Substructural Epistemic Logics

Igor Sedlár

Comenius University in Bratislava, Slovakia

– Workshop on Resource-Bounded Agents, ESSLLI 2015, Barcelona, 13. 8. 2015 –

Main Points

- A substructural epistemic logic of belief supported by evidence
- Evidence as a resource used in justifying beliefs and actions
- Support is not necessarily closed under classical consequence (resource-boundedness)

Overview

- Substructural epistemic logics – recent history and my contribution
- Motivating scenarios involving facts, evidence and beliefs
- Details of my approach
- Technical results – axiomatization and definability (briefly)
- Conclusion

Modal Epistemic Logic

Definition 1.1 (Language $\mathcal{L}_\Box$)

- $p, \neg\varphi, \varphi \wedge \psi$;
- $\Box\varphi$ as "the agent believes that φ".

Definition 1.2 (Models for $\mathcal{L}_\Box$)

$M = \langle P, E, V \rangle$

- P is a non-empty set ("possible worlds");
- $E \subseteq P \times P$ ("epistemic accessibility");
- $V(p) \subseteq P$ for every variable p.

Truth and validity:

- $M, w \models \Box\varphi$ iff $M, v \models \varphi$ for all v such that wEv;
- $M \models \varphi$ iff $M, w \models \varphi$ for all $w \in P$.

The Logical Omniscience Problem (Hintikka, 1962, 1975)

Fact 1.3

$$M \models \left(\bigwedge_{0 \leq i \leq n} \varphi_i \right) \rightarrow \psi \quad \Rightarrow \quad M \models \left(\bigwedge_{0 \leq i \leq n} \Box \varphi_i \right) \rightarrow \Box \psi$$

One Solution – Epistemic FDE (Levesque, 1984)

Definition 1.4 (Compatibility Models)

$M = \langle P, W, E, C, V \rangle$

- $C \subseteq P \times P$ ("compatibility") (Berto, 2015; Dunn, 1993);
- $W \subseteq P$ such that $u \in W$ only if $uCx \leftrightarrow u = x$ ("worlds").

Truth and validity:

- $M, x \models \neg\varphi$ iff $M, y \not\models \varphi$ for all y such that xCy;
- $M \models \varphi$ iff $M, w \models \varphi$ for all $w \in W$.

One Solution – Epistemic FDE (Levesque, 1984)

Definition 1.4 (Compatibility Models)

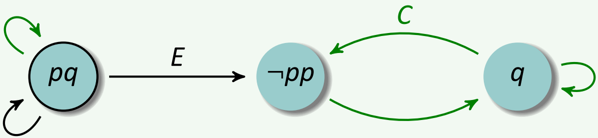
$M = \langle P, W, E, C, V \rangle$

- $C \subseteq P \times P$ ("compatibility") (Berto, 2015; Dunn, 1993);
- $W \subseteq P$ such that $u \in W$ only if $uCx \leftrightarrow u = x$ ("worlds").

Truth and validity:

- $M, x \models \neg\varphi$ iff $M, y \not\models \varphi$ for all y such that xCy;
- $M \models \varphi$ iff $M, w \models \varphi$ for all $w \in W$.

Example 1.5



A Solution – Sources (Bílková et al., 2015)

Definition 1.6 (Epistemic Models)

$F = \langle P, \leq, L, R, S, C, V \rangle$

- L is a $\leq$-closed subset of P ("logical states")
- $R \subseteq P^3$ ("pooling of information") (Beall et al., 2012)
- $S \subseteq P^2$ ("sources")

Truth and validity

- $M, x \models \varphi \to \psi$ iff for all y, z, if $M, y \models \varphi$ and $Rxyz$, then $M, z \models \psi$;
- $M, x \models \Box\varphi$ iff there is ySx such that $M, y \models \varphi$;
- $M \models \varphi$ iff $M, x \models \varphi$ for all $x \in L$.

Some Problems of (Bílková et al., 2015)

- No explicit counterpart of the possible states of the environment;
- Models only "explicit" knowledge, construed as support by a source;

Some Problems of (Bílková et al., 2015)

- No explicit counterpart of the possible states of the environment;
- Models only “explicit” knowledge, construed as support by a source;

My Contribution

- Non-classical logics for evidence-based belief;
- A combination of modal substructural logics with normal modal logics based on a functional treatment of sources;
- General completeness theorem.


