Introduction to Model Theory

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Answers to exercises from handout 1

1. Prove that if h_1 is a homomorphism from A to B and h_2 is a homomorphism from B to C, then their composition is a homomorphism from A to C.

   We need to show that $h = h_2 \circ h_1$ is a homomorphism, i.e.:

   - for each constant c, $h(c^A) = c^C$.
     Since h_1 is a homomorphism, $h_1(c^A) = c^B$. Similarly, $h_2(c^B) = c^C$, which together gives $h_2(h_1(c^A)) = c^C$.
   - for each n-ary relation symbol R and every n-tuple of elements of $dom(A)$, if $\langle a_1, \ldots, a_n\rangle \in R^A$ then $\langle h(a_1), \ldots, h(a_n)\rangle \in R^C$.
     Since h_1 is a homomorphism, if $\langle a_1, \ldots, a_n\rangle \in R^A$, then $\langle h_1(a_1), \ldots, h_1(a_n)\rangle \in R^B$. Since h_2 is a homomorphism, if $\langle h_1(a_1), \ldots, h_1(a_n)\rangle \in R^B$, then $\langle h_2(h_1(a_1)), \ldots, h_2(h_1(a_n))\rangle \in R^C$.
   - for each n-ary function symbol f in L and every n-tuple of elements of $dom(A)$, $h(f^A(a_1, \ldots, a_n)) = f^C(h(a_1), \ldots, h(a_n))$.
     Since h_1 is a homomorphism, $h_1(f^A(a_1, \ldots, a_n)) = f^B(h_1(a_1), \ldots, h_1(a_n))$.
     $h_2(h_1(f^A(a_1, \ldots, a_n))) = h_2(f^B(h_1(a_1), \ldots, h_1(a_n))) =$
     $= f^C(h_2(h_1(a_1)), \ldots, h_2(h_1(a_n))))$.

2. Give a truth definition for $\forall x\varphi$ given that $\forall x\varphi$ should be logically equivalent to $\neg\exists x\neg\varphi$.

   $A, s \models \forall x\varphi$ iff for all assignments s' which differ from s at most in the value assigned to x, $A, s' \models \varphi$.

3. Write a first order formula which is only true

   (a) on structures with more than three elements:
   $\exists x\exists y\exists z(\neg(x = y) \wedge \neg(x = z) \wedge \neg(y = z))$.

   (b) on structures with infinitely many elements:
   $\forall x\exists y R(x, y) \wedge$
   $\forall x\forall y\forall z(R(x, y) \wedge R(y, z) \rightarrow R(x, z)) \ \wedge \ \forall x\neg R(x, x)$.

   (c) on structures with even number of elements: impossible.

4. Prove that only the values assigned by s to the free variables of φ matter for deciding whether $A, s \models \varphi$. Namely, if there are two assignments s and z which agree on the values assigned to the free variables of φ, then $A, s \models \varphi$ iff $A, z \models \varphi$.

   First we need to show that if two assignments agree on the variables which occur in a term t, they assign the same value to t: $v^s_A(t) = v^z_A(t)$. This is proved by induction on the construction of the term.

   - if t is a variable x (which obviously occurs in t), s and z agree by assumption: $v^s_A(x) = s(x) = z(x) = v^z_A(x)$;
   - the value of constants does not depend on assignment:
     $v^s_A(c) = v^z_A(x) = c^A$;
   - the value of complex terms depends on the interpretation of function symbols (which does not depend on assignment) and the values of arguments (which by the inductive hypothesis are the same for s and z): $v^s_A(f(t_1, \ldots, t_n) = f^A(v^s_A(t_1), \ldots, v^s_A(t_n)) = f^A(v^z_A(t_1), \ldots, v^z_A(t_n)) = v^z_A(f(t_1, \ldots, t_n)$.

Now we prove by induction that the same formulas are satisfied by s and z.

