A Logic of Belief with a Complexity Measure

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Challenges for modeling belief systems

A belief system might contain:

- a contradictory proposition: $\mathbf{B}\big(\alpha \wedge (\alpha \to \beta) \wedge \neg\beta\big)$
- an inconsistent set of propositions: $\mathbf{B}(\alpha \to \beta)$, $\mathbf{B}(\neg\beta)$, $\mathbf{B}\alpha$

A belief system should fail to satisfy the following conditions:

- *Omnidoxasticity*: an agent may fail to believe a valid proposition, e.g., $\neg\mathbf{B}\big((\alpha \to \beta) \to (\neg\beta \to \neg\alpha)\big)$
- *Closure under implication*: an agent may fail to use the *modus ponens* rule over his beliefs, e.g., $\mathbf{B}\alpha$, $\mathbf{B}(\alpha \to \beta)$, $\neg\mathbf{B}\beta$
- *Closure under valid implication* (i.e. *consequential closure*): an agent may fail to believe a logical consequence of her beliefs, e.g., $\mathbf{B}(\alpha \to \beta)$, $\neg\mathbf{B}(\neg\beta \to \neg\alpha)$

Problems with existing approaches

Existing approaches can be roughly classified as:

- Coarse-grained: most approaches involving only possible worlds; e.g., they cannot distinguish $\{\alpha, \alpha \rightarrow \beta\}$ belief set from $\{\alpha, \alpha \rightarrow \beta, \beta\}$;
- Fine-grained (i.e. syntactic): most approaches with an awareness operator or impossible worlds; e.g., even $\{\alpha, \beta, \alpha \wedge \beta\}$ and $\{\alpha, \alpha \wedge \beta\}$ belief sets might be different;
- Resource-bounded agents (RBAs): a rule-based agent lacks some resources to be an ideal reasoner. From cognitive perspectives, often essential resources are deprived of (e.g., a complete set of rules [Konolige,84], the format of rules [Jago,09]) or resources are measured in an unrealistic way (e.g., #steps [Jago,09], [Elgot-Drapkin,88]).

Current approach

The current approach falls in the logics with rule-based and RBAs, where each agent has a certain amount of resource that is some function over her reasoning skills and available time for reasoning.

Two types of beliefs are considered:

- Initial belief – an explicit belief of [Levesque,84], i.e. a belief that is actively held to be true by an agent;
- Potential belief – a belief at which an agent has a resource to arrive based on his initial beliefs.

An amount of resources required to arrive at a belief α is determined by a (cognitively relevant) complexity measure, which measures a complexity of a reasoning process that is necessary to be carried out for obtaining α.

Outline

The rest of the presentation is structured as follows:

- Abstract complexity measure (ACM)
- Logic of belief with a complexity measure (LBC)
- Concrete complexity measure (CCM)
- Tableau belief logic (TABL)
- Related work
- Conclusion & References

Language of beliefs

Let $\mathcal{L}$ be a propositional language with the standard logical connectives $\vee, \wedge, \rightarrow, \neg$ and a constant false proposition `f`.

An equivalence relation $\approx$ over $\mathcal{L}$ holds between $\alpha, \beta \in \mathcal{L}$ iff α can be obtained by shuffling positions of β's conjuncts and disjuncts and using the idempotence property of $\wedge$ and $\vee$:

$$p \wedge q \wedge \neg(q \vee p \vee q) \approx q \wedge \neg(q \vee p) \wedge p$$

Let $\mathcal{L}^{\approx}$ be the language representing beliefs.

Abstract complexity measure ($\textsc{ACM}$)

Let an abstract complexity measure be a partial function $c(\alpha\,|\,X) \in R$, where R is a partially ordered set (with the least $\bot$ and the greatest $\top$ elements) and a monoid (with a commutative $\oplus$ operation and an identity $\bot$), s.t. $r_1 < r_1 \oplus r_2$ if $r_2 \neq \bot$.