WHY, GOD,
WHY?

Motivations

Example 2.1

Alice notices rain beating on her windowpane. She has her radio on, and news has just come on. Interestingly enough, the forecast for today calls for 'sunny and pleasant' weather.

Motivations

Example 2.1

Alice notices rain beating on her windowpane. She has her radio on, and news has just come on. Interestingly enough, the forecast for today calls for ‘sunny and pleasant’ weather.

Motivations

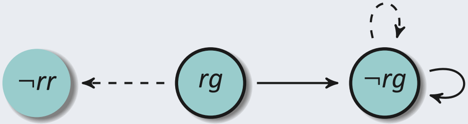
Example 2.2

Alice is listening to the radio and she does not notice the rain outside. The weather forecast for today is 'sunny and pleasant'. The forecast makes her believe that it is not raining. She also mishears a report about an accident that occurred on a canal. She thinks it took place on the canal surrounding Groningen's city center. Alice now believes that Groningen's city center is surrounded by a canal.

Motivations

Example 2.2

Alice is listening to the radio and she does not notice the rain outside. The weather forecast for today is ‘sunny and pleasant’. The forecast makes her believe that it is not raining. She also mishears a report about an accident that occurred on a canal. She thinks it took place on the canal surrounding Groningen’s city center. Alice now believes that Groningen’s city center is surrounded by a canal.



Motivations

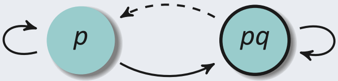
Example 2.3

Beth is taking a course in first-order logic. Let p represent a sound and complete axiomatization and q a rather complicated first-order theorem. q is true in every possible world and it follows from Beth's beliefs that p. But assume that Beth has never actually proved q. Her evidential situation does not support q, nor the fact that q follows from Beth's beliefs.

Motivations

Example 2.3

Beth is taking a course in first-order logic. Let p represent a sound and complete axiomatization and q a rather complicated first-order theorem. q is true in every possible world and it follows from Beth's beliefs that p. But assume that Beth has never actually proved q. Her evidential situation does not support q, nor the fact that q follows from Beth's beliefs.



Motivations

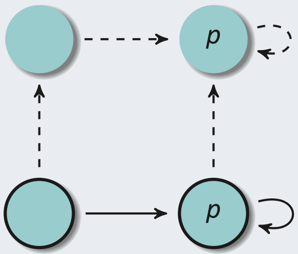
Example 2.4

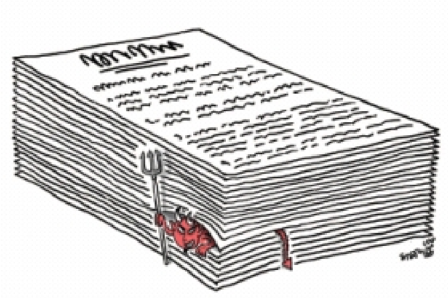
Assume that Carol is conducting an experiment. Carol's evidential situation may be seen as comprising of her background knowledge, the lab, the experiment and its results, together with Carol's interpretation of the results. Assume that, in fact the experiment does not support a conclusion p, but Carol assumes that it does. As a result, Carol believes that p.

Motivations

Example 2.4

Assume that Carol is conducting an experiment. Carol's evidential situation may be seen as comprising of her background knowledge, the lab, the experiment and its results, together with Carol's interpretation of the results. Assume that, in fact the experiment does not support a conclusion p, but Carol assumes that it does. As a result, Carol believes that p.





The Language $\mathcal{L}_B$

Definition 3.1

$$\varphi ::= p \mid \top \mid \bot \mid t \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \otimes \varphi \mid \varphi \rightarrow \varphi \mid \Box\varphi \mid A\varphi$$

- $\Box\varphi$ as "The agent implicitly believes that φ";
- $A\varphi$ as "The body of evidence available to the agent supports φ"; and
- $B\varphi =_{def} \Box\varphi \wedge A\varphi$ (Fagin and Halpern, 1988).

