Semantics for Dynamic Syntactic Epistemic Logics

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Abstract

Traditional epistemic logic assumes that agents' knowledge is closed under logical consequence. Many attempts to solve this logical omniscience problem weaken the closure condition by assuming that agents are ignorant of certain logical rules. Duc (1997) avoids the apparent paradox of non-omniscience and non-ignorance by introducing propositions on the form "ϕ is true after some train of thought of agent i" explicitly into the language. A logic $DES4_n$ for this language is presented as a dynamic version of $S4_n$. $DES4_n$ describes agents who do not necessarily know any ($S4_n$) consequence of their knowledge now, but can get to know any such consequence in the future. Duc does not, however give a semantics for $DES4_n$. In this paper we provide a semantics, for $DES4_n$ and some weaker systems, and prove soundness and completeness. A key assumption is that an agent can only know a finite number of formulae at each time. The semantics is based on Kripke models, where each world syntactically assigns a finite number of formulae to each agent and the transitions model steps of reasoning.

Introduction

Traditional epistemic logic, defining knowledge as truth in all indiscernible worlds in a Kripke structure, assumes that agents' knowledge is closed under logical consequence. For example, if an agent knows both the formula ϕ and the formula ψ, it is assumed that he also knows the formula $\phi \wedge \psi$. Or, if he knows the rules of chess he knows whether white has a winning strategy. The fact that this is not a description of real agents has been called the logical omniscience problem (Hintikka 1975). Many attempts to solve this logical omniscience problem weaken the closure condition by assuming that agents are ignorant of certain logical rules (see e.g. (Moreno 1998; Sim 1997; Fagin *et al.* 1995) for surveys). Duc (1997) avoids the apparent paradox of non-omniscience and non-ignorance by introducing propositions on the form "ϕ is true after some train of thought of agent i" explicitly into the language. Rather than modeling such "trains of thought" explicitly, he adds a generic operator $\langle F_i \rangle$, for each agent i, to the language. The intended meaning of a formula such as $\langle F_i \rangle K_i(\phi \wedge \psi)$ is that agent i can get to know the formula $\phi \wedge \psi$ some time

in the future. Duc presents a formal logical system $DES4_n$ for this language, intended to be a "dynamic" version of $S4_n$. $DES4_n$ does not have the usual closure properties for knowledge, such as the K axiom; it describes agents who do not necessarily know any consequence of their knowledge now but can get to know any such consequence in the future.

Duc does not, however, present a semantics for $DES4_n$. In (Duc 1995), he describes semantics and proves completeness for a simpler system, however according to that semantics an agent may know infinitely many formulae at a single point in time. While it is not mentioned explicitly in (Duc 1997) that the agents described by $DES4_n$ are assumed to know only finitely many formulae at each point in time, it is an implicit consequence of the assumption that an agent does not know anything automatically and that getting to know any formula requires a (finite) "train of thought".

In this paper we interpret the language of $DES4_n$ in Kripke structures. Rather than using the transitions to model indiscernibility, we use them to model steps of reasoning. Knowledge is modeled by syntactic assignments (see e.g. (Fagin *et al.* 1995; Ågotnes & Walicki 2005b)), describing for each agent i a set of formulae X_i known by i in a given state (under certain restrictions prescribed by $DES4_n$). A crucial point is that we require the set X_i for each agent i in each state to be *finite*. Thus we provide a natural semantics, both for $DES4_n$ and for some weaker systems, and prove soundness and completeness.

In the next section, we formally introduce the logic $DES4_n$. Then we present our semantics for dynamic epistemic logics. In the sections that follow, we show soundness and completeness of both $DES4_n$ and some related systems. The latter include both weaker systems, and general systems containing axioms expressing that an agent knows an inference rule. Finally, we discuss related work and conclude.

System $DES4_n$

The logic $DES4_n$ (Duc 1997) is a formal system in the language $\mathcal{LDE}$, which is a propositional language with epistemic operators for expressing that something is known now and temporal operators for expressing what will be the case in the future. $\mathcal{LDE}$ is parameterised by a set of primitive propositions Θ and a number of agents n. We write $Agt = \{1, \dots, n\}$ for the set of agents.

Language

First, the traditional language of epistemic logic is the language of propositional logic extended with a operator K_i for each agent i. $\mathcal{L}_n(\Theta)$, or just $\mathcal{L}$, is defined as follows.

$$\Theta \subset \mathcal{L}$$

If $i \in Agt$ and $\phi \in \mathcal{L}$ then $K_i\phi \in \mathcal{L}$

If $\phi \in \mathcal{L}$ then $\neg\phi \in \mathcal{L}$

If $\phi, \psi \in \mathcal{L}$ then $(\phi \to \psi) \in \mathcal{L}$

An operator K_i is called an epistemic operator. A formula $\phi \in \mathcal{L}$ is an *objective formula* if it has no occurrence of an epistemic operator.

The language $\mathcal{LDE}$ adds an operator $\langle F_i \rangle$ for each agent i. The use of the operator is restricted, however: it is not allowed within the scope of an epistemic operator. $\mathcal{LDE}_n(\Theta)$, or just $\mathcal{LDE}$, is defined as follows.

$$\mathcal{L} \subseteq \mathcal{LDE}$$

If $\phi \in \mathcal{LDE}$ then $\neg\phi \in \mathcal{LDE}$

If $\phi, \psi \in \mathcal{LDE}$ then $(\phi \to \psi) \in \mathcal{LDE}$

If $i \in Agt$ and $\phi \in \mathcal{LDE}$ then $\langle F_i \rangle\phi \in \mathcal{LDE}$

The usual derived propositional connectives are used. The operator $[F_i]$ is defined as a dual to $\langle F_i \rangle$ in the usual way: $[F_i]\phi \equiv \neg\langle F_i \rangle\neg\phi$. A *maximal* set of $\mathcal{LDE}$ formulae is a set containing either ϕ or $\neg\phi$, for any $\phi \in \mathcal{LDE}$.

The intended meaning of $\langle F_i \rangle\phi$ and $[F_i]\phi$ are that ϕ is true after some train of thought of i, and after every train of thought of i, respectively. Thus, the "F" in the operator can be read as standing for "Future".

A formula starting with an epistemic operator is an *epistemic atom*. We will sometimes use $At(\phi)$ to denote the set of primitive propositions and epistemic atoms which occur outside the scope of any epistemic operators in a formula ϕ. For example, $At((\langle F_i \rangle K_i K_i(p \wedge q)) \to p) = \{K_i K_i(p \wedge q), p\}$ – this set does *not* contain $K_i(p \wedge q)$ nor q since both the epistemic atom $K_i(p \wedge q)$ and the primitive proposition q only occur within the scope of an epistemic operator.

