ is false, so $\neg R(x, y)$ is true.

2. Construct some interpretation where $\forall x \forall y(R(x, y) \supset R(y, x))$ is true.
   Answer: Infinitely many correct answers are possible. The minimal one is a model with one element in the domain, for example $D = \{1\}$. It doesn't even matter whether $\langle 1, 1\rangle \in I(R)$ or not. Since there is only one value that can be assigned to x and y, the implication holds trivially.

3. Construct some interpretation where $\forall x \forall y(R(x, y) \supset R(y, x))$ is false.
   Answer: Infinitely many correct answers are possible. The minimal one is a model with two elements in the domain, for example $D = \{1, 2\}$ where $\langle 1, 2\rangle \in I(R)$ and $\langle 2, 1\rangle \notin I(R)$. *You don't have to come up with minimal counterexamples in the exam, unless asked explicitly.*

4. Let $J_1 = (D_1, I_1)$, where $D_1 = \{a, b\}$, $I_1(f)$ is the identity function ($I_1(f)(a) = a$ and $I_1(f)(b) = b$), and $I_1(R) = \{\langle a, a\rangle, \langle b, a\rangle\}$. An assignment μ_1 is such that $\mu_1(x) = a$.

   Does it hold that $J_1, \mu_1 \models \exists x R(x, f(x))$?
   Answer: Yes, we don't even need to find a different value for x, $\mu_1(x)$ works because it assigns a to x, $f(x)$ is also a because f is identity function, and $\langle a, a\rangle \in I(R)$.

5. Come up with an interpretation which makes $\forall x \exists y R(x, f(y))$ true.
Answer: Infinitely many correct answers are possible. J_1 above works because when x is a, we can find a value for y, which is a. When x is b, then the value for y is a.

6. Come up with an interpretation which makes $\forall x \exists y R(x, f(y))$ false.
Answer: Infinitely many correct answers are possible. We can change J_1 so that $I(R)$ only contains $\langle a, a \rangle$. Then there is no value for y which would work when x has value b.

7. (difficult - only do this if you actually like it). Find an interpretation where the three sentences below are true together. Is there a finite interpretation (one with a finite domain D) where they are all true?

   (a) $\forall x \neg R(x, x)$
   (b) $\forall x \exists y R(x, y)$
   (c) $\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \supset R(x, z))$

   Answer: Infinitely many correct answers to the first part of the question are possible. For example, D can be all natural numbers, and R can be interpreted as $<$. For the second part of the question: it is not possible to have a finite interpretation which satisfies all three sentences. Consider an arbitrary element a_1 in our hypothetical finite interpretation. It is not possible that $R(a_1, a_1)$ (I will write this instead of $\langle a_1, a_1 \rangle \in I(R)$ for brevity) holds, because of the first sentence (irreflexivity). So there should be a_2 such that $R(a_1, a_2)$ holds (because of the second sentence, seriality). Because $\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \supset R(x, z))$ (transitivity) is true, it is not possible that $R(a_2, a_1)$ holds, because otherwise $R(a_1, a_1)$ would hold, contradicting the first sentence. So we need some element a_3 to satisfy $R(a_2, a_3)$ (seriality), where a_3 is different from a_1 and a_2, and so on. We can show by induction that for any finite set of elements D, we either don't have an R-successor for some element in it, violating seriality, or we have a cycle $a_1 R a_2 R a_3 R ... R a_1$ which by transitivity means we have $R(a_1, a_1)$, violating irreflexivity.