Substructural Frames

Definition 3.2 (Weakly Commutative Simple Frames)

$F = \langle P, \leq, L, R, C\rangle$

- $\langle P, \leq\rangle$ is a poset with a non-empty domain P;
- $L \subseteq P$ is $\leq$-closed ($x \in L$ and $x \leq y$ only if $y \in L$);
- $x \leq y \iff (\exists z \in L).Rzxy$
- $Rxyz$ and $x' \leq x$ and $y' \leq y$ and $z \leq z' \Longrightarrow Rx'y'z'$
- $Rxyz \Longrightarrow Ryxz$
- Cxy and $x' \leq x$ and $y' \leq y \Longrightarrow Cx'y'$
- $Cxy \Longrightarrow Cyx$

Substructural Models

Definition 3.3

$M = \langle F, V \rangle$, $V(p)$ is $\leq$-closed

- $x \models p$ iff $x \in V(p)$
- $x \models t$ iff $x \in L$
- $x \models \neg\varphi$ iff for all y, Cxy implies $y \not\models \varphi$
- $x \models \varphi \otimes \psi$ iff there are y, z such that $Ryzx$ and $y \models \varphi$ and $z \models \psi$
- $x \models \varphi \rightarrow \psi$ iff for all y, z, if $Rxyz$ and $y \models \varphi$, then $z \models \psi$
- φ is L-valid in M ($M \models^L \varphi$) iff $x \models \varphi$ for all $x \in L$

Commutative distributive non-associative full Lambek calculus with a simple negation **DFNLe**.

Worlds

Definition 3.4

$w \in P$ is a world in F iff (for all x, y)

1. Cww
2. Cwx implies $x \leq w$
3. $Rwww$
4. $Rwxy$ implies $x \leq w \leq y$
5. $Rxyw$ implies $x \leq w$ and $y \leq w$

Worlds

Definition 3.4

$w \in P$ is a world in F iff (for all x, y)

1. Cww
2. Cwx implies $x \leq w$
3. $Rwww$
4. $Rwxy$ implies $x \leq w \leq y$
5. $Rxyw$ implies $x \leq w$ and $y \leq w$

Lemma 3.5 (Extensionality and Logicality of Worlds)

1. $worlds \subseteq L$
2. $w \models \neg\varphi$ *iff* $w \not\models \varphi$
3. $w \models \varphi \rightarrow \psi$ *iff* $w \not\models \varphi$ *or* $w \models \psi$
4. $w \models \varphi \otimes \psi$ *iff* $w \models \varphi$ *and* $w \models \psi$

Evidence Frames

Definition 3.6

$\mathfrak{F} = \langle F, W, E, |\cdot| \rangle$

- $W \subseteq P$ is a set of worlds in F
- Exy and $x' \leq x$ and $y \leq y' \implies Ex'y'$
- Exy and $Wx \implies Wy$
- $x \leq y \implies |x| \leq |y|$

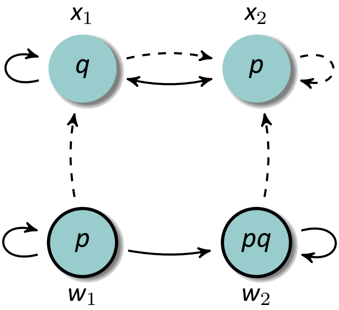
Evidence Models

Definition 3.7

$\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, $V(p)$ is $\leq$-closed

- $x \models \Box\varphi$ iff for all y, Exy implies $y \models \varphi$
- $x \models A\varphi$ iff $|x| \models \varphi$
- φ is valid in $\mathfrak{M}$ ($\mathfrak{M} \models \varphi$) iff $x \models \varphi$ for all $x \in W$

$\bigwedge \varphi_{(n)} \to \psi$ and $\bigwedge B\varphi_{(n)} \to B\psi$



w_i are "local". $x_i \leq y$ iff $x_i = y$, $Rx_1x_1x_1$, $Rx_1x_2x_2$ and $Rx_2x_1x_2$, while Cx_ix_j for all $i,j \in \{1,2\}$. $L = \{w_1, w_2, x_1\}$.

