

# Embedding Coalition Logic in the Minimal Normal Multi-Modal Logic with Intersection

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**Abstract** We present a satisfiability-preserving embedding of coalition logic into a normal modal logic. An advantage of standard, normal, modal logics is a well understood theoretical foundation and the availability of tools for automated verification and reasoning. The target logic is multi-modal  $K$  with intersection of modalities, interpreted over standard Kripke models corresponding to game structures. There is a restriction: we consider only game structures that are injective. We argue that this is a minor limitation, e.g., because coalition logic cannot discern between injective and non-injective game structures. We give a complete axiomatisation of the corresponding models, as well as a characterisation of key complexity problems. We also prove a representation theorem identifying the effectivity functions corresponding to injective games.

## 1 Introduction

Logics interpreted in *game structures* have received considerable interest in recent years. For several, including computational, reasons most such logics are modal logics [16]. One of the most popular approaches is reasoning about *coalitional ability*. Examples of logics in this category include *Coalition Logic* (CL) [24] and *Alternating-time Temporal Logic* (ATL) [3], and many extensions of these, which are interpreted in *game structures*. These logics have *coalition operators* of the form  $[C]$  where  $C$  is a set of agents (a *coalition*), and  $[C]\phi$  means that  $C$  can make  $\phi$  true by

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choosing some joint action. On the other hand, it has been shown [22, 23, 5, 6] that standard propositional dynamic logic (PDL) [15] is natural for reasoning about games. An advantage of using normal modal logics like PDL is that they are theoretically well understood, with a range of mathematical and computational tools available. Clearly, understanding the relationship between these two approaches are of interest. For example, van Benthem [7] states that “an explicit action/strategy version of coalition logic” is an interesting open problem.

In this paper we take a clue from van Benthem [7]: that operators for *intersection* of strategies can be used to express coalitional ability. In particular, we observe that *standard PDL with intersection* can be used to express coalition operators. We study in detail a minimal variant, multi-modal  $K$  with intersection of modalities ( $K_n^\cap$ ), interpreted in game structures, and define a satisfiability-preserving embedding of CL into  $K_n^\cap$ .  $K_n^\cap$  is a fragment of Boolean Modal Logic [10] which has been extensively studied (and implemented) as a variant of propositional dynamic logic with intersection and also by researchers in description logic (see for example [20]).

Although other logics that are normal modal logics that can express coalition operators have been studied recently [8, 19], these logics are not standard PDL. The logic in [19] is *based* on PDL but also has non-PDL operators. The focus in the current paper is on reasoning about joint action in game structures using (a fragment of) *standard PDL with intersection*. Also, we study many types of research questions not considered in [19], and vice versa. See Section 7 for a further discussion about related work.

The main contributions of the paper are threefold. First, we give an interpretation of  $K_n^\cap$  over models corresponding to game structures and give a sound and complete axiomatisation as well as characterisations of key complexity problems. There is a limitation: this restricted model class does in fact not correspond to *all* game structures, only to *injective* game structures [12], i.e., game structures where two different strategy profiles never lead to the same outcome state. However, we argue that this is a minor limitation:

- Coalition Logic cannot discern between injective and non-injective game structures.
- Injectivity is a very common assumption in game theory. Indeed, the notion of outcome states is often (as in the standard textbook [21]) dispensed with altogether, and preferences defined directly over strategy profiles – implicitly defining an injective game.

Second, we show that coalition operators can be expressed in  $K_n^\cap$ , and that both the model checking and satisfiability problems for coalition logic can be reduced to the corresponding problems in  $K_n^\cap$ . In particular, we show that coalition logic is embedded in  $K_n^\cap$ , in the following sense: we present a satisfiability-preserving translation between coalition logic and  $K_n^\cap$ . Third, we prove a variant of Pauly’s *representation theorem* [24, 14] for injective games: we characterise the class of effectivity functions that correspond to injective games.

We argue that the logic we study is interesting for several reasons. As a normal modal logic it has a well understood theoretical foundation, and it is well supported

by tools for automated verification and reasoning as a fragment of standard computer science logics such as PDL. For example, model checking can be done using standard model checkers for PDL (with intersection). In contrast, CL is a non-normal modal logic, with only special purpose tool support (mainly tools developed for ATL) available. Our logic can also be seen as providing an additional set of model-checking and theorem-proving tools for CL. Last but not least, from a theoretical viewpoint, the logic establishes new connections between coalition logic and normal modal logics [17, 11].

The paper is organised as follows. We start with introducing CL and  $K_n^\cap$ . We then, in Section 3, discuss the restriction of CL to injective game structures and prove the representation theorem. In Section 4 we define the interpretation of  $K_n^\cap$  in game structures, together with translations from CL and the embedding result, before axiomatisation and complexity are discussed in Sections 5 and 6. We discuss related work and conclude in Section 7.

