Proof system

- Can we characterise the set of all valid formulas of basic modal logic syntactically?
- It turns out, there is an axiomatic system which is complete for basic modal logic: all valid formulas are provable (and all provable formulas are valid).
- There are also various natural deduction and tableau systems, but it is easier to prove completeness for the axiomatic system.

Axiom system K

- A formula ϕ is derivable in K ($\vdash_K \phi$) there is a sequence of formulas $\phi_1, \ldots, \phi_n$, such that $\phi_n = \phi$ and each formula ϕ_i is either an axiom of K or is obtained from the previous formulas by one of the inference rules of K.
- Axioms of K:
  - classical tautologies
  - $\Box(\phi \to \psi) \to (\Box \phi \to \Box\psi)$
- Rules of K:
  - modus ponens: from ϕ and $\phi \to \psi$ derive ψ
  - necessitation: from ϕ derive $\Box\phi$

Provable and consistent formulas

- A formula ϕ is provable (in K) $\vdash_K \phi$
- A formula ϕ is consistent (in K) if its negation is not provable.
- We'll see that provability in K and validity in basic modal logic coincide, as do consistency in K and satisfiability in basic modal logic.

Exercises

- Prove that the following are theorems of K:
  - $\Diamond(\phi \vee \psi) \to (\Diamond\phi \vee \Diamond\psi)$
  - $\Diamond(\phi \wedge \psi) \to \Diamond\phi$

Completeness of K

- For every formula ϕ of basic modal logic, $\vdash_K \phi$ if, and only if, ϕ is valid in basic modal logic ($\models \phi$).
- First we prove soundness: $\vdash_K \phi$ implies $\models \phi$
- Then completeness proper: $\models \phi$ implies $\vdash_K \phi$

Soundness of K

- $\vdash_K \phi$ implies $\models \phi$
- Proof by induction on the derivation of ϕ:
  - axioms are valid
  - for every rule, if the premises are valid, then the conclusion is valid.

Completeness of K

- $\models \phi$ implies $\vdash_K \phi$
- We will prove: if ϕ is consistent, then ϕ is satisfiable:

not ($\vdash_K \neg\phi$) implies $\not\models \neg\phi$

- By contraposition, $\models \neg\phi$ implies $\vdash_K \neg\phi$
- Since we are quantifying over all formulas and we are in classical logic where $\neg\neg\phi = \phi$, this implies

if $\models \phi$ then $\vdash_K \phi$

Plan of the proof

- Take a consistent formula ϕ (such that not ($\vdash_K \neg\phi$)
- Build a model where ϕ is true in one of the worlds (this going to be a model built out of ϕ's subformulas)
- Hence ϕ is satisfiable.
- In addition, we build a *finite* model for ϕ, of bounded size ($2^{|\phi|}$ to be precise).

Decidability as a consequence

- A consequence of our proof is that the satisfiability problem for basic modal logic is decidable!
- Given that every satisfiable formula ϕ has a model of size $2^{|\phi|}$, we only need to check all models exponential in the size of ϕ and if none satifies ϕ then it is not satisfiable.
- This algorithm has double exponential complexity, in fact it can be improved to PSPACE.

Building blocks

- Let Subf(ϕ) be the set of all subformulas of ϕ.
- In addition, let us close Subf(ϕ) under single negations; that is, if $\psi \in$ Subf(ϕ) and ψ is not of the form $\neg\chi$, then add $\neg\psi$ to the set NegSubf(ϕ).
- Note that NegSubf(ϕ) is finite (size linear in $|\phi|$).

Example: if ϕ is $p \to \neg\Box q$ then

- Subf(ϕ) = $\{p \to \neg\Box q, p, \neg\Box q, \Box q, q\}$
- NegSubf(ϕ) = $\{p \to \neg\Box q, p, \neg\Box q, \Box q, q, \neg(p\to\neg\Box q), \neg p, \neg q\}$

Atoms (maximal consistent sets)

- Given a set of formulas S which is closed under single negations, an atom over S is a subset of S such that it is consistent and adding another formula from S to it will make it inconsistent.
- Some properties of a maximal consistent set A over S:
  - for every $\psi\in$ S: either $\psi \in$ A or $\neg\psi \in$ A.
  - for every $(\phi \to \psi)\in$ S: $(\phi \to \psi)\in$ A iff $\neg\phi \in$ A or $\psi\in$ A
- Proof of properties by propositional reasoning (left as an exercise).

Defining the model for ϕ

- Given a consistent formula ϕ:
  - Take NegSubf(ϕ)
  - Produce the set At(ϕ) of all possible atoms over NegSubf(ϕ).
  - Note that ϕ itself belongs to at least one atom because it is consistent
  - Also note that there are at most $2^{|\phi|}$ atoms.
- Now let W = At(ϕ) and for every $p \in$ Subf(ϕ) and every atom $A \in$ W, V(p,A) = true iff $p \in A$.