Some Valid Schemas

1. Propositional tautologies (in $\mathcal{L}_B$) and Modus Ponens
2. $(\varphi \otimes \psi) \leftrightarrow (\varphi \wedge \psi)$
3. $t \leftrightarrow \top$
4. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
5. $\varphi \,/\, \Box\varphi$
6. $\top \to A\top$ and $A\bot \to \bot$
7. $\left(\bigwedge A\varphi_{(n)}\right) \leftrightarrow A\left(\bigwedge \varphi_{(n)}\right)$
8. $\left(\bigvee A\varphi_{(n)}\right) \leftrightarrow A\left(\bigvee \varphi_{(n)}\right)$
9. If $\mathfrak{M} \models^{L} \bigwedge \varphi_{(n)} \to \bigvee \psi_{(m)}$, then $\mathfrak{M} \models \bigwedge A\varphi_{(n)} \to \bigvee A\psi_{(m)}$ and $\mathfrak{M} \models \bigwedge B\varphi_{(n)} \to B \bigvee \psi_{(m)}$

Axiomatization of **K** $+$ **DFNLe**

l-axioms

- $\varphi \to \varphi$
- $\varphi \wedge \psi \to \varphi$ and $\varphi \wedge \psi \to \psi$
- $\varphi \to \varphi \vee \psi$ and $\psi \to \varphi \vee \psi$
- $\varphi \to \top$ and $\bot \to \varphi$
- $\varphi \wedge (\psi \vee \chi) \to (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
- $\top \to A\top$ and $A\bot \to \bot$

r-axioms

- propositional tautologies in $\mathcal{L}_B$
- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- $\varphi \wedge \psi \leftrightarrow \varphi \otimes \psi$
- $t \leftrightarrow \top$

l-rules

- $\varphi, \varphi \to \psi \,/\, \psi$
- $\varphi \to \psi, \psi \to \chi \,/\, \varphi \to \chi$
- $\chi \to \varphi, \chi \to \psi \,/\, \chi \to (\varphi \wedge \psi)$
- $\varphi \to \chi, \psi \to \chi \,/\, (\varphi \vee \psi) \to \chi$
- $\varphi \to (\psi \to \chi) \,//\, (\psi \otimes \varphi) \to \chi$
- $\varphi \to (\psi \to \chi) \,//\, \psi \to (\varphi \to \chi)$
- $t \to \varphi \,//\, \varphi$
- $\varphi \to \neg\psi \,//\, \psi \to \neg\varphi$
- $\bigwedge \varphi_{(n)} \to \bigvee \psi_{(m)} \,/$ $\bigwedge A\varphi_{(n)} \to \bigvee A\psi_{(m)}$, for $n, m \geq 1$
- $\bigwedge \varphi_{(n)} \to \psi \,/\, \bigwedge \Box\varphi_{(n)} \to \Box\psi$, for $n \geq 1$

r-rules

- Modus Ponens
- $\varphi \,/\, \Box\varphi$

Proofs

are ordered couples of sequences of $\mathcal{L}_B$-formulas:

1. If $\overrightarrow{\chi_n} \mid \overrightarrow{\chi_m}$ is a proof and φ is a l-axiom, then $\overrightarrow{\chi_n}\varphi|\overrightarrow{\chi_m}$ is a proof ($n, m \geq 0$)
2. If $\overrightarrow{\chi_n} \mid \overrightarrow{\chi_m}$ is a proof and φ is a r-axiom, then $\overrightarrow{\chi_n}|\overrightarrow{\chi_m}\varphi$ is a proof ($n, m \geq 0$)
3. If $\overrightarrow{\chi_n} \mid \overrightarrow{\chi_m}$ is a proof such that $\overrightarrow{\chi_n}$ contains $\varphi_1, \ldots, \varphi_n$ and $\varphi_1, \ldots, \varphi_n \,/\, \psi$ is a l-rule, then $\overrightarrow{\chi_n}\psi|\overrightarrow{\chi_m}$ is a proof
4. If $\overrightarrow{\chi_n} \mid \overrightarrow{\chi_m}$ is a proof such that $\overrightarrow{\chi_m}$ contains $\varphi_1, \ldots, \varphi_n$ and $\varphi_1, \ldots, \varphi_n \,/\, \psi$ is a r-rule, then $\overrightarrow{\chi_n}|\overrightarrow{\chi_m}\psi$ is a proof
5. If $\overrightarrow{\chi_n}\psi|\overrightarrow{\chi_m}$ is a proof, then $\overrightarrow{\chi_n}\psi|\overrightarrow{\chi_m}\psi$ is a proof ("the jump rule")

φ is provable ($\vdash \varphi$) iff there is a proof $\overrightarrow{\chi_n}|\overrightarrow{\chi_m}\varphi$.