The complexity measure c satisfies the following properties:

(1) $c(\alpha\,|\,X) \in R$ iff $X \models \alpha$
(2) $c(\alpha\,|\,X) = \bot$ if $\alpha \in X$
(3) $c(\alpha\,|\,Y) \leq c(\alpha\,|\,X)$ if $X \subseteq Y$
(4) $c(\alpha\,|\,X) \leq c(\alpha \wedge \beta\,|\,X)$
(5) $c(\mathtt{f}\,|\,X \cup \{\alpha, \neg\alpha\}) = \bot$
(6) $c(\alpha\,|\,X \cup Y) \leq c(\alpha\,|\,Y \cup \{\beta\}) \oplus c(\beta\,|\,X)$

The following properties are derivable:

$c(\alpha \mid \{\alpha, \neg\alpha\}) = \bot$ $\quad$ $c(\alpha \mid \{\neg\alpha\}) \uparrow$
$c(\alpha \mid \{\alpha \wedge \beta\}) = \bot$ $\quad$ possibly $c(\alpha \wedge \beta \mid \{\alpha, \beta\}) \neq \bot$
$c(\alpha\,|\,\{\gamma\}) \leq c(\alpha\,|\,\{\beta\}) \oplus c(\beta\,|\,\{\gamma\})$

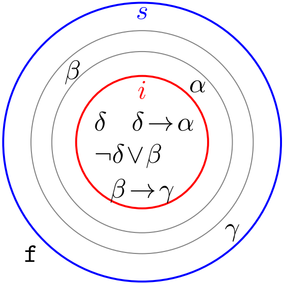
Belief state

An r-belief state $\mathcal{B}^r = \langle i^r, s^r \rangle$ is a pair of *initial* and *potential* belief sets.

An initial belief set i^r is:

- r-consistent, i.e. $c(\mathtt{f} \mid i^r) \nleq r$;
- $\wedge$-set, i.e. $\alpha, \beta \in i^r$ iff $\alpha \wedge \beta \in i^r$.

A potential belief set s^r contains all and only beliefs r-obtainable from i^r, i.e. $s^r = \{\alpha \mid c(\alpha \mid i^r) \leq r\}$.



Properties of a belief state

An r-belief state $\mathcal{B}^r = \langle i^r, s^r \rangle$:

- $c(\mathtt{f} \mid i^r) \not\leq r$ r-consistent
- $\alpha, \beta \in i^r$ iff $\alpha \wedge \beta \in i^r$ $\wedge$-set
- $s^r = \{\alpha \mid c(\alpha \mid i^r) \leq r\}$ r-obtainable

Several properties of an r-belief state for any $r \in R$:

- $i^r \subseteq s^r$ since if $\alpha \in i^r$, $c(\alpha \mid i^r) = \bot \leq r$
- $i^r = \emptyset$ is possible since $c(\mathtt{f} \mid \emptyset) \not\leq r$ as $\emptyset \not\models \mathtt{f}$
- $\mathtt{f} \notin s^r$ since i^r is r-consistent
- $\alpha, \beta \in s^r$ if $\alpha \wedge \beta \in s^r$ semi-$\wedge$-set since $c(\alpha \mid i^r) \leq c(\alpha \wedge \beta \mid i^r) \leq r$
- $\{\alpha, \neg\alpha\} \not\subseteq i^r$ otherwise $c(\mathtt{f} \mid i^r) = \bot \leq r$
- $\{\alpha, \neg\alpha\} \subseteq s^r$ is possible

Logic of belief with the ACM (LBC)

Let $\mathcal{L}_{\mathbf{IP}}$ be a standard non-nested extension of a propositional language $\mathcal{L}$ with initial $\mathbf{I}$ and potential $\mathbf{P}$ belief operators.