An earlier version of this paper was presented at the CLIMA 2011 workshop [1].

## 2 Background

In this section we provide some background on Coalition Logic (CL) [24] and multi-modal  $K$  with intersection of modalities.

### 2.1 Coalition Logic

Let  $N = \{1, \dots, g\}$  be a finite set of agents, and  $\Theta$  a set of propositional variables. The language of  $\mathcal{L}_{CL}(N, \Theta)$  of CL [24] is defined as follows:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid [C]\phi$$

where  $p \in \Theta$  and  $C \subseteq N$ . A set  $C \subseteq N$  is called a *coalition*. Derived propositional connectives are defined as usual. We write  $\bar{C}$  for  $N \setminus C$  and sometimes abuse notation and write a singleton coalition  $\{i\}$  as  $i$ . The language can be interpreted over *concurrent game structures* (CGSs) [3]. A CGS over  $N$  and  $\Theta$  is a tuple  $M = \langle S, V, Act, d, \delta \rangle$  where

- $S$  is a set of *states*;
- $V$  is a *valuation function*, assigning a set  $V(s) \subseteq \Theta$  to each state  $s \in S$ ;
- $Act$  is a set of *actions*;
- For each  $i \in N$  and  $s \in S$ ,  $d_i(s) \subseteq Act$  is a non-empty set of actions available to agent  $i$  in  $s$ .  $D(s) = d_1(s) \times \dots \times d_g(s)$  is the set of *full joint actions* in  $s$ . When  $C \subseteq N$ ,  $D_C(s) = \times_{i \in C} d_i(s)$  is the set of *C-actions* in  $s$ . If  $\mathbf{a} \in D_C(s)$ ,  $a_i$  ( $i \in C$ ) denotes the action taken from  $d_i(s)$ . When  $C_1 \cap C_2 = \emptyset$  and  $\mathbf{a} \in D_{C_1}(s)$

and  $\mathbf{b} \in D_{C_2}(s)$ , then  $(\mathbf{a}, \mathbf{b})$  denotes the  $C_1 \cup C_2$  action  $\mathbf{c}$  with  $c_i = a_i$  for  $i \in C_1$  and  $c_i = b_i$  for  $i \in C_2$ .

- $\delta$  is a *transition function*, mapping each state  $s \in S$  and full joint action  $\mathbf{a} \in D(s)$  to a state  $\delta(s, \mathbf{a}) \in S$ .

Let  $\mathcal{M}_{cgs}(N, \Theta, Act)$  be the class of CGSSs over  $N$  and  $\Theta$  having  $Act$  as the set of actions.

A CGS can be seen as a state-transition system where the edges are labelled with full joint actions, but also, alternatively, as an assignment of a *strategic game form* to each state. A strategic game form is a tuple  $G = \langle N, \{\Sigma_i : i \in N\}, S, o \rangle$  where for each  $i \in N$ ,  $\Sigma_i$  is a set of *strategies* for  $i$ , and  $o : \times_{j \in N} \Sigma_j \rightarrow S$  is an *outcome function*. For brevity we will in the following refer to a strategic game form as a *game form* or sometimes (abusing the terminology slightly) simply a *game*. When  $o$  is injective, we say that the game form  $G$  is injective. We write  $\Sigma_C$  for  $\times_{i \in C} \Sigma_i$ ; the set of *strategy profiles* for coalition  $C$ . When  $\sigma_C \in \Sigma_C$ , we use  $(\sigma_C)_i$  to denote the strategy for agent  $i$ . When  $C_1 \cap C_2 = \emptyset$  and  $\sigma_{C_1} \in \Sigma_{C_1}$  and  $\sigma_{C_2} \in \Sigma_{C_2}$ , then  $(\sigma_{C_1}, \sigma_{C_2})$  denotes the strategy profile  $\sigma$  for group  $C_1 \cup C_2$  such that  $(\sigma)_i = (\sigma_{C_1})_i$  for  $i \in C_1$  and  $(\sigma)_i = (\sigma_{C_2})_i$  for  $i \in C_2$ . A CGS can be seen as an assignment of a game  $G(s) = \langle N, \{\Sigma_i^s : i \in N\}, S, o^s \rangle$  to each state  $s \in S$ , where  $\Sigma_i^s = d_i(s)$  and  $o^s(\mathbf{a}) = \delta(s, \mathbf{a})$ .

