Brachman and Levesque, chapter 2 exercise 4 .

Question .

Formulate the requirements below as sentences of first order logic and show that the two of them cannot be true together in any interpretation. (This is the barber's paradox by Bertrand Russell.)

1. Anyone who does not shave himself must be shaved by the barber [note the barber - some unique person, not a barber].
2. Whomever the barber shaves, must not shave himself.

Hints: introduce a constant for the barber and a binary predicate $Shaves(x, y)$.

Answer .

Translation:

1. $\forall x(\neg Shaves(x, x) \supset Shaves(barber, x))$
2. $\forall x(Shaves(barber, x) \supset \neg Shaves(x, x))$

Suppose some interpretation (D, I) satisfies both 1 and 2. There is an object in D which is the meaning of 'barber' in that interpretation, $I(barber)$. Let us call it b. This b either shaves himself or not, or in logic speak the pair (b, b) is either in the interpretation $I(Shaves)$ of the predicate $Shaves$, or not.

First suppose it is: $(b, b) \in I(Shaves)$. Then since the second sentence is true in (D, I), for every assignment v:

$$(D, I), v \models Shaves(barber, x) \supset \neg Shaves(x, x)$$

in particular for v which assigns b to x. (In plain language, again, since the sentence is true for all x, it must be true for the barber). Since

$$(D, I), v \models Shaves(barber, x)$$

(because $(b, b) \in I(Shaves)$), we have

$$(D, I), v \models \neg Shaves(x, x)$$

which means $(b, b) \notin I(Shaves)$. A contradiction.

So suppose $(b, b) \notin I(Shaves)$. Since the first sentence is true in (D, I), for every assignment v:

$$(D, I), v \models \neg Shaves(x, x) \supset Shaves(barber, x)$$

in particular this holds for v with $v(x) = b$. Since $(b, b) \notin I(Shaves)$,

$$(D, I), v \models \neg Shaves(x, x)$$

and since the implication is true,

$$(D, I), v \models Shaves(barber, x)$$

but the latter means $(b, b) \in I(Shaves)$. A contradiction again, so we cannot make both sentences true.