For a fixed ACM, semantics of $\mathcal{L}_{\mathbf{IP}}$ wrt a model $\mathcal{M} = \langle V, \mathcal{B}_1^{r_1}, \ldots \mathcal{B}_n^{r_n} \rangle$, where V is an interpretation function over $\mathcal{L}$ and $B_k^{r_k}$ is a belief state for the k^{th} agent:

$$\begin{array}{lll} \mathcal{M} \models \alpha & \text{iff} & V(\alpha) = 1 \\ \mathcal{M} \models \mathbf{I}_k\alpha & \text{iff} & \alpha \in i^{r_k} \\ \mathcal{M} \models \mathbf{P}_k\alpha & \text{iff} & \alpha \in s^{r_k} \qquad (\text{iff} \quad c(\alpha \mid i^{r_k}) \leq r_k) \\ \mathcal{M} \models \psi & \multicolumn{2}{l}{\text{defined recursively in the standard way}} \end{array}$$

Validity for LBC is defined in a standard way:
$\models \psi$, iff for any model $\mathcal{M}$, $\mathcal{M} \models \psi$.

Valid formulas: $\models \mathbf{I}\alpha \to \mathbf{P}\alpha$, $\models \neg\mathbf{I}\mathtt{f} \wedge \neg\mathbf{P}\mathtt{f}$, and $\models \neg(\mathbf{I}\alpha \wedge \mathbf{I}\neg\alpha)$

Tableau system for LBC

The set of tableau rules $\mathcal{R}$ for LBC consists of standard complete set of propositional rules and several rules for $\mathbf{I}$ and $\mathbf{P}$ operators:

$$\frac{\neg\mathbf{I}(\alpha\wedge\beta)}{\neg\mathbf{I}\alpha \qquad \neg\mathbf{I}\beta}(\neg\mathbf{I}\wedge) \qquad \frac{\mathbf{B}(\alpha\wedge\beta)}{\begin{array}{c}\mathbf{B}\alpha\\ \mathbf{B}\beta\end{array}}(\mathbf{B}\wedge),\quad \text{where } \mathbf{B}\in\{\mathbf{P},\mathbf{I}\}$$

$$\begin{array}{c}\boxed{\text{unobtainability}}\\ \dfrac{\begin{array}{c}\mathbf{I}_k\alpha_1\\ \vdots\\ \mathbf{I}_k\alpha_n\\ \neg\mathbf{P}_k\beta\end{array}}{c(\beta\mid X)\not\leq r^k}(\mathbf{I}\neg\mathbf{P})\end{array} \qquad \begin{array}{c}\boxed{\text{consistency}}\\ \dfrac{\begin{array}{c}\mathbf{I}_k\alpha_1\\ \mathbf{I}_k\alpha_2\\ \vdots\\ \mathbf{I}_k\alpha_n\end{array}}{c(\mathtt{f}\mid X)\not\leq r^k}(\mathbf{I})\end{array}$$

where $X=\mathcal{S}_\wedge(\{\alpha_i\}_{i=1}^n)$

Tableau system for LBC (2)

potential compatibility

$$\frac{\begin{array}{c}\mathbf{I}_k\alpha_1\\ \vdots\\ \mathbf{I}_k\alpha_n\\ \mathbf{P}_k\beta\end{array}}{\begin{array}{c}c(\mathtt{f}\mid Y)\uparrow \quad \text{if } r^k=\top\\ c(\mathtt{f}\mid Y)\neq\bot \text{ otherwise}\end{array}}(\mathbf{IP})$$

checking a constraint

$$\frac{\text{a constraint on } c(\alpha\mid X)}{\begin{array}{c}\text{check the constraint;}\\ \text{if it fails, then } \times\end{array}}(c)$$

$$c(\mathtt{f}\mid X)\leq c(\mathtt{f}\mid Y)\oplus c(\beta\mid X)=\bot\oplus r^k=r^k$$

where $X=\mathcal{S}_\wedge(\{\alpha_i\}_{i=1}^n)$, and $Y=X\cup\{\beta\}$

Theorem (soundness & completeness)

Given an ACM c, the tableau method represents a sound and complete proof procedure for LBC

Concrete complexity measure (CCM)

One way to define a concrete complexity measure is to *measure* the proofs of one's favorite proof system.

