Handout for G53KRR lecture on resolution .

1. Reducing a first order sentence to clausal normal form.

1. eliminate $\supset$ and $\equiv$ using

$$(\alpha \supset \beta) \equiv (\neg\alpha \vee \beta)$$

$$(\alpha \equiv \beta) \equiv ((\alpha \supset \beta) \wedge (\beta \supset \alpha))$$

2. move $\neg$ inward so that it appears only in front of an atom, using

$$\neg\neg\alpha \equiv \alpha$$

$$\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$$

$$\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta)$$

$$\neg\forall x\alpha \equiv \exists x\neg\alpha$$

$$\neg\exists x\alpha \equiv \forall x\neg\alpha$$

3. ensure that each quantifier has a distinct variable by renaming:

$$\forall x\alpha \equiv \forall y\alpha(x/y)$$

$$\exists x\alpha \equiv \exists y\alpha(x/y)$$

where y does not occur in $\forall x\alpha$ and $\exists x\alpha$ and $\alpha(x/y)$ means α with all occurrences of x replaced by y.

4. eliminate existentials using Skolemisation:

if $\exists x\alpha$ is not in the scope of any universal quantifiers, then we replace $\exists x\alpha$ with $\alpha(a)$ where a is a new constant called a Skolem constant. (It should be different for every existential quantifier).

if $\exists x\alpha$ is in the scope of universal quantifiers $\forall x_1, \dots, \forall x_n$ (and these are all universal quantifiers it is in the scope of):

$$\forall x_1(\dots\forall x_2 \dots \forall x_n(\dots\exists x\alpha)\dots)$$

then replace $\exists x\alpha$ by $\alpha(x/f(x_1,\dots,x_n))$ where f is a Skolem function (again use a different Skolem function for every existential quantifier). Example: $\exists x_1\exists x_2\forall y\exists z P(x_1,x_2,y,z)$ becomes $\forall y P(c_1,c_2,y,f(y))$.

5. Move universals outside the scope of $\wedge$ and $\vee$ using the following equivalences (provided x is not free in α):

$$(\alpha \wedge \forall x\beta) \equiv \forall x(\alpha \wedge \beta)$$

$$(\alpha \vee \forall x\beta) \equiv \forall x(\alpha \vee \beta)$$

6. We now got $\forall x_1 \ldots \forall x_n \alpha$ where α does not contain quantifiers. Reduce α to CNF as before using distributivity:

$$\alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$$

7. Collect terms:

$$(\alpha \vee \alpha) \equiv \alpha$$

$$(\alpha \wedge \alpha) \equiv \alpha$$

8. Reduce to clausal form by dropping universal quantifiers and conjunctions and making disjunctions into clauses (lists of literals):

$$\forall x((P(x) \vee \neg R(a, f(b, x))) \wedge Q(x, y))$$

becomes

$$[P(x), \neg R(a, f(b, x))], [Q(x, y)]$$

9. Actually need also to do *factoring*: if a clause contains two literals ρ_1 and ρ_2 which unify in both directions (there is a substitution θ such that $\rho_1\theta = \rho_2$ and θ' such that $\rho_2\theta' = \rho_1$) then replace them with a single literal. For example, $[P(x), P(y)]$ becomes $[P(x)]$.

2. Substitution and unification.

A substitution θ is a finite set of pairs $\{x_1/t_1, \ldots, x_n/t_n\}$ where x_i are distinct variables and t_i are arbitrary terms (could all be the same variable y, or $f(x, y, b)$ or whatever). If ρ is a literal then $\rho\theta$ is a literal which results from simultaneously substituting each x_i in ρ by t_i. Same for clauses: if c is a clause the $c\theta$ is the result of applying the substitution to all literals in c.

For example, if $\theta = \{x/a, y/g(x, b, z)\}$ then

$$[P(x), \neg R(a, f(b, x))]\theta = [P(a), \neg R(a, f(b, a))]$$

$$[Q(x, y)]\theta = [Q(a, g(x, b, z))]$$

θ unifies (is a unifier for) two literals ρ_1 and ρ_2 if $\rho_1\theta = \rho_2\theta$. For example, $P(x, f(x))$ and $P(y, f(a))$ are unified by $\theta = \{x/a, y/a\}$.

3. General rule of resolution:

$$\frac{c_1 \cup \{\rho_1\} \quad c_2 \cup \{\neg\rho_2\}}{(c_1 \cup c_2)\theta}$$

where θ unifies ρ_1 and ρ_2: $\rho_1\theta = \rho_2\theta$. We also assume that we renamed all variables in $c_1 \cup \{\rho_1\}$ and $c_2 \cup \{\neg\rho_2\}$ so that each clause has its distinct variables.

Example:

$$\frac{[\neg Man(x), Mortal(x)] \quad [Man(socrates)]}{[Mortal(socrates)]}$$

using $\theta = \{x/socrates\}$. (So $[Mortal(x)]\theta = [Mortal(socrates)]$.)