Theorem 4.1

$\vdash \varphi$ iff $\mathfrak{M} \models \varphi$ for all $\mathfrak{M}$.

Definability

Schema	Property
$\Box\varphi \to A\varphi$	$Wx \wedge \lvert x\rvert \leq y \to Sxy$
$A\varphi \to \Box\varphi$	$Wx \wedge Sxy \to \lvert x\rvert \leq y$
$A\varphi \to \varphi$	$Wx \to \lvert x\rvert \leq x$
$A\varphi \to AA\varphi$	$Wx \to \lvert x\rvert \leq \lvert x\rvert^2$
$A\varphi \to \neg A\neg\varphi$	$Wx \to C\lvert x\rvert x\rvert$
$\neg A\varphi \to A\neg A\varphi$	$Wx \wedge C\lvert x\rvert y \to \lvert y\rvert \leq \lvert x\rvert$
$A\varphi \to \Box A\varphi$	$Wx \wedge Sxy \to \lvert x\rvert \leq \lvert y\rvert$
$\neg A\varphi \to \Box\neg A\varphi$	$Wx \wedge Sxy \to \lvert y\rvert \leq \lvert x\rvert$
$\Box\varphi \to A\Box\varphi$	$Wx \wedge S\lvert x\rvert y \to \exists z.(Sxz \wedge z \leq y)$
$\neg\Box\varphi \to A\neg\Box\varphi$	$Wx \wedge Sxy \wedge C\lvert x\rvert z \to \exists u.(Szu \wedge u \leq y)$
$\Box\varphi \to \varphi$	$Wx \to Sxx$
$\Box\varphi \to \Box\Box\varphi$	$Wx \wedge Sxy \wedge Syz \to Sxz$
$\neg\Box\varphi \to \Box\neg\Box\varphi$	$Wx \wedge Sxy \wedge Sxz \to Syz$

More Details On

- definability
- strong completeness of extensions
- non-classical relational belief revision
- outline of informational dynamics

Can Be Found In

- "Substructural Epistemic Logics", to appear in the *Journal of Applied Non-Classical Logics*,
- "Epistemic Extensions of Modal Distributive Substructural Logics", to appear in the *Journal of Logic and Computation*,
- "Information, Awareness and Substructural Logics", in *Proc. of WoLLIC 2013.*

Future Work

- A fuller development of substructural models of information dynamics and action;
- Group-epistemic modalities in the substructural setting;
- Combinations with related approaches;
- Applications in Phil, CS etc.

THANK YOU!

References I

Jc Beall, Ross Brady, J. Michael Dunn, A. P. Hazen, Edwin Mares, Robert K. Meyer, Graham Priest, Greg Restall, David Ripley, John Slaney, and Richard Sylvan. On the ternary relation and conditionality. *Journal of Philosophical Logic*, 41:595–612, 2012. doi: 10.1007/s10992-011-9191-5.

Francesco Berto. A Modality Called 'Negation'. *Mind*, 2015. doi: 10.1093/mind/fzv026. To appear.

Marta Bílková, Ondrej Majer, and Michal Peliš. Epistemic logics for sceptical agents. *Journal of Logic and Computation*, 2015. doi: 10.1093/logcom/exv009. To appear.

J. Michael Dunn. Star and perp: Two treatments of negation. *Philosophical Perspectives*, 7:331–357, 1993. doi: 10.2307/2214128.

Ronald Fagin and Joseph Y. Halpern. Belief, awareness, and limited reasoning. *Artificial Intelligence*, 34:39–76, 1988. doi: 10.1016/0004-3702(87)90003-8.

Jaakko Hintikka. *Knowledge and Belief. An Introduction to the Logic of the Two Notions*. Cornell University Press, Ithaca, 1962.

Jaakko Hintikka. Impossible possible worlds vindicated. *Journal of Philosophical Logic*, 4:475–484, 1975. doi: 10.1007/BF00558761.

Hector Levesque. A logic of implicit and explicit belief. In *Proc. of AAAI 1984*, pages 198–202, 1984.