Let R be a standard complete set of propositional tableau rules plus several admissible rules. For example, some members of R:

$$\frac{\alpha \vee \beta}{\alpha \qquad \beta}(\vee) \quad \frac{\begin{matrix}\alpha \vee \beta \\ \neg\alpha\end{matrix}}{\beta}(\vee_{\neg}) \quad \frac{\begin{matrix}\alpha \rightarrow \beta \\ \alpha\end{matrix}}{\beta}(\rightarrow) \quad \frac{\begin{matrix}\alpha \\ \neg\alpha\end{matrix}}{\mathtt{f}}(\mathtt{f}) \quad \frac{\alpha \wedge \beta}{\begin{matrix}\alpha \\ \beta\end{matrix}}(\wedge)$$

Let $\mathcal{C}$ be a cost assignment that assigns *cognitively relevant costs* to the consequent formulas of tableau rules; e.g., $\mathcal{C}(\vee, L1) = 1$:

$$\frac{\alpha \vee \beta\,_x}{\alpha_{x+1} \qquad \beta_{x+1}}(\vee) \quad \frac{\begin{matrix}\alpha \vee \beta\,_x \\ \neg\alpha_y\end{matrix}}{\beta_{x+y+2}}(\vee_{\neg}) \quad \frac{\begin{matrix}\alpha \rightarrow \beta\,_x \\ \alpha_y\end{matrix}}{\beta_{x+y+1}}(\rightarrow) \quad \frac{\begin{matrix}\alpha_x \\ \neg\alpha_y\end{matrix}}{\mathtt{f}_{x+y}}(\mathtt{f}) \quad \frac{\alpha \wedge \beta\,_x}{\begin{matrix}\alpha_x \\ \beta_x\end{matrix}}(\wedge)$$

Cost of a tableau proof

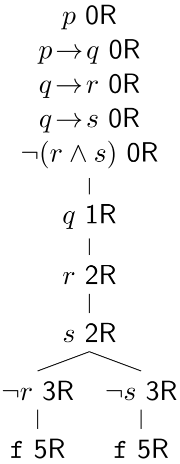
Calculating a cost of $r \wedge s$ with respect to $\{p,\ p \to q,\ q \to r,\ q \to s\}$:

Tableau rules with costs:

$$\frac{\begin{array}{c}\alpha \to \beta\ {}_x \\ \alpha_y\end{array}}{\beta_{x+y+1}}(\to)$$

$$\frac{\neg(\alpha \wedge \beta)\ {}_x}{\neg\alpha_{x+3} \qquad \neg\beta_{x+3}}(\neg\wedge)$$

$$\frac{\begin{array}{c}\alpha_x \\ \neg\alpha_y\end{array}}{\mathtt{f}_{x+y}}(\mathtt{f})$$



p 0R

$p \to q$ 0R

$q \to r$ 0R

$q \to s$ 0R

$\neg(r \wedge s)$ 0R

q 1R

r 2R

s 2R

$\neg r$ 3R $\qquad$ $\neg s$ 3R

$\mathtt{f}$ 5R $\qquad$ $\mathtt{f}$ 5R

The tableau costs 10R

Tableau rules with costs (fixed)

