Handout 2
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Introduction to Model Theory
Natasha Alechina

Maps and formulas they preserve.

Additional truth definitions:

- $A, s \models R(t_1, \ldots, t_n)$ iff $\langle v^s_A(t_1), \ldots, v^s_A(t_n)\rangle \in R^A$.
- $A, s \models \neg\varphi$ iff $A, s \not\models \varphi$
- $A, s \models \varphi \wedge \psi$ iff $A, s \models \varphi$ and $A, s \models \psi$
- $A, s \models \varphi \vee \psi$ iff $A, s \models \varphi$ or $A, s \models \psi$
- $A, s \models \varphi \rightarrow \psi$ iff $A, s \not\models \varphi$ or $A, s \models \psi$
- $A, s \models \exists x\varphi$ iff for some assignment s' which differs from s at most in the value assigned to x, $A, s' \models \varphi$.
- $A, s \models \forall x\varphi$ iff for every assignment s' which differs from s at most in the value assigned to x, $A, s' \models \varphi$.

A sentence is a formula with no free variables.

Two structures are elementarily or first order equivalent if they satisfy the same first order sentences.

Theorem 1 *If A and B are structures in the same signature L and h an isomorphism from A to B, then for all first order formulas φ in the signature L, $A \models \varphi$ iff $B \models \varphi$.*

Proof hint. Prove by induction: for any assignment s, $A, s \models \varphi$ iff $B, hs \models \varphi$, where $hs(x) = h(s(x))$. First show that for every term t, h sends the value of t in A under s to the value of t in B under hs.

Theorem 1 implies that isomorphic structures are first order equivalent.

The reverse is true for finite structures:

Theorem 2 *If A and B are two finite elementarily equivalent structures, then there is an isomorphism between A and B.*

Exercise (*): give a counterexample for infinite structures.

Additional piece of notation (formula with parameters) $A \models \varphi[a_1, \ldots, a_n]$ for $A, s \models \varphi(x_1, \ldots, x_n)$ with $s(x_i) = a_i$.

Proof hint. The property of having exactly n elements is first-order expressible, so A and B should have the same number of elements. Suppose $A = \{a_1, \ldots, a_n\}$. There are $n!$ possible bijections h between A and B. If none of them is an isomorphism, then for every h there is an atomic or negated atomic formula $\varphi_h(x_1, \ldots, x_n)$ such that $\varphi_h[a_1, \ldots, a_n]$ is true on A and $\varphi_h[h(a_1), \ldots, h(a_n)]$ false on B. Then $\exists x_1 \ldots \exists x_n(\bigwedge \neg(x_i = x_j) \wedge \bigwedge \varphi_h(x_1, \ldots, x_n))$ is true on A and false on B.

A positive formula is built from atomic formulas using conjunction, disjunction and existential and universal quantifiers only (no negation).

An existential formula is built from atomic formulas and their negations using conjunction, disjunction and existential quantification only (no universal quantifiers and negations apart from on atomic formulas).

A universal formula is built from atomic formulas and their negations using conjunction, disjunction and universal quantification only (no existential quantifiers and negations apart from on atomic formulas).

A Horn formula is a disjunction of atomic and negated atomic formulas where at most one disjunct is non-negative and all variables are universally quantified.

A more convenient representation of Horn formulas is

$$\forall x_1 \ldots \forall x_n(\alpha_1 \wedge \ldots \wedge \alpha_m \to \alpha)$$

where α_i are atomic formulas with free variables among $x_1, \ldots, x_n$.

Theorem 3 *If A and B are structures in the same signature L and h a homomorphism from A to B, then for all positive existential first order sentences φ in the signature L, if $A \models \varphi$, then $B \models \varphi$.*

Proof hint. Same as for isomorphisms, but instead of $\Leftrightarrow$ we have $\Rightarrow$. Can show that if A satisfies an atomic formula, then B does, if A satisfies a conjunction, disjunction or an existential formula, then B does. *Exercise: give a counterexample for a formula containing a negation and for a universally quantified formula.*

Theorem 4 *If B is a substructure of A, then for all universal first order sentences φ, if $A \models \varphi$, then $B \models \varphi$.*

Theorem 5 *If A is embedded in B, then for all existential first order sentences φ, if $A \models \varphi$, then $B \models \varphi$.*

Direct product of structures $A_1, \ldots, A_n$ is a structure $A = A_1 \times \ldots \times A_n$ with

- $dom(A) = dom(A_1) \times \ldots \times dom(A_n)$
- $c^A = \langle c^{A_1}, \ldots, c^{A_n} \rangle$
- Interpretation of relation symbols: let $s_1, \ldots, s_k$ be elements of $dom(A)$ (sequences of elements of $A_1, \ldots, A_n$). Refer to the ith element of the sequence s_j by s_j^i. Then $\langle s_1, \ldots, s_k \rangle \in R^A$ iff for every i, $1 \leq i \leq n$, $\langle s_1^i, \ldots, s_k^i \rangle \in R^{A_i}$.
- $f^A(s_1, \ldots, s_k) = \langle f^{A_1}(s_1^1, \ldots, s_k^1), \ldots, f^{A_n} s_1^n, \ldots, s_k^n) \rangle$.

Theorem 6 *Let $A_1, \ldots, A_n$ be structures in the same signature, and A their direct product. Then if fr every i, A_i satisifes a Horn formula φ, then $A \models \varphi$.*

Exercise. Show that the property "At any moment of time, if the value of a is positive, then either the value of b is positive or the value of c is positive" is not definable by a Horn formula. (For simplicity, assume that you have unary relation symbols P_a, P_b and P_c where $P_a(x)$ means "a is positive at the moment x".)