Brachman and Levesque, Chapter 3 exercise 1. An exam question may look like this (although obviously not exactly this question) and will be worth 25 points.

Question Consider the following piece of knowledge:

Tony, Mike and John belong to the Alpine Club. Every member of the Alpine Club who is not a skier is a mountain climber. Mountain climbers do not like rain, and anyone who does not like snow is not a skier. Mike dislikes whatever Tony likes, and likes whatever Tony dislikes. Tony likes rain and snow.

(a) Prove that the given sentences logically entail that there is a member of Alpine Club who is a mountain climber but not a skier.

(b) Suppose we had been told that Mike likes whatever Tony dislikes, but we had not been told that Mike dislikes whatever Tony likes. Prove that the resulting set of sentences no longer logically entails that there is a member of Alpine Club who is a mountain climber but not a skier.

Answer The answer given during one of the lectures used the following predicates and constants:

$Member$ unary predicate meaning a member of the Alpine Club

$Skier$ unary predicate meaning a skier

$Climber$ unary predicate meaning a climber

$Likes$ binary predicate where $Likes(x, y)$ means that x likes y

constants $tony$, $mike$, $john$, $rain$, $snow$

1. Translation into first order logic. I give sentences names so that it is easy to refer to them later.

Tony, Mike and John belong to the Alpine Club.

S1 $Member(tony)$

S2 $Member(mike)$

S3 $Member(john)$

It is not a mistake to translate them as one conjunction $Member(tony) \wedge Member(mike) \wedge Member(john)$ but more awkward to work with.

Every member of the Alpine Club who is not a skier is a mountain climber.

S4 $\forall x(Member(x) \wedge \neg Skier(x) \supset Climber(x))$

Mountain climbers do not like rain

S5 $\forall x(Climber(x) \supset \neg Like(x, rain))$

and anyone who does not like snow is not a skier.

S6 $\forall x(\neg Like(x, snow) \supset \neg Skier(x))$

Mike dislikes whatever Tony likes

S7 $\forall x(Like(tony, x) \supset \neg Like(mike, x))$

and likes whatever Tony dislikes.

S8 $\forall x(\neg Like(tony, x) \supset Like(mike, x))$

Tony likes rain and snow.

S9 $Like(tony, rain)$

S10 $Like(tony, snow)$

2. Proving that S1-S10 logically entail $\exists x(Member(x) \wedge Climber(x) \wedge \neg Skier(x))$.

Consider any interpretation (D, I) where S1-S10 are true. We have to show that

$$(D, I) \models \exists x(Member(x) \wedge Climber(x) \wedge \neg Skier(x))$$

The way to do this is to prove that there is some object $d \in D$ such that if an assignment v assigns d to x, then

$$(D, I), v \models Member(x) \wedge Climber(x) \wedge \neg Skier(x)$$

in other words, find an object $d \in D$ such that

$$d \in I(Member),\ d \in I(Climber), d \notin I(Skier)$$

Our only hold on which objects exist in D is that we know that D contains interpretations of *tony*, *mike*, *john*, *rain* and *snow*: $I(tony) \in D$, $I(mike) \in D$, and so on. We know of some of the properties of those objects because (D, I) satisfies the sentences S1-S10. For example from S1,

$$(D, I) \models Member(tony)$$

from the truth conditions we know that $I(tony) \in I(Member)$ (the object which is called 'tony' in (D, I), belongs to the set of things which are considered Club Members in (D, I)).

For one of those objects, we need to prove that it is in $I(Member)$, in $I(Climber)$, and is not in $I(Skier)$. Clearly *john*, *rain* and *snow* are non-starters.

Let us check if it could be true that $I(tony) \in I(Climber)$. From S9, we know that $(I(tony), I(rain)) \in I(Like)$. If $I(tony)$ were in $I(Climber)$, then there is an assignment v which assigns $I(tony)$ to x such that

$$(D, I), v \not\models Climber(x) \supset \neg Like(x, rain)$$

because $Climber(x)$ is true under v and $\neg Like(x, rain)$ is false. But this contradicts sentence S5 being true (it says that $Climber(x) \supset \neg Like(x, rain)$ is true for all assignments). So Tony can't be a climber.

Our last hope is Mike. From S10 we know that $(I(tony), I(snow)) \in I(Like)$. From S7 we know that for every assignment v,

$$(D, I), v \models Like(tony, x) \supset \neg Like(mike, x)$$

in particular if $v(x) = I(snow)$, we get that $(D, I), v \models \neg Like(mike, x)$. So $(I(mike), I(snow)) \notin I(Like)$. From S6 we know that for every v,

$$(D, I), v \models \neg Like(x, snow) \supset \neg Skier(x)$$

in particular if $v(x) = I(mike)$ this should also be true. So $I(mike) \notin I(Skier)$. Finally, from S4 we know that the set of members who are not skiers is included in the set of climbers, so since $I(mike) \in I(Member)$ and $I(mike) \notin I(Skier)$ then $I(mike) \in I(Climber)$. We have found an object with desired properties: if $v(x) = I(mike)$,

$$(D, I), v \models Member(x) \wedge Climber(x) \wedge \neg Skier(x)$$

so

$$(D, I) \models \exists x(Member(x) \wedge Climber(x) \wedge \neg Skier(x))$$

3. Suppose we do not have S7, only S1-S6 and S8-S10. Prove that $\exists x(Member(x) \wedge Climber(x) \wedge \neg Skier(x))$ no longer follows.

To do this, we have to produce an interpretation where S1-S6 and S8-S10 are true and the last sentence is false. The interpretation could be like this (there are other possible ones):

$D = \{t, m, j, s, r\}$

Interpretation:

$I(tony) = t, I(mike) = m, I(john) = j, I(snow) = s, I(rain) = r$

$I(Member) = \{t, m, j\}$

$I(Skier) = \{t, m, j\}$

$I(Climber) = \{\}$

$I(Like) = \{(t, s), (t, r), (m, s), (m, r), (m, m), (m, t), (m, j), (j, s)\}$ (that is, Tony likes rain and snow as before, Mike likes every single object in the universe, John likes snow, and rain and snow don't have any feelings about things).

Now S1-S3 are obviously true. S4 is trivially true because there is no member who is not a skier. S5 is also trivially true because there are no climbers. S6 is true because the only skiers we have are t, m, j and they all like snow. S8 is true because Mike likes everything. S9 and S10 are true because we included $(t, s), (t, r)$ in $I(Like)$.

Finally, the sentence $\exists x(Member(x) \wedge Climber(x) \wedge \neg Skier(x))$ is false because there are no climbers so we cannot find an assignment to x which would make $Climber(x)$ true.