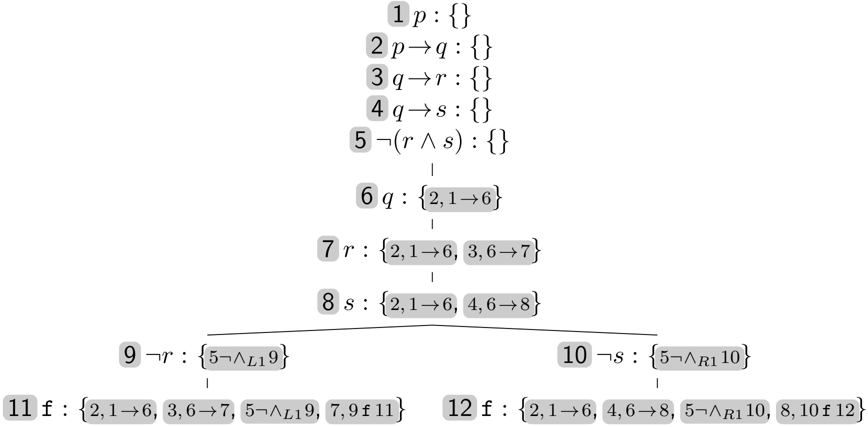
$$\frac{\begin{array}{c} \alpha \to \beta_{\,x} \\ \alpha_y \end{array}}{\beta_{\,x+y+1}}(\to) \qquad \frac{\begin{array}{c} l_1\ \alpha \to \beta : x \\ l_2\ \alpha : y \end{array}}{l_3\ \beta : x \cup y \cup \{l_1, l_2 \to l_3\}}(\to)$$

$$\frac{\neg(\alpha \wedge \beta)_{\,x}}{\neg\alpha_{x+3} \qquad \neg\beta_{x+3}}(\neg\wedge) \qquad \frac{l_1\ \neg(\alpha \wedge \beta) : x}{l_2\ \neg\alpha : x \cup \{l_1 \neg\wedge_{L1} l_2\} \qquad l_3\ \neg\beta : x \cup \{l_1 \neg\wedge_{R1} l_3\}}(\neg\wedge)$$

$$\frac{\begin{array}{c} \alpha_x \\ \neg\beta_y \end{array}}{\mathtt{f}_{x+y}}(\mathtt{f}) \qquad \frac{\begin{array}{c} l_1\ \alpha : x \\ l_2\ \neg\beta : y \end{array}}{l_3\ \mathtt{f} : x \cup y \cup \{l_1, l_2\, \mathtt{f}\, l_3\}}(\mathtt{f})$$

Cost of a tableau proof (fixed)

Calculating a cost of $r \wedge s$ with respect to $\{p,\ p \to q,\ q \to r,\ q \to s\}$:



1 $p : \{\}$

2 $p \to q : \{\}$

3 $q \to r : \{\}$

4 $q \to s : \{\}$

5 $\neg(r \wedge s) : \{\}$

6 $q : \{2, 1 \to 6\}$

7 $r : \{2, 1 \to 6,\ 3, 6 \to 7\}$

8 $s : \{2, 1 \to 6,\ 4, 6 \to 8\}$

9 $\neg r : \{5 \neg\wedge_{L1} 9\}$

10 $\neg s : \{5 \neg\wedge_{R1} 10\}$

11 $\mathtt{f} : \{2, 1 \to 6,\ 3, 6 \to 7,\ 5 \neg\wedge_{L1} 9,\ 7, 9\, \mathtt{f}\, 11\}$

12 $\mathtt{f} : \{2, 1 \to 6,\ 4, 6 \to 8,\ 5 \neg\wedge_{R1} 10,\ 8, 10\, \mathtt{f}\, 12\}$

The tableau costs as much as $\{2, 1 \to 6,\ 3, 6 \to 7,\ 5 \neg\wedge_{L1} 9,\ 7, 9\, \mathtt{f}\, 11,\ 4, 6 \to 8,\ 5 \neg\wedge_{R1} 10,\ 8, 10\, \mathtt{f}\, 12\}$ rule applications together, i.e. 9R.

Tableau cost function

A cost of a tableau proof t, denoted as $\texttt{cost}_{\mathcal{C}}(t)$, is not defined if t is open; otherwise the cost of t is a cost of a set of rule applications that introduce `f` on each branch.

A tableau cost function $C^R_{\mathcal{C}}(\alpha \mid X)$ is defined as the cost of the cheapest tableau built over $\{\neg\alpha\} \cup X$:

$$C^R_{\mathcal{C}}(\alpha \mid X) = \min_{t \in \mathfrak{T}} \texttt{cost}_{\mathcal{C}}(t)$$

where $\mathfrak{T}$ is a set of all tableaux built over $\{\neg\alpha\} \cup X$ wrt R rules.