Intuitively, the CL expression  $[C]\phi$  means that the coalition  $C$  is *effective* for formula  $\phi$ , i.e., that they can ensure that  $\phi$  holds in the next state no matter what the other agents do. Formally, a formula  $\phi$  is interpreted in a state  $s$  of CGS  $M$  as follows:

$$\begin{aligned} M, s &\models p \Leftrightarrow p \in V(s) \\ M, s &\models \neg\phi \Leftrightarrow M, s \not\models \phi \\ M, s &\models (\phi_1 \wedge \phi_2) \Leftrightarrow (M, s \models \phi_1 \text{ and } M, s \models \phi_2) \\ M, s &\models [C]\psi \Leftrightarrow \exists \mathbf{a}_C \in D_C(s) \forall \mathbf{a}_{\bar{C}} \in D_{\bar{C}}(s), M, \delta(s, (\mathbf{a}_C, \mathbf{a}_{\bar{C}})) \models \psi \end{aligned}$$

As effectivity is the only property of games relevant for the interpretation of coalition logic, we can in fact abstract away all other aspects of game structures. An *effectivity function* [24] over  $N$  and a set of states  $S$  is a function  $E$  that maps any coalition  $C \subseteq N$  to a set of sets of states  $E(C) \subseteq 2^S$ . Given a strategic game form  $G$ , the ( $\alpha$ -)effectivity function  $E_G$  of  $G$  is defined as follows:

$$X \in E_G(C) \text{ iff } \exists \sigma_C \in \Sigma_C \forall \sigma_{\bar{C}} \in \Sigma_{\bar{C}}, o(\sigma_C, \sigma_{\bar{C}}) \in X.$$

Which effectivity functions are the effectivity functions of strategic game forms? In [24] it is claimed that an effectivity function  $E$  is the  $\alpha$ -effectivity function of a strategic game form iff  $E$  is *playable*:

1.  $(X \in E(C) \text{ and } X \subseteq Y \text{ and } Y \subseteq S) \Rightarrow Y \in E(C)$  (*outcome monotonicity*);
2.  $S \setminus X \notin E(\emptyset) \Rightarrow X \in E(N)$  (*N-maximality*);
3.  $\emptyset \notin E(C)$  (*Liveness*);
4.  $S \in E(C)$  (*Safety*);
5.  $(C \cap D = \emptyset \text{ and } X \in E(C) \text{ and } Y \in E(D)) \Rightarrow X \cap Y \in E(C \cup D)$  (*superadditivity*).

However, it has recently been showed [14] that this claim is in fact not correct: there are playable effectivity functions over infinite sets, which are not  $\alpha$ -effectivity functions of any strategic game forms. In [14] the result is also corrected: an effectivity function  $E$  is said to be *truly playable* iff it is playable and  $E(\emptyset)$  has a *complete nonmonotonic core*. The *nonmonotonic core*  $E^{nc}(C)$  of  $E(C)$ , for  $C \subseteq N$ , is defined as follows:  $E^{nc}(C) = \{X \in E(C) : \neg \exists Y (Y \in E(C) \text{ and } Y \subset X)\}$ .  $E^{nc}(C)$  is *complete* if for every  $X \in E(C)$  there exists  $Y \in E^{nc}(C)$  such that  $Y \subseteq X$ . The corrected representation theorem [14] shows that  $E$  is the  $\alpha$ -effectivity function of a strategic game form iff  $E$  is truly playable. A *coalition model* is a tuple  $\mathcal{M} = \langle S, E, V \rangle$  where  $E$  gives a truly playable effectivity function  $E(s)$  for each state  $s \in S$ , and  $V$  is a valuation function. The coalition logic language can alternatively and equivalently be interpreted in a coalition model, as follows:

$$\mathcal{M}, s \models [C]\phi \text{ iff } \phi^{\mathcal{M}} \in E(s)(C)$$

where  $\phi^{\mathcal{M}} = \{t \in S : \mathcal{M}, t \models \phi\}$ . It is easy to see that the two semantics coincide:  $M, s \models \phi$  iff  $\mathcal{M}_{\alpha}, s \models \phi$  for all  $\phi$ , where  $M = (S, V, Act, d, \delta)$  and  $\mathcal{M}_{\alpha} = (S, E_{\alpha}, V)$  and  $E_{\alpha}(s) = E_{G(s)}$ .

## 2.2 Multi-modal $K$ with Intersection of Modalities

Given a finite set of atomic modalities  $\Pi^0$  of cardinality  $n$  and a countably infinite set of propositional variables  $\Theta$ , the formulae  $\phi \in \mathcal{L}_K^{\cap}(\Pi^0, \Theta)$  and modalities  $\pi \in \Pi$  of multi-modal  $K$  with intersection of modalities ( $K_n^{\cap}$ ) are defined as follows:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid [\pi]\phi \quad \pi ::= a \mid \pi \cap \pi$$

where  $p \in \Theta$  and  $a \in \Pi^0$ . As usual,  $\langle \pi \rangle \phi$  is defined as  $\neg[\pi]\neg\phi$ , and derived propositional connectives are defined as usual.