Axiomatic system

In the definition of the system $DES4_n$ for the language $\mathcal{LDE}$, a sublanguage $\mathcal{L}_E^+ \subseteq \mathcal{L}$ of *persistent formulae* is used. $\mathcal{L}_E^+$ is the least set satisfying the following conditions:

If ϕ is an objective formula then $\phi \in \mathcal{L}_E^+$

If $\phi, \psi \in \mathcal{L}_E^+$ then $(\phi \wedge \psi) \in \mathcal{L}_E^+$

If $\phi, \psi \in \mathcal{L}_E^+$ then $(\phi \vee \psi) \in \mathcal{L}_E^+$

If $i \in Agt$ and $\phi \in \mathcal{L}_E^+$ then $K_i\phi \in \mathcal{L}_E^+$

$DES4_n$ has the following axiom schemata:

*PC*1.	$\phi \to (\psi \to \phi)$
*PC*2.	$(\phi \to (\psi \to \gamma)) \to ((\phi \to \psi) \to (\phi \to \gamma))$
*PC*3.	$(\neg\psi \to \neg\phi) \to (\phi \to \psi)$
*TL*1.	$[F_i](\phi \to \psi) \to ([F_i]\phi \to [F_i]\psi)$
*TL*2.	$[F_i]\phi \to [F_i][F_i]\phi$
*DE*1.	$K_i\phi \wedge K_i(\phi \to \psi) \to \langle F_i \rangle K_i\psi$
*DE*2.	$K_i\phi \to \phi$
*DE*3.	$K_i\phi \to [F_i]K_i\phi$, if $\phi \in \mathcal{L}_E^+$
*DE*4.	$\langle F_i \rangle K_i(\phi \to (\psi \to \phi))$
*DE*5.	$\langle F_i \rangle K_i((\phi \to (\psi \to \gamma)) \to ((\phi \to \psi) \to (\phi \to \gamma)))$
*DE*6.	$\langle F_i \rangle K_i((\neg\psi \to \neg\phi) \to (\phi \to \psi))$
*DE*7.	$\langle F_i \rangle K_i(K_i\phi \to \phi)$
*DE*8.	$K_i\phi \to \langle F_i \rangle K_i K_i\phi$, if $\phi \in \mathcal{L}_E^+$

and the following inference rules:

*R*1.	From $\phi, \phi \to \psi$, prove ψ
*R*2.	From ϕ, prove $[F_i]\phi$

where $i \in agt$. We use $\vdash_{DES4_n}$ to denote derivability in $DES4_n$.

PC1–PC3 and R1 axiomatise the underlying propositional logic. TL1–TL2 and R2 axiomatise the $\langle F_i \rangle$ operators, essentially as a temporal "future" operator satisfying axioms of the modal system $K4$. DE1 and DE4–DE6 say that an agent can reason perfectly in propositional logic. DE2 say that an agent can never get to know anything false or inconsistent, and DE7 that he can get to know this after some reasoning. DE3 says that an agent always reasons monotonically. Only persistent formulae are included in this definition of monotone reasoning, however. In particular, negative epistemicals (e.g. $\neg K_i\phi$) are not persistent. Finally, DE8 says that an agent can do positive introspection. In the next section we make these semantic conditions implicit in the logic $DES4_n$ explicit.

Examples of $DES4_n$ theorems illustrating that the logic describes agents who are non-omniscient but nevertheless are rational and non-ignorant are:

$$K_i(\phi \wedge \psi) \to \langle F_i \rangle K_i\phi$$
$$K_i\neg\neg\phi \to \langle F_i \rangle K_i\phi$$
$$(K_i\phi \wedge K_i\psi) \to \langle F_i \rangle K_i(\phi \wedge \psi)$$

Note that there are no axioms in $DES4_n$ connecting the knowledge of different agents. From this point of view, the system could have been presented for a single agent. We continue to study the multi-agent case, both because the system was originally introduced in this form, and because there is an obvious option of extending such a system with axioms saying, e.g., 'agent i eventually communicates all his knowledge to agent j':

$$K_i\phi \to \langle F_i \rangle K_j\phi$$

Structures and Interpretation

By viewing the epistemic atoms as primitives and the $\langle F_i \rangle$ operators as diamonds, we can interpret the language $\mathcal{LDE}$

as a modal language in Kripke structures. As discussed in the introduction, we are mainly interested in structures in which only a finite number of epistemic atoms are true in each state. Let

$$ATOMS = \Theta \cup \{K_i\phi : K_i\phi \in \mathcal{LDE}\}$$

We will henceforth refer to members of *ATOMS* as *atoms*.

Definition 1 *A model is a tuple $M = (W, R_1, \ldots, R_n, V)$, where W is a non-empty set of states, R_i a binary relation over W for each $i \in Agt$, and V a function $V : W \to \wp(ATOMS)$ such that $\{\phi : K_i\phi \in V(w)\}$ is finite for every w and i. The set $\{\phi : K_i\phi \in V(w)\}$ of formulae known by agent i in state w is called agent i's* epistemic state *in w. The class of all models is denoted $\mathcal{M}(\Theta, n)$ (or just $\mathcal{M}$).*

A *general model* is a model without the requirement that epistemic states are finite:

Definition 2 *A general model is a tuple $M = (W, R_1, \ldots, R_n, V)$ where W is a non-empty set of states, R a binary relation over W for each i, and V a function $V : W \to \wp(ATOMS)$. The class of all general models is denoted $\mathcal{M}^{gen}(\Theta, n)$ (or just $\mathcal{M}^{gen}$).*

When R_i is a binary relation over W and $w, v \in W$, we will freely use different notations $R_i wv$, $wR_i v$, $R_i(w, v)$, $(w, v) \in R_i$, to denote the fact that R_i relates w to v.

The interpretation of $\mathcal{LDE}$ in (general) models is defined as usual in modal logic (see, e.g., (Blackburn, de Rijke, & Venema 2001)). When $M = (W, R_1, \ldots, R_n, V)$ and $w \in W$:

$$
\begin{array}{lll}
M, w \models p & \Leftrightarrow & p \in V(w), \text{ when } p \in \Theta \\
M, w \models K_i\alpha & \Leftrightarrow & K_i\alpha \in V(w) \\
M, w \models \neg\phi & \Leftrightarrow & M, w \not\models \phi \\
M, w \models \phi \to \psi & \Leftrightarrow & M, w \models \phi \text{ implies that } M, w \models \psi \\
M, w \models \langle F_i \rangle \phi & \Leftrightarrow & \exists_{(w,w') \in R_i} M, w' \models \phi
\end{array}
$$

Clearly, under this interpretation, restrictions must be imposed on the class of models in order to get completeness of $DES4_n$. In the next sections we show completeness of $DES4_n$ and of weaker logics with respect to the proper model classes. First, we present some tools that will be useful

Canonical Models

Observe that by viewing the $\langle F_i \rangle$ operators as modal diamonds, and epistemic atoms $K_i\phi$ as atomic propositions, $DES4_n$ is "almost" a normal modal logic: it contains propositional logic, and has the *TL*1 axiom for each $[F_i]$; however, it does not fully obey the rule of uniform substitution. In the following section, we will also discuss some other systems over the language $\mathcal{LDE}$, some of which are normal. We will make use of the fact all of these systems have a *canonical model*, in which every consistent formula is satisfiable in some state. This can be proved for all the systems considered in this paper, in exactly the same way as for normal modal logics. For reference, we briefly review the main definitions and results (we refer to (Blackburn, de Rijke, & Venema 2001) for further details).