Tableau cost function as a CCM

The tableau cost function $C^{R}_{\mathcal{C}}$ has all the properties of the ACM:

(1) $c(\alpha\,|\,X)\downarrow \qquad$ iff $X \models \alpha$

(2) $c(\alpha\,|\,X) = 0 \qquad$ if $\alpha \in X$

(3) $c(\alpha\,|\,Y) \leq c(\alpha\,|\,X) \qquad$ if $X \subseteq Y$

(4) $c(\alpha\,|\,X) \leq c(\alpha \wedge \beta\,|\,X)$

(5) $c(\mathtt{f}\,|\,X \cup \{\alpha, \neg\alpha\}) = 0$

(6) $c(\alpha\,|\,X \cup Y) \leq c(\alpha\,|\,Y \cup \{\beta\}) + c(\beta\,|\,X)$

if there is a cut rule in R: $\dfrac{}{\beta \qquad \neg\beta}(cut)$

and the cost assignment $\mathcal{C}$ assigns costs as follows:

$$\frac{\begin{array}{c}\alpha_x\\ \neg\alpha_y\end{array}}{\mathtt{f}_{x+y}}(\mathtt{f}) \qquad \frac{\alpha\wedge\beta\;_x}{\begin{array}{c}\alpha_x\\ \beta_x\end{array}}(\wedge) \qquad \frac{}{\beta_0 \qquad \neg\beta_0}(cut)$$

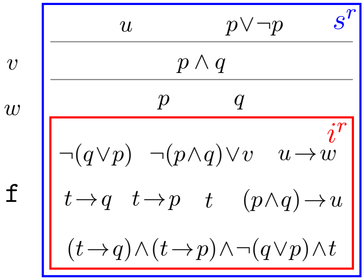
Tableau belief logic (TABL)

If we assume that the ACM $c = C_{\mathcal{C}}^{R}$ in BLC, then we will get a concrete instance of BLC — a tableau belief logic.

$$\boxed{\text{checking constraints}}$$

$$\frac{\text{a constraint on } c(\alpha \mid X)}{\begin{array}{c}\text{Check the constraint on } C_{\mathcal{C}}^{R}(\alpha \mid X);\\ \text{if it fails, then close the branch}\end{array}}(c)$$

Properties of TABL



s^r

u $\quad p\vee\neg p$

v $\quad p\wedge q$

w $\quad p \quad q$

i^r

f

$\neg(q\vee p)\quad \neg(p\wedge q)\vee v\quad u\rightarrow w$

$t\rightarrow q\quad t\rightarrow p\quad t\quad (p\wedge q)\rightarrow u$

$(t\rightarrow q)\wedge(t\rightarrow p)\wedge\neg(q\vee p)\wedge t$

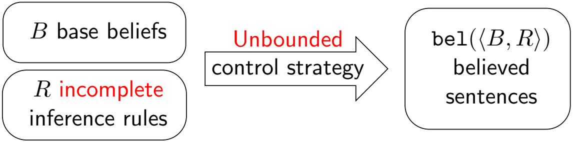
$\neg(p\wedge q)\leftrightarrow(\neg p\vee\neg q)$

- a contradictory belief: $\mathbf{P}\big((t\rightarrow q)\wedge(t\rightarrow p)\wedge\neg(q\vee p)\wedge t\big)$
- an inconsistent set of beliefs;
- no omnidoxasticity: $\neg\mathbf{P}\big(\neg(p\wedge q)\leftrightarrow(\neg p\vee\neg q)\big)$
- no closure under implication: $\mathbf{P}u,\ \mathbf{P}(u\rightarrow w),\ \neg\mathbf{P}w$
- no closure under valid implication.

- It can model RBAs with different intelligence, where r parameter will stand for intelligence measure (a perfect reasoner is obtained in a straightforward way: $r=\infty$);
- The logic permits the framing effect.