- $A, s \models R(t_1, \ldots, t_n)$ iff $\langle v_A^s(t_1), \ldots, v_A^s(t_n)\rangle \in R^A$ iff $\langle v_A^z(t_1), \ldots, v_A^z(t_n)\rangle \in R^A$ iff $A, z \models R(t_1, \ldots, t_n)$ (since $v_A^s(t_i) = v_A^z(t_i)$ by the argument above, and R^A does not depend on assignment).
- $A, s \models \neg\varphi$ iff $A, s \not\models \varphi$. The size of φ is less than the size of $\neg\varphi$, so by the induction hypothesis the latter holds iff $A, z \not\models \varphi$ iff $A, z \models \neg\varphi$.
- $A, s \models \varphi \wedge \psi$ iff $A, s \models \varphi$ and $A, s \models \psi$ by the inductive hypothesis iff $A, z \models \varphi$ and $A, z \models \psi$ iff $A, z \models \varphi \wedge \psi$.
- $A, s \models \exists x\varphi$ iff for some assignment s' which differs from s at most in the value assigned to x, $A, s' \models \varphi$. Consider an assignment z' which is the same as z except $z'(x) = s'(x)$. Now s' and z' agree on the free variables of φ, so by the inductive hypothesis $A, z' \models \varphi$. So $A, z \models \exists x\varphi$. The same argument applis to show that if $A, z \models \exists x\varphi$, then $A, s \models \exists x\varphi$.

5. Prove that isomorphic structures satisfy the same first order formulas.

   Prove by induction: for any assignment s, $A, s \models \varphi$ iff $B, hs \models \varphi$, where $hs(x) = h(s(x))$.

   First show that for every term t, h sends the value of t in A under s to the value of t in B under hs, that is $h(v_A^s(t)) = v_B^{hs}(t)$.

   - for variables, this is immediate: $h(v_A^s(x)) = h(s(x)) = v_B^{hs}(x)$.
   - $h(v_A^s(c)) = h(c^A) = c^B = v_B^{hs}(c)$.
   - $h(v_A^s(f(t_1, \ldots, t_n)) = f^B(h(v_A^s(t_1)), \ldots, h(v_A^s(t_n))) = f^B(v_B^{hs}(t_1), \ldots, v_B^{hs}(t_n)) = v_B^{hs}(f(t_1, \ldots, t_n))$.

   Now for formulas, again by induction:

   - $A, s \models R(t_1, \ldots, t_n)$ iff $\langle v_A^s(t_1), \ldots, v_A^s(t_n)\rangle \in R^A$ iff (by the definition of isomorphism) $\langle h(v_A^s(t_1)), \ldots, h(v_A^s(t_n))\rangle \in R^B$ iff (since $h(v_A^s(t_i)) = v_B^{hs}(t_i)$) $\langle v_B^{hs}(t_1)), \ldots, v_B^{hs}(t_n))\rangle \in R^B$ iff $B, hs \models R(t_1, \ldots, t_n)$.
   - $A, s \models \neg\varphi$ iff $A, s \not\models \varphi$ iff (by the inductive hypothesis) $B, hs \not\models \varphi$ iff $B, hs \models \neg\varphi$.
   - $A, s \models \varphi \wedge \psi$ iff $A, s \models \varphi$ and $A, s \models \psi$ by the inductive hypothesis iff $B, hs \models \varphi$ and $B, hs \models \psi$ iff $B, hs \models \varphi \wedge \psi$.
   - $A, s \models \exists x\varphi$ iff for some assignment s' which differs from s at most in the value assigned to x, $A, s' \models \varphi$. Consider an assignment hs' which is $h \circ s'$. By the inductive hypothesis, $A, s' \models \varphi$ iff $B, hs' \models \varphi$ hence $B, hs \models \exists x\varphi$.

     Assume $B, hs \models \exists x\varphi$, then for some assignment z which differs from hs at most in the value for x, $B, z \models \varphi$. Since h is a bijection, there is a unique element $a \in A = h^{-1}(z(x))$. Consider the assignment s' which is the same as s but $s'(x) = a$. By the inductive hypothesis, $A, s' \models \varphi$, so $A, s \models \exists x\varphi$.

6. Can you say anything similar about homomorphic structures: e.g. if there is a homomorphism from A to B and ψ is a certain kind of formula, then if A satisfies ψ, so does B?

   If φ is a formula build from atomic formulas using $\wedge$, $\vee$ and $\exists$, and h is a homomorphism, then if $A, s \models \varphi$, then $B, hs \models \varphi$. The proof is much as above, but we can only show that:

- for atomic formulas, if A, s satisfy the formula, then B, hs do (not vice versa, because of the definition of homomorphism);
- for conjunctions and disjunctions: if A, s satisfy the formula, so do B, hs.
- for existential quantifiers: if A, s satisfy the formula, then B, hs do (cannot use the fact about bijection and the inductive hypothesis to prove the reverse).