A formula $\phi \in \mathcal{LDE}$ is *consistent* in a logical system L iff not $\phi \to \bot$ is derivable in L, where $\bot$ is some propositional contradiction. Given one of our logics L over the language

$\mathcal{LDE}$ (such as, e.g., $L = DES4_n$), the canonical model for L is the general model

$$M^c = (W^c, R_1^c, \ldots, R_n^c, V^c)$$

where

- W^c is the set of all maximal L-consistent sets of formulae
- For each $i \in \{1, \ldots, n\}$, $R_i^c(w, v)$ iff for every formula ϕ, if $\phi \in v$ then $\langle F_i \rangle \phi \in w$
- For every atom $\phi \in ATOMS$, $\phi \in V^c(w)$ iff $\phi \in w$

The *truth lemma* holds for all the systems L considered in this paper: for all $w \in W^c$ and $\phi \in \mathcal{LDE}$

$$M^c, w \models \phi \Leftrightarrow \phi \in w$$

It is important to keep in mind that a canonical model of a logic L is a *general model*, and not necessarily a proper *model*, since the epistemic states can possibly be infinite.

Preservation

In completeness proofs we will often need to transform one general model into another while preserving satisfaction of certain formulae. We here describe two such transformations. First, we review the well known concept of *bisimulation*, and, second, we describe a more general type of transformation which we call Σ-*bisimulation*.

Bisimulation   A brief review if the concept of a bisimulation (see, e.g., (Blackburn, de Rijke, & Venema 2001) for further details):

Definition 3 *Let $M = (W, R_1, \ldots, R_n, V)$ and $M' = (W', R_1', \ldots, R_n', V')$ be two general models. A relation $Z \subseteq W \times W'$ is a bisimulation between M and M' iff the following three conditions hold:*

1. *For all $w \in W$, $w' \in W'$: if wZw' then $V(w) = V'(w')$ (w and w' satisfy the same atoms)*
2. *For all $w \in W$, $w' \in W'$, $i \in Agt$: if wZw' and wR_iv then there is a $v' \in W'$ such that vZv' and $R_i'w'v'$*
3. *For all $w \in W$, $w' \in W'$, $i \in Agt$: if wZw' and $w'R_i'v'$ then there is a $v \in W$ such that vZv' and R_iwv*

It is well known that bisimulation preserves satisfiability: if Z is a bisimulation between M and M' and wZw', then $M, w \models \phi$ iff $M', w' \models \phi$ for any formula $\phi \in \mathcal{LDE}$.

Σ-bisimulation   We define Σ-simulation and show a corresponding preservation result.

Definition 4 *Let $\Sigma \subseteq ATOMS$, and let $M = (W, R_1, \ldots, R_n, V)$ and $M' = (W', R_1', \ldots, R_n', V')$ be two general models. A relation $Z \subseteq W \times W'$ is a Σ-bisimulation between M and M' iff the following condition holds, in addition to conditions 2 and 3 for standard bisimulation:*

1. *For all $w \in W$, $w' \in W'$: if wZw' then $V(w) \cap \Sigma = V'(w') \cap \Sigma$ (w and w' satisfy the same atoms from Σ)*

We write $M \overset{\Sigma}{\longleftrightarrow} M'$ to denote the fact that there is a Σ-bisimulation between M and M'; we write $w \overset{\Sigma}{\longleftrightarrow} w'$ to denote the fact that there is a Σ-bisimulation Z between M and M' such that wZw'.

It is easy to see that when $\Sigma = ATOMS$, Σ-bisimulation and bisimulation coincides.

The following theorem shows that Σ-bisimulation preserves satisfaction of formulae not containing atoms not in Σ.

When $\Sigma \subseteq ATOMS$, let $\mathcal{LDE}(\Sigma)$ denote the subset of $\mathcal{LDE}$ not containing formulae with atoms in $ATOMS \setminus \Sigma$, i.e., $\phi \in \mathcal{LDE}(\Sigma)$ iff $\phi \in \mathcal{LDE}$ and $At(\phi) \subseteq \Sigma$.

Theorem 1 *Let w be a state in a general model M, and w' a state in a general model M'.*

$$w \overset{\Sigma}{\longleftrightarrow} w' \Rightarrow \forall_{\phi \in \mathcal{LDE}(\Sigma)}(M, w \models \phi \Leftrightarrow M', w' \models \phi)$$

Proof. Let $M = (W, R_1, \ldots, R_n, V)$ and $M' = (W', R'_1, \ldots, R'_n, V')$. If we assume that $w \overset{\Sigma}{\longleftrightarrow} w'$, we can prove that the equivalence holds for any $\phi \in \mathcal{LDE}(\Sigma)$ by structural induction. For the base case, $\phi \in ATOMS$ and $\phi \in \Sigma$. $M, w \models \phi$ iff $\phi \in V(w)$ iff $\phi \in V(w) \cap \Sigma$ iff, by (1) in Definition 4, $\phi \in V'(w') \cap \Sigma$ iff $M', w' \models \phi$. The inductive step can be shown in exactly the same way as for standard bisumulation. $\square$

Theorem 1 formally shows the intuitive property that satisfaction of a formula does not depend on the valuation of atoms not mentioned in the formula.

Weaker and Related Systems

Before we look at the model class for $DES4_n$, we discuss some other systems which are interesting for the dynamics of syntactic knowledge including some systems strictly weaker than $DES4_n$. In these other systems, the operator $\langle F_i \rangle$ will no longer necessarily mean "some time in the future" as in $DES4_n$. For example, the models for a system might not necessarily be transitive. A general informal interpretation of $\langle F_i \rangle \phi$ is that ϕ is true after some epistemic action performed by i. Furthermore, the K_i operator will not necessarily mean "knowledge" as in $DES4_n$. In particular, the knowledge axiom $DE2$ might not hold. A general interpretation of $K_i\phi$ is that ϕ is *believed* by i.

The minimal logic for $\mathcal{M}$

First of all, let us consider the minimal logic of $\mathcal{M}$, that is, the set of formulae valid in all models of $\mathcal{M}$, without any conditions on the accessibility relation. Although $\mathcal{M}$ is a proper subset of all Kripke structures over the given atomic propositions, it turns out to be just the basic multi-modal logic K_n.

The logical system K_n consists of the axiom schemata

(Prop) ϕ, when ϕ is a substitution instance of a propositional tautology

(K) $[F_i](\phi \rightarrow \psi) \rightarrow ([F_i]\phi \rightarrow [F_i]\psi)$

and the rules

(Modus Ponens) From $\phi, \phi \rightarrow \psi$, prove ψ

(Gen) From ϕ, prove $[F_i]\phi$

By *weak completeness*, we mean the property that every valid formula is a theorem, i.e., $\models \phi \Rightarrow \vdash \phi$ for any formula ϕ.