Motivating/related work

Konolige's deduction model of belief [Konolige,84]:



B base beliefs

R incomplete inference rules

Unbounded control strategy

$\texttt{bel}(\langle B, R\rangle)$ believed sentences

Although a belief state is closed under derivation, consequential closure is avoided if R is incomplete.

But it is necessary that an agent is unable to use a certain boolean rule in order to prevent him from believing all prop. tautologies.

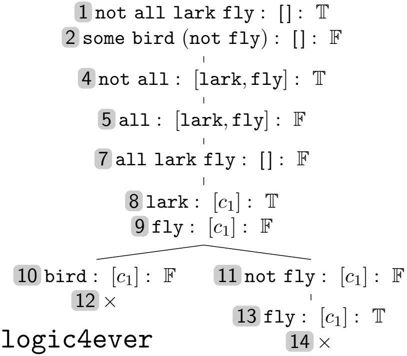
"Probably the chief motivation for requiring derivational closure is that it simplifies the technical task of formalizing the deduction model."

"it makes difference to the control strategy as to whether a sentence is a member of the base set, or obtained at some point in a derivation. One cannot simply say "Agent S believes P," because such a statement doesn't give enough information about P to be useful. If P is derived at the very limit of deductive resources, then nothing will follow from it;" [Konolige,84]

Motivating/related work

The program Towards Logics that Model Natural Reasoning aims to develop "a general theory of the natural logic behind human reasoning and human information processing by studying formal logics that operate directly on linguistic representations" [Muskens,11].

An analytic tableau system for Natural Logic [Muskens,10; Abzianidze,15] can reason over linguistic expressions:



1 `not all lark fly` : [] : $\mathbb{T}$
2 `some bird (not fly)` : [] : $\mathbb{F}$
4 `not all` : [`lark`, `fly`] : $\mathbb{T}$
5 `all` : [`lark`, `fly`] : $\mathbb{F}$
7 `all lark fly` : [] : $\mathbb{F}$
8 `lark` : [c_1] : $\mathbb{T}$
9 `fly` : [c_1] : $\mathbb{F}$
10 `bird` : [c_1] : $\mathbb{F}$
12 ×
11 `not fly` : [c_1] : $\mathbb{F}$
13 `fly` : [c_1] : $\mathbb{T}$
14 ×

`http://tinyurl.com/logic4ever`

Conclusion

Pros

- The model takes into account complexity of reasoning processes that makes it cognitively relevant and realistic;
- LBC offers further options, whether choosing a different formal language or a different proof theory;
- Pairing tableau proofs of Natural Logic with results of the experiments on reasoning [Chater&Oaksford,99] might give promising clues about the cost assignment.

Future work

- Modeling higher-order beliefs requires changes in ACM and in the model of LBC (e.g., a resource assignment for agents);
- For $C^R_{\mathcal{C}}$ needs to be shown whether there is always a cheapest tableau that is cut-free;
- Investigate other proof procedures for CCM as agents are not always reasoning in a refutation style.

Thank you

References

- Abzianidze, L.: A Tableau Prover for Natural Logic and Language. In proceedings of Empirical Methods in Natural Language Processing (2015).
- Elgot-Drapkin, J.,J.: Step-logic: reasoning situated in time. Doctoral Dissertation, University of Maryland, MD (1988).
- Jago, M.: Epistemic Logic for Rule-Based Agents, Journal of Logic, Language and Information 18(1), 131-158 (2009).
- Konolige, K.: A deduction model of belief and its logics, Technical Note 326, SRI International, Melon Park, CA (1984).
- Levesque, H.J.: A logic of implicit and explicit belief, Proceedings AAAI-84, 198–202 (1984).
- Muskens, R.: An Analytic Tableau System for Natural Logic. Logic, Language and Meaning. LNCS, 6042, pp.104-113 (2010)
- Muskens, R.: Towards Logics that Model Natural Reasoning. Program Description (2011)
- Chater, N., Oaksford, M.: The Probability Heuristics Model of Syllogistic Reasoning. Cognitive Psychology, 38, pp.191258 (1999)