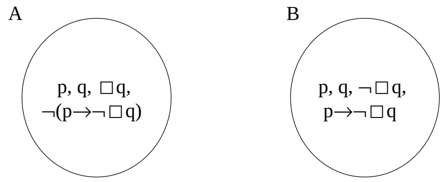
Defining R in our model

- Note that atoms are finite
- For an atom A, denote the conjunction of atoms in A by ∧A.
- For any two atoms A,B ∈ W:

R(A,B) if ∧A ∧ ◊ ∧B is consistent

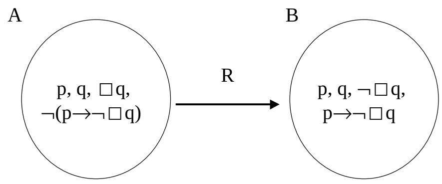
- Intuitively, we insert R between any two sets of formulas A and B where it would not lead to problems (inserting R means that we should be able to add formulas from B prefixed by a diamond, to A).

Example



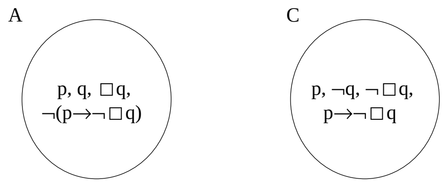
NegSubf(ϕ) = {p → ¬□ q, p, ¬□ q, □ q, q, ¬(p→¬□q), ¬ p, ¬q}

Example



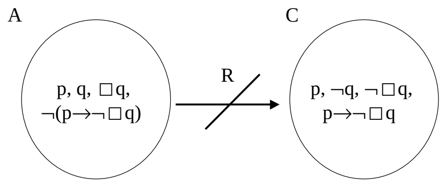
NegSubf(ϕ) = {p → ¬□ q, p, ¬□ q, □ q, q, ¬(p→¬□q), ¬ p, ¬q}

Example



NegSubf(ϕ) = {p → ¬□ q, p, ¬□ q, □ q, q, ¬(p→¬□q), ¬ p, ¬q}

Example



Not safe: □q is in A and ¬q is in C; □q and ◊¬q = ¬□ q are inconsistent.

Truth lemma

- In the model M we just built, we will show that for any ψ∈ NegSubf(ϕ) and for any atom A ∈ W,

ψ∈ A iff M,A |= ψ

- This will show that the formula ϕ is satisfied in at least one world of M.

Proof of the Truth Lemma

Proof goes by induction on the complexity of ψ

- Basis of induction: prove for $\psi = p$
- Inductive step: assume this holds for less complex formulas; show for
  - $\psi = \neg \psi_1$
  - $\psi = \psi_1 \rightarrow \psi_2$
  - $\psi = \Box \psi_1$

Proof of the Truth Lemma continued

- Basis: M, A |= p iff V(p,A) = true iff $p \in A$;
- Inductive step:

  M,A |= $\neg \psi$ iff M,A |≠ ψ iff $\psi \notin A$ (inductive hypothesis) iff $\neg \psi \in A$ (maximal consistent set)

  M,A |= $\chi \rightarrow \psi$ iff either M,A |≠ χ or M,A |= ψ iff $\chi \notin A$ or $\psi \in A$ iff $\chi \rightarrow \psi \in A$.

  Remains to show:

  M,w |= $\Box\psi$ iff $\Box\psi \in A$.

Last step: ⇐

- Remains to show:

  M,w |= $\Box\psi$ iff $\Box \psi \in A$.
- Suppose $\Box \psi \in A$. We need to show that for all atoms B such that R(A,B), B |= ψ, which by the inductive hypothesis is the same as $\psi \in$ B. Reasoning by contradiction, assume there is a B such that $\psi \notin B$ hence $\neg\psi \in$ B. R(A,B) means that $\bigwedge A \wedge \Diamond \bigwedge B$ is consistent. However $\bigwedge A \wedge \Diamond \bigwedge B$ implies $\Box \psi \wedge \Diamond \neg\psi$ which is inconsistent, so R(A,B) does not hold.

Last step ⇒

- Remains to show:

  M,w |= $\Box\psi$ iff $\Box \psi \in A$.
- Suppose $\Box \psi \notin$ A. We need to show that M,A |≠ $\Box \psi$. To do this we need to construct an atom B such that R(A,B) and $\psi \notin B$ ($\neg\psi \in$ B). Since $\Box \psi \notin$ A, $\neg \Box \psi \in A$, which is the same as $\Diamond \psi \in A$. So $\bigwedge A \wedge \Diamond \psi$ is consistent; we just need to expand $\psi$to a maximal consistent set.
- We do this by enumerating all the formulas in NegSubf(ϕ): $\phi_1, \ldots, \phi_n$. and assembling the atom B by adding either ϕ_i or $\neg \phi_i$ to ψ. For every B', either $\bigwedge A \wedge \Diamond(B' \wedge \phi_i)$ is consistent or $\bigwedge A \wedge \Diamond(B' \wedge \neg \phi_i)$.