Theorem 2 *K_n is sound and weakly complete with respect to $\mathcal{M}$.*

Proof. It suffices to show that any K_n consistent formula ϕ is satisfied in $\mathcal{M}$. Let $M^c = (W^c, R_1^c, \ldots, R_n^c, V^c)$ be the canonical model for K_n. ϕ is true in at least one of the states in M^c. Let $M^f = (W^f, R_1^f, \ldots, R_n^f, V^f)$ be as follows: $W^f = W^c$, $R_i^f = R_i^c$ and for every $w \in W^f$, $V^f(w) = V^c(w) \cap At(\phi)$, where $At(\phi)$ is the set of atoms occurring outside the scope of an epistemic operator in ϕ. Since M^c and M^f are $At(\phi)$-bisimilar, for every world w, $M^c, w \models \phi$ iff $M^f, w \models \phi$. $V^f(w)$ is finite for each w, since there are only finitely many atoms in ϕ. Thus, $M^f \in \mathcal{M}$ and ϕ is satisfied in M^f. $\square$

Note that strong completeness, i.e., that $\Gamma \models \phi$ implies that $\Gamma \vdash \phi$ for any set of formulae Γ and any formula ϕ, is impossible since the logic is not compact, due to the requirement that only finitely many atoms are true in any given state. This holds already for the epistemic fragment (Ågotnes & Walicki 2005a): a counter example to compactness is the theory $\Gamma = \{K_i\phi : \phi \in \mathcal{LDE}\}$. Γ is not satisfiable in $\mathcal{M}$, but each of its finite subsets is.

Some standard conditions on R_i

In the next two sections, we consider imposing additional conditions on the accessibility relation R_i. First, in this section, we look at the standard conditions, where the correspondence between a condition and a modal axiom is well known from modal logic. However, the conditions have a distinct meaning in the context of the dynamics of syntactic knowledge. In the following section, we will look at conditions on R_i which are specific to our logic, and reflect the knowledge of inference rules.

Unbounded Reasoning Many syntactic approaches to epistemic logic (Elgot-Drapkin *et al.* 1999; Alechina, Logan, & Whitsey 2004; Ågotnes & Walicki 2004) are based on the view that reasoning does not have an end point, but goes on indefinitely. This explains logical non-omniscience without sacrificing rationality: an agent can eventually get to know *any* particular fact it is able to compute, but can never get to know all of them at the same time. In the models $\mathcal{M}$, the assumption that an agent should be able to do any reasoning at all in a given state of the system is not made. Here, we restrict the logic by adding this assumption.

Semantically, it corresponds to requiring that the accessibility relations are serial. A *serial* model is a model $(W, R_1, \ldots, R_n, V)$ where the accessibility relations are serial, i.e. where for each world $w \in W$ there exists a $u \in W$ such that $R_i wu$. The class of all serial models is denoted $\mathcal{M}^s$.

Proof-theoretically, the assumption of unbounded reasoning corresponds to adding the axiom schema

(D) $[F_i]\phi \rightarrow \langle F_i \rangle \phi$

The modal system KD_n is K_n extended with the D axiom. Soundness and (weak) completeness of KD_n with respect to $\mathcal{M}^s$ follow from Theorem 3 below.

Deterministic Reasoning The models $\mathcal{M}$ are models of nondeterministic reasoning, in the sense that an agent may have several possible transitions from one state. Here, we look at the special case when reasoning is deterministic, i.e. when there is at most (or *exactly*, in the case of unbounded reasoning) one possible next state for each state. Formally, a *deterministic model* is one in which the accessibility relations are partial functions. The set of all deterministic models is

$$\mathcal{M}^d = \{(W, R_1, \ldots, R_n, V) \in \mathcal{M} :$$
$$\forall i \forall w, v \in W((R_i wu \ \& \ R_i wv) \Rightarrow u = v)\}$$

and the class of all deterministic serial models is $\mathcal{M}^{ds} = \mathcal{M}^d \cap \mathcal{M}^s$.

Proof-theoretically, we add the axiom schema

(F) $\langle F_i \rangle \phi \rightarrow [F_i] \phi$

The modal systems KF_n and KDF_n are K_n and KD_n extended with the F axiom, respectively. Soundness and (weak) completeness of KF_n and KDF_n with respect to $\mathcal{M}^s$ and $\mathcal{M}^{ds}$, respectively, follows from Theorem 3 below.

Transitivity As mentioned, an informal interpretation of $\langle F_i \rangle \phi$ in a model in $\mathcal{M}$ is that ϕ will be true after i has performed some action. If we want "action" to mean an arbitrary finite number of actions, and thus $\langle F_i \rangle \phi$ to mean that i can make ϕ be true at some point in the future, such as in the logic $DES4_n$, we can require that the accessibility relations are transitive. Let $\mathcal{M}^t$ be the class of all transitive models.

Syntactically, this corresponds to the axiom schema

(4) $\langle F_i \rangle \langle F_i \rangle \phi \rightarrow \langle F_i \rangle \phi$

The modal system $K4_n$ is K_n extended with the 4 axiom. Soundness and (weak) completeness of $K4_n$ with respect to $\mathcal{M}^t$ follows from Theorem 3 below.

Let P be a property of the accessibility relation (e.g. transitivity), and ϕ_P a modal formula. If the canonical model for any normal modal logic L containing ϕ_P has property P, and ϕ_P is valid on any class of models with property P, then ϕ_P is *canonical* for P (cf. (Blackburn, de Rijke, & Venema 2001), p.206). For example, the 4 axiom is canonical for transitivity.

Theorem 3 *Let ϕ_P be canonical for P. Then a logic of syntactic knowledge $K_n + \phi_P$ is sound and weakly complete with respect to the subclass of $\mathcal{M}$ satisfying the condition P.*

Proof. By the assumption of the theorem, the canonical model for $K_n + \phi_P$ satisfies P. However, as in the proof of Theorem 2, we need to show that this model can be transformed into a model from $\mathcal{M}$, where each state validates only a finite number of epistemic atoms. It is easy to check that the method described in the proof of Theorem 2 does not change the properties of the accessibility relations, so the resulting model is in $\mathcal{M}$ and it still satisfies P. $\square$

Knowing inference rules

In this section, we introduce a new kind of condition on the accessibility relations, which connects the presence of epistemic atoms in a state with the availability of transitions, and corresponds to knowing an inference rule. We show how

such conditions correspond to certain axioms. An example of such an axiom is DE1 in $DES4_n$.