A (Kripke) model for the language  $\mathcal{L}_K^{\cap}(\Pi^0, \Theta)$  is a tuple  $M = \langle S, V, \{R_{\pi} : \pi \in \Pi\} \rangle$  where

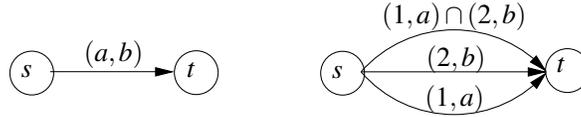
- $S$  is a set of *states*;
- $V : \Theta \rightarrow 2^S$  is a *valuation function*;
- For each  $\pi \in \Pi$ ,  $R_{\pi} \subseteq S \times S$
- $R_{\pi_1 \cap \pi_2} = R_{\pi_1} \cap R_{\pi_2}$  (INT)

The interpretation of a formula in a state of a model is defined as follows (other clauses as usual):

$$M, s \models [\pi]\phi \text{ iff } \forall s' \in S \text{ such that } (s, s') \in R_{\pi}, M, s' \models \phi$$

### 3 Injective Games

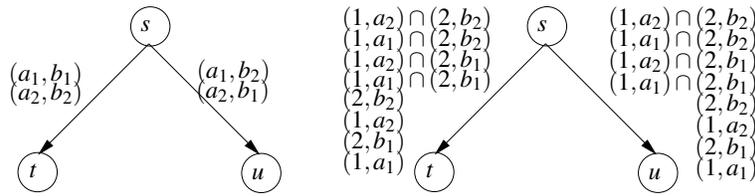
The idea of interpreting the  $K_n^\cap$  language in game structures is very simple: view an action by a single agent as an atomic non-deterministic PDL action, where the non-determinism is due to the actions chosen by the other agents. More formally: interpret a full joint action  $\langle a_1, \dots, a_g \rangle$  in a CGS as a set of  $g$  different (“atomic”) transitions, one for each agent-action combination. This gives us a  $K_n^\cap$  model, where the atomic modalities are agent-action pairs. Full joint actions can be recovered by taking the intersection between the relations for two or more atomic modalities for different agents.



**Fig. 1** CGS (left) and  $K_n^\cap$  model (right).

For example, consider the CGS on the left in Figure 1. The corresponding  $K_n^\cap$  model is shown to the right. Coalition operators can now be captured approximately as follows. If we for example want to say that there exists a joint action by agents 1 and 2, all executions of which result in the outcome  $p$ , in such a  $K_n^\cap$  model, we can say something like  $\bigvee_{a,b \in Act} [(1, a) \cap (2, b)]p$  (in addition we must check that the actions  $a$  and  $b$  are actually available in the current state, but that is straightforward).

However, there is a problem with this approach. Consider the CGS to the left in Figure 2. The approach above gives us the  $K_n^\cap$  model to the right in the figure. This model has four atomic transitions from  $s$  to  $t$ : two labelled  $(1, a_1)$  and  $(2, b_1)$  which correspond to the full joint action  $\langle a_1, b_1 \rangle$ , and two labelled  $(1, a_2)$  and  $(2, b_2)$  which correspond to  $\langle a_2, b_2 \rangle$ . The full joint actions can be recovered by intersection of the atomic transitions, but the problem is that too much is “recovered” in this way: we get the spurious transitions  $\langle a_1, b_2 \rangle$  and  $\langle a_2, b_1 \rangle$  which are not present between these states in the original model.



**Fig. 2** CGS (left) and  $K_n^\cap$  model (right). An arrow with more than one label represents a transition for each label.

The problem is that by decomposing a full joint action into individual actions, we lose information about which *combinations* of actions relate the two states. That information is crucial, e.g., for the interpretation of coalition operators. We call a CGS without two or more different full joint actions between the same two states, i.e., with an injective  $\delta$ , *injective* (following [12]). Injective CGSS do not suffer this problem.

It is relatively straightforward to see that any CGS is equivalent, in the sense of satisfying the same coalition logic formulae, to an injective CGS: take the *tree-unfolding* of the model (see [2] for relevant definitions of tree-unfoldings, bisimulations, and invariance under bisimulation, for the ATL language which contains the CL language). The tree-unfolding, however, is a model with infinitely many states, which may be a problem, e.g., if we want to do model checking. Fortunately it turns out that every finite CGS (finite state space) is CL-equivalent to a finite, and even “small”, injective CGS. The following theorem follows immediately from a result by Goranko [12, Proposition 12] (with some minor changes and amendments).

**Theorem 1.** *For every CGS  $M = \langle S, V, Act, d, \delta \rangle$  there is an injective CGS  $M'$  with states  $S'$  such that  $S \subseteq S'$  and for all CL formulae  $\phi$  and states  $s \in S$ ,  $M, s \models \phi$  iff  $M', s \models \phi$ . Moreover, if  $M$  is finite, then  $|S'| \leq |S| + |\