Consider the following natural class of conditions on the accessibility relations, which we will call *addition conditions*. These conditions have the following form: if agent i believes formulae $\phi_1, \ldots, \phi_m$, then agent i can reach a state where it believes formulae $\psi_1, \ldots, \psi_k$:

$$K_i \phi_1, \ldots, K_i \phi_m \in V(w) \Rightarrow$$
$$\exists w'(R_i(w, w') \wedge K_i \psi_1, \ldots, K_i \psi_k \in V(w'))$$

An example of such a condition is

K_iMP $K_i \phi, K_i(\phi \rightarrow \psi) \in V(w) \Rightarrow$
$$\exists w'(R_i(w, w') \wedge K_i \psi \in V(w'))$$

Theorem 4 *Any set of addition conditions of the form*

$$K_i \phi_1, \ldots, K_i \phi_m \in V(w) \Rightarrow$$
$$\exists w'(R_i(w, w') \wedge K_i \psi_1, \ldots, K_i \psi_k \in V(w'))$$

is axiomatisable by adding to K_n axiom schemata of the form

$$K_i \phi_1 \wedge \ldots \wedge K_i \phi_m \rightarrow \langle F_i \rangle (K_i \psi_1 \wedge \ldots \wedge K_i \psi_k)$$

Proof. Soundness is straightforward. For completeness, consider a general canonical model where the axioms hold. In the general canonical model, if $K_i \phi_1, \ldots, K_i \phi_m \in V(w)$, there is an R_i-accessible state w' with $K_i \psi_1, \ldots, K_i \psi_k \in w'$, and the addition condition holds. Now we need to produce a model for a consistent formula ϕ with finite epistemic states, where the semantic condition still holds. We modify the proof of Theorem 2 as follows. Take a world w which satisfies ϕ. Unravel the sub-model generated by w. Now we have a tree model whose root w satisfies ϕ. Instead of intersecting $V(w)$ with $At(\phi)$ as before, we define a set $At^k(\phi)$, which is a set of atoms with which we intersect $V(v)$ for the states v which are at the distance k from the root of the tree. $At^0(\phi) = At(\phi)$, and $At^k(\phi)$ is $At^{k-1}(\phi)$ closed under a single application of addition conditions. For example, if the condition is **K_iMP**, then $At^1(\phi)$ will contain, in addition to the formulae from $At(\phi)$, all formulae $K_i \chi$ such that $K_i \psi$, $K_i(\psi \rightarrow \chi)$ are in $At(\phi)$. Note that $At^k(\phi)$ is finite for every k, and the intersection of $At^k(\phi)$ with $V(v)$ is finite. However, $At^k(\phi)$ contains all formulae which are required by the addition conditions to be true in a given state. It is straightforward to show that the two models are $At(\phi)$-bisimilar, so ϕ is still satisfied at the root of the model. $\square$

Semantics for $DES4_n$

We define a class of models, and show that $DES4_n$ is sound and complete with respect to this class. The completeness proofs in the previous subsections illustrated techniques for dealing with the requirement that an agent can only know finitely many formulae in each state. Compared to these previous proofs, however, the proof for $DES4_n$ is in addition to the finiteness requirement complicated by the facts that first, for each agent in each state the union of the formulae the agent knows in accessible states is infinite (DE4–DE7), and,

second, the knowledge of an agent in a state must include the persistent knowledge of the agent in all predecessors of the state (DE3). These requirements rule out image-finite models, and thus the finite model property, as well as models with infinite preimages. In particular, it is not *prima facie* clear that the axioms of $DES4_n$ are compatible with finite epistemic states at all.

Let Ax_i be the set of formulae DE4–DE7 prescribe that agent i must know in some accessible state:

$$Ax_i = \left\{ \begin{array}{l} \phi \to (\psi \to \phi), \\ (\phi \to (\psi \to \gamma)) \to ((\phi \to \psi) \to (\phi \to \gamma)), \\ (\neg\psi \to \neg\phi) \to (\phi \to \psi), \\ K_i\alpha \to \alpha \\ : \phi, \psi, \gamma \in \mathcal{LDE}, \alpha \in \mathcal{L} \end{array} \right\}$$

Let $\mathcal{M}^{DEL} \subset \mathcal{M}$ (respectively, $\mathcal{M}^{gen,DEL} \subset \mathcal{M}^{gen}$) be the class of models $M = (W, R_1, \ldots, R_n, V)$ satisfying the following conditions for each agent i and each state $w \in W$:

D1 R_i is transitive

D2 If $R_i(w, v)$, $K_i\phi \in V(w)$ and $\phi \in \mathcal{L}_E^+$, then $K_i\phi \in V(v)$ (Monotonicity)

D3 For every ϕ, if $K_i\phi \in V(w)$, then $M, w \models \phi$ (Knowledge)

D4 If $K_i\phi$, $K_i(\phi \to \psi) \in V(w)$, then $\exists v\, (R_i(w, v)\ \&\ K_i\psi \in V(v))$

D5 For every $\tau \in Ax_i$, $\exists v\, (R_i(w, v)\ \&\ K_i\tau \in V(v))$

D6 If $K_i\phi \in V(w)$ and $\phi \in \mathcal{L}_E^+$, then $\exists v\, (R_i(w, v)\ \&\ K_iK_i\phi \in V(v))$.

As an intermediate result, we first prove soundness and completeness of $DES4_n$ with respect to the class of general models satisfying conditions **D1-D6**, $\mathcal{M}^{gen,DEL}$. A similar result was proved in (Duc 1995), as we discuss in the following section on "Related Work". We then prove the main, and more difficult, result: completeness with respect to $\mathcal{M}^{DEL}$.

Theorem 5 *$DES4_n$ is sound and strongly complete with respect to $\mathcal{M}^{gen,DEL}$.*

Proof. First, let us consider soundness. We want to prove that for every set of formulae Γ and formula ϕ_0, if $\Gamma \vdash_{DES4_n} \phi_0$, then $\Gamma \models_{gen,DES4_n} \phi_0$. The proof is by induction on the length of the derivation of ϕ_0 from Γ. Clearly, the inference rules preserve validity. We need to show that every instance of an axiom schema of $DES4_n$ is valid on the class $\mathcal{M}^{gen,DEL}$. The axiom schemata $PC1$–$PC3$ are valid classical tautologies. It is well known that $TL1$ is valid in all Kripke models and that $TL2$ is valid in all models with a transitive accessibility relation. $DE1$ is valid because of condition **D4**: if some state w satisfies $K_i\phi$ and $K_i(\phi \to \psi)$, then by **D4**, w has a successor which satisfies $K_i\psi$, so w satisfies $\langle F_i \rangle K_i\psi$. $DE2$ is valid because of the condition **D3**: if $K_i\phi$ is true in w, then ϕ has to be true in w. $DE3$ is valid due to the Monotonicity condition **D2**. $DE4$–$DE7$ are valid because of **D5**. Finally, $DE8$ is valid because of **D6**.

The proof of completeness proceeds in a standard way. Suppose $\Gamma_0 \nvdash_{DES4_n} \phi_0$; we show that then there exists a model and a state where Γ_0 is satisfied and ϕ_0 is not, so $\Gamma_0 \nvDash_{gen,DES4_n} \phi_0$. In other words, we show how to construct a satisfying model for $\Gamma_0 \cup \{\neg\phi_0\}$ provided

this set is $DES4_n$-consistent. For convenience, we will refer to this set as Γ. Any $DES4_n$-consistent set can be extended to a maximal consistent set in a standard way. Let $M^c = (W^c, R_1^c, \ldots, R_n^c, V^c)$ be the canonical model for $DES4_n$, as defined earlier. Recall that M^c is a general model; we must show that $M^c \in \mathcal{M}^{gen,DEL}$. It is straightforward to show that each R_i is transitive, because each state in the model contains $TL2$. Now we have to show that conditions **D2-D6** hold. Suppose for some w, $K_i\phi$ and $K_i(\phi \to \psi) \in V^c(w)$. This means, by the definition of V^c, that $K_i\phi, K_i(\phi \to \psi) \in w$. Since w is maximal, it also contains $K_i\phi \wedge K_i(\phi \to \psi)$ and $K_i\phi \wedge K_i(\phi \to \psi) \to \langle F_i \rangle K_i\psi$ ($DE1$). Since it is closed under inference, it also contains $\langle F_i \rangle K_i\psi$. By the truth lemma, $\langle F_i \rangle K_i\psi$ is true in w. Hence w has a successor v where $K_i\psi$ is true, which by the truth lemma implies $K_i\psi \in V^c(v)$. Analogously, we can show that **D3-D6** hold. This completes the argument that $M^c \in \mathcal{M}^{gen,DEL}$; since Γ is consistent, it is contained in one of the states in M^c, and by the truth lemma, is satisfied there. $\square$

Theorem 6 *$DES4_n$ is sound and weakly complete with respect to $\mathcal{M}^{DEL}$.*

Proof. The proof of soundness is identical to the proof in Theorem 5.

For completeness, assume that ϕ_0 is a $DES4_n$-consistent formula. We will construct a model satisfying ϕ_0, in several stages. First we will construct a general model M^c satisfying ϕ_0. Then we will transform this model into a proper model with finite epistemic states.

Let $M^c = (W^c, R_1^c, \ldots, R_n^c, V^c)$ be as in the proof of Theorem 5 (the canonical model for $DES4_n$). We showed that M^c satisfies **D1-D6**, but it may have infinite epistemic states and thus not be in the class $\mathcal{M}^{DEL}$. At least one of the states in M^c satisfies ϕ_0; let us call that state w_0. We will now unravel M^c around w_0, and then take the transitive closure; and we will end up with a transitive tree (general) model M^t. The details are as follows, cf. (Blackburn, de Rijke, & Venema 2001), pp. 220–221 for further discussion. Let $M^g = (W^g, R_1^g, \ldots, R_n^g, V^g)$ be the (general) submodel of M^c *generated* by w_0: it is the smallest submodel of M^c such that $w_0 \in W^g$ and $u \in W^g$ whenever $w \in W^g$ and $R_i^c wu$ for some $i \in Agt$. The *unraveling* of the general model M^g around $w_0 \in W^g$ is the general model $M' = (W', R_1', \ldots, R_n', V')$ such that:

- W' is the set of all finite sequences $(w_0, w_1, \ldots, w_m)$, $m \geq 0$, such that $w_1, \ldots, w_m \in W^g$ and $R_{i_1}^g(w_0, w_1), R_{i_2}^g(w_1, w_2), \ldots, R_{i_n}^g(w_{m-1}, w_m)$ for some $i_1, \ldots, i_n \in Agt$

- $R_i'(w_1, \ldots, w_k)(v_1, \ldots, v_l)$ iff $(w_1, \ldots, w_k) = (w_1, \ldots, v_{l-1})$ and $R_i^g v_{l-1} v_l$

- $V'((w_0, w_1, \ldots, w_m)) = V^g(w_m)$

The result of unraveling around w_0 is an intransitive tree where w_0 is the root. The *transitive unraveling* $M^t = (W^t, R_1^t, \ldots, R_n^t, V^t)$ is obtained by taking the transitive closure of each of the relations in the unraveling M'. Note the

following properties of the transitive unraveling:

$$R_i^t(w_0, \ldots, w_k)(v_0, \ldots, v_l) \Rightarrow R_i^c w_k v_l \quad (1)$$

$$R_i^t r_1 r_2 \Rightarrow r_1 \text{ is a (proper) prefix of } r_2 \quad (2)$$

We argue that (1) and (2) hold. Let $r_1 = (w_0, \ldots, w_k)$ and $r_2 = (v_0, \ldots, v_l)$. If $R_i^t r_1 r_2$, then there exist a path $s_0, \ldots, s_j \in W^t$, $j \geq 0$, of R_i^t steps between the two sequences; i.e., such that $R_i^t r_1 s_0, \ldots, R_i^t s_j r_2$. In turn, this means that there exist $u_0, \ldots, u_j \in W^g$ such that

$$s_0 = (w_0, \ldots, w_k, u_0) \qquad\qquad R_i^g w_k u_0$$
$$\vdots$$
$$s_j = (w_0, \ldots, w_k, u_0, \ldots, u_j) \qquad R_i^g u_{j-1} u_j$$
$$(v_0, \ldots, v_l) = (w_0, \ldots, w_k, u_0, \ldots, u_j, v_l) \quad R_i^g u_j v_l$$

The previous equation shows that (2) holds (r_1 is a *proper* prefix of r_2 since R_i^t, and thus R_i^t, is irreflexive). Since R_i^c is transitive and includes R_i^g, it also follows that $R_i^c w_k v_l$ and thus (1) holds.

We now show that M^t and M^c are bisimilar. Let $Z \subseteq W^t \times W^c$ be defined as follows:

$$(w_0, \ldots, w_m) Z w_m$$

We show that Z is a bisimulation between M^t and M^c, by the three required conditions:

1. $V^t((w_0, \ldots, w_k)) = V'((w_0, \ldots, w_k)) = V^g(w_k) = V^c(w_k)$

2. Immediate by (1).

3. We must show that if $(w_0, \ldots, w_k) \in W^t$ and $R_i^c w_k v$ for some $v \in W^c$, then there exists a sequence $(v_0, \ldots, v_l) \in W^t$ with $v_l = v$ such that $R_i^t(w_0, \ldots, w_k)(v_0, \ldots, v_l)$. This holds immediately by taking $(v_0, \ldots, v_l) = (w_0, \ldots, w_k, v)$: $w_k \in W^g$ and thus $v \in W^g$ and thus $R_i^g w_k v$, and $R_i^t(w_0, \ldots, w_k)(w_0, \ldots, w_k, v)$.

Since M^c and M^t are bisimilar, for every state w reachable from w_0, $M^c, w \models \phi$ iff $M^t, (s, w) \models \phi$, for every formula ϕ and sequence $s \in W^t$ (we abuse the notation here, and use (s, w) to denote the sequence resulting from concatenating the element w to the sequence s). This includes the special case when s is empty, $w = w_0$, and $\phi = \phi_0$. Since for every state (s, w) in M^t there is a bisimilar state w in M^c, we can also show that all of the conditions **D1 - D6** are satisfied in M^t. **D1** (transitivity) holds immediately for M^t. **D2** (monotonicity) holds because if $K_i \phi \in V^t((s, w))$, and $\phi \in \mathcal{L}_E^+$, and $R_i^t((s, w), (s', v))$, then $K_i \phi \in V^c(w)$ (by construction of V^t), $R_i^c(w, v)$ (by (1)), and $K_i \phi \in V^c(v)$ (because M^c satisfies **D2**). By the bisimulation between (s', v) and v, $K_i \phi \in V^t((s', v))$. Similarly for **D3**: suppose $K_i \phi \in V^t((s, w))$. Then $K_i \phi \in V^c(w)$. Then, since M^c satisfies **D3**, $M^c, w \models \phi$. By the bisimulation, $M^t, (s, w) \models \phi$. **D4-D6** hold because for every $(s, w) \in W^t$, if (s, w) satisfies a certain formula, then it has a bisimilar state w in W^c, which satisfies the same formula and has a successor v which satisfies another required formula; by the bisimulation, (s, w) then also has a successor (s, w, v) which satisfies the same formula.

So, the root w_0 of M^t satisfies ϕ_0; conditions **D1-D6** continue to hold. Now we are going to transform M^t into a proper model, by intersecting the epistemic state in all states in M^t with a finite set of formulae; we need to do this in such a way that ϕ_0 is still satisfied at the root and conditions **D1-D6** continue to hold. More precisely, the epistemic states at level k in the tree model M' (where w_0 is at level 0, and its one step successors at level 1, etc.) are going to be intersected with a finite set of formulae L_k, defined inductively as follows. Note that there are infinitely many levels in the tree due to seriality imposed by *DE4–DE7*. In the following definition, given a set of atoms $X \subseteq ATOMS$, $Cl(X)$ denotes the closure of X under nested epistemic atoms: $Cl(X)$ is the least set such that (i) $X \subseteq Cl(X)$ and (ii) if $K_i \phi \in Cl(X)$ then $At(\phi) \subseteq Cl(X)$ (where $At(\phi)$ as usual denotes the members of *ATOMS* occurring outside the scope of an epistemic operator in ϕ).

$$L_0 = Cl(At(\phi_0))$$

$L_k = Cl(Inf(L_{k-1} \cup \{K_1 \tau_{k-1}^1, \ldots, K_n \tau_{k-1}^n\}))$ when $k \geq 1$, where, for each agent $i \in Agt$, $\tau_1^i, \tau_2^i, \ldots$ is some enumeration of the (countable) set Ax_i, and $Inf(X)$, for some set of atoms X, is the set containing X and formulae derived from X by a single application of the following rules:

$$\frac{K_i \phi, K_i(\phi \to \psi)}{K_i \psi} \qquad \frac{K_i \phi}{K_i K_i \phi}$$

Note that if X is finite, so is $Cl(Inf(X))$, and thus that L_k is finite for each k.

Consider the resulting model $M = (W, R_1, \ldots, R_n, V)$: $W = W^t$, $R_i = R_i^t$ and

$$V(w_0, \ldots, w_k) = V^t(w_0, \ldots, w_k) \cap L_k$$

We need to prove the following statements about M:

Truth $M, w_0 \models \phi_0$

Finiteness Epistemic states in M are finite.

D1 - D6 Conditions **D1-D6** are satisfied.

Here is the proof for each of these statements.

Truth M and M^t are $At(\phi_0)$-bisimilar (Def. 4): take Z to be the identity relation on W^t. Theorem 1 says that formulae only containing atoms from $At(\phi_0)$ are preserved under $At(\phi_0)$-bisimulation. Since $\phi_0 \in \mathcal{LDE}(At(\phi_0))$ and $M^t, w_0 \models \phi_0$, it follows that $M, w_0 \models \phi_0$.

Finiteness $At(\phi_0)$ is finite, and each epistemic state is a subset of a finite set L_k.

D1 the accessibility relations in M are transitive, because they are transitive in M^t.

D2 Let $R_i(w, v)$, $\phi \in \mathcal{L}_E^+$, $K_i \phi \in V(w)$. We need to show that $K_i \phi \in V(v)$. We have that $R_i^t w v$, so by (2) the sequence w is a prefix of the sequence v; say $w = (v_0, \ldots, v_k)$ and $v = (v_0, \ldots, v_l)$ with $k < l$. $K_i \phi \in V^t(w)$, and since Monotonicity holds in M^t, $K_i \phi \in V^t(v)$. We also have that $K_i \phi \in L_k$, and since $L_k \subseteq L_l$, $K_i \phi \in L_l$. Thus, $K_i \phi \in V(v) = V^t(v) \cap L_l$.

D3 Let $K_i \phi \in V(w) = V^t(w) \cap L_k$, where $w = (w_0, \ldots, w_k)$. We must show that $M, w \models \phi$. First, we claim that

$$Cl(\{\phi'\}) \subseteq L_k \Rightarrow (M, w \models \phi' \Leftrightarrow M^t, w \models \phi')$$

for all $\phi' \in \mathcal{L}$. The argument is an easy induction over the structure of ϕ'. We have that $M^t, w \models K_i\phi$ and thus $M^t, w \models \phi$ by Knowledge for M^t. We also have that $K_i\phi \in L_k$ and thus that $Cl(\{\phi\}) \subseteq L_k$. It follows that $M, w \models \phi$.

D4 Let $K_i\phi, K_i(\phi \rightarrow \psi) \in V(w) = V^t(w) \cap L_k$, where $w = (w_0, \ldots, w_k)$. By **D4** for M^t, there is a $v = (v_0, \ldots, v_l)$ such that $R_i^t wv$ and $K_i\psi \in V^t(v)$. By (2) $l > k$, and since $K_i\phi, K_i(\phi \rightarrow \psi) \in L_k$ it follows that $K_i\psi \in L_l$ by construction of L_{k+1}. Thus, $K_i\psi \in V(v) = V^t(v) \cap L_l$. Since $R_i^t wv$, $R_i wv$.

D5 Let $\tau \in Ax_i$ and $w = (w_0, \ldots, w_k) \in W$. We must show that there is a $v = (v_0, \ldots, v_l) \in W$ such that $R_i wv$ and $K_i\tau \in V(v) = V^t(v) \cap L_l$. τ is one of the elements in the enumeration of Ax_i in the definition of L_k, say $\tau = \tau_j^i$. We first show that for any $u \in W$:

$$u = (u_0, \ldots, u_m), m \geq j \Rightarrow \exists u' \in W \begin{cases} R_i uu' \\ K_i\tau \in V(u') \end{cases} \tag{3}$$

By **D5** for M^c, there is a $u_{m+1} \in W^c$ such that $R_i^c u_m u_{m+1}$ and $K_i\tau \in V^c(u_{m+1})$. Let $u' = (u_0, \ldots, u_m, u_{m+1})$. $R_i^g u_m u_{m+1}$ ($u_m \in W^g$), so $R_i' uu'$ and thus $R_i^t uu'$ and $R_i uu'$. $K_i\tau \in V^g(u_{m+1})$, so $K_i\tau \in V'(u') = V^t(u')$. $K_i\tau = K_i\tau_j^i \in L_j$, and, since $m \geq j$, $L_j \subseteq L_{m+1}$, and we have that $K_i\tau_j^i \in V^t(u') \cap L_{m+1} = V(u')$ and (3) holds. Now in the case that $k \geq j$, we are done by (3). Let $k < j$. R_i^g is serial (**D5** holds for M^g), so there exist $w_{k+1}, \ldots, w_j \in W^g$ such that $R_i^g w_k w_{k+1}, \ldots, R_i^g w_{j-1} w_j$. Thus, $R_i' w(w_0, \ldots, w_k, w_{k+1}), \ldots, R_i'(w_0, \ldots, w_{j-1})(w_0, \ldots, w_{j-1} w_j)$. By transitivity, $R_i^t w(w_0, \ldots, w_j)$ and thus $R_i w(w_0, \ldots, w_j)$. By (3) there is a u' such that $R_i(w_0, \ldots, w_j)u'$ and $K_i\tau \in V(u')$, and by transitivity again we get that $R_i wu'$ and we are done.

D6 Let $w = (w_0, \ldots, w_k) \in W$, $\phi \in \mathcal{L}_E^+$ and $K_i\phi \in V(w) = V^t(w) \cap L_k$. $K_i\phi \in V^t(w) = V'(w)$. Since M' satisfies **D6**, there is a $v \in W'$ such that $R_i' wv$ and $K_i K_i\phi \in V'(v)$. It must be the case that $v = (w_0, \ldots, w_k, w_{k+1})$ for some $w_{k+1} \in W'$. It follows that $R_i^t wv$ and $K_i K_i\phi \in V^t(v)$. Again, it follows that $R_i wv$. By construction of L_{k+1}, since $K_i\phi \in L_k$, we have that $K_i K_i\phi \in L_{k+1}$. Thus, $K_i K_i\phi \in V(v) = V^t(v) \cap L_{k+1}$.

$\square$

Related Work

In the paper (Duc 1995), which appeared earlier than (Duc 1997), Duc gives a semantics for and proves the soundness and completeness of a logic similar to $DES4_n$. We presently give a brief review of these results, and compare them to the ones we have presented in this paper. The logic presented in (Duc 1995), called *Basic Dynamic-Epistemic Logic (BDE)*, is defined over the (least) language which contains Kp when p is a formula of propositional logic, and is closed under the usual propositional connectives and under the formation rule: if ϕ is a formula then $\langle F \rangle \phi$ is a formula. BDE contains the following axioms and rules:

A1 ϕ, when ϕ is a substitutional instance of a propositional tautology

A2 $[F](\phi \rightarrow \psi) \rightarrow ([F]\phi \rightarrow [F]\psi)$

A3 $[F]\phi \rightarrow [F][F]\phi$

A4 $Kp \wedge K(p \rightarrow q) \rightarrow \langle F \rangle Kq$

A5 $Kp \rightarrow [F]Kp$

A6 $\langle F \rangle Kp$ when p is an propositional tautology

R1 From $\phi, \phi \rightarrow \psi$, prove ψ

R2 From ϕ, prove $[F]\phi$

Thus, BDE differs from $DES4_n$ in the following aspects: (i) the language does not contain atomic propositions as formulae, (ii) it is a single agent, rather than a multi agent, logic, (iii) knowledge can not be nested and (iv) knowledge is not veridical ($DES4_n$ axioms $DE2$ and $DE7$ do not hold). Note the following consequences of (iii): first, all formulae ϕ occurring in an expression $K\phi$ are *persistent* in the $DES4_n$ sense, and, second, introspection is impossible ($DES4_n$ axiom $DE8$ does not hold). Thus, BDE can be seen as the single agent fragment of $DES4_n$ without $DE2$, $DE7$ and $DE8$. The semantics given to BDE is in fact essentially our general models with the conditions **D1**, **D2**, **D4** and **D5** (the latter with a slightly different version of Ax_i), interpreting the language of BDE in exactly the same way as we have interpreted $\mathcal{LDE}$. It is shown that BDE is sound and complete with respect to this semantics. However, the semantics of BDE does not require that the agent only knows finitely many formulae in each state which, as we have argued, is implicit in the motivation for $DES4_n$ (and for BDE as well). Like our Theorem 5, the completeness result for BDE holds directly by the standard modal logic argument by canonical models. Our main result, Theorem 6, on the other hand, is *not* trivial to prove – as a result of the finiteness assumption.

Another closely related approach is *Descriptive Dynamic Logic DDL* introduced in (Sierra *et al.* 1996). It is intended to model deductive capabilities of multiple knowledge bases, where a different language L_k and a set of inference rules is associated with each knowledge base unit k (the unit can also be seen as a reasoning agent). Atomic propositions of *DDL* are 'quoted' formulae of the object language, $k : [\phi]$ (where ϕ is an object language formula and k is the unit), very similar to the languages considered in this paper. The only difference is that in (Sierra *et al.* 1996), each language L_k is assumed to be finite. Program modalities of *DDL* correspond to inference rule applications (they are parameterised by a rule and the formulae involved in the rule application).

A general model of syntactic knowledge is discussed in (Fagin *et al.* 1995), but this model does not describe the *dynamics* of syntactic knowledge, i.e., how knowledge can evolve over time. Of approaches describing syntactic knowledge, Konolige (1984) was as far as we know the first to propose a model of a reasoner which is explicitly parameterised by some inference rules. The reasoner's knowledge is, however, assumed to be closed under these inference rules. Similarly, logics of algorithmic knowledge (Pucella 2004) assume closure of the agent's belief under the set of

agent's reasoning rules. The logic of awareness (Fagin & Halpern 1987) relaxes this assumption by requiring that the agent only believes the formulae in the intersection of the closure of its belief set with an arbitrary set of formulae - an awareness filter. However, formulae in the awareness set are believed instantly, and the dynamics of beliefs is not modeled.

Another closely related area of work are active logics (Elgot-Drapkin *et al.* 1999), which study development of agent's belief sets over discrete time. Unlike in $DES4_n$, active logics only consider linear time, however. These logics use a first order meta language which is interpreted in models of predicate logic.

Recent work in dynamic and temporal epistemic logic (van Ditmarsch, van der Hoek, & Kooi) has a similar motivation to our work (modeling knowledge change) but assumes that the agent's knowledge is deductively closed before and after the update.

Finally, the present work is based on the authors' respective work on logics for syntactic knowledge, (Ågotnes & Walicki 2005a; Alechina, Logan, & Whitsey 2004).

Conclusions and Future Work

We have presented a general semantics for the dynamics of finite syntactic knowledge, and shown that $DES4_n$ is sound and complete with respect to a proper class of models. Furthermore, we have shown soundness and completeness of some intermediate systems by showing that standard results from modal logic are transferable to the case when only a finite number of atomic propositions (of a certain type) can be true in the same state. Finally, we have completely axiomatised a notion of knowing an inference rule in syntactic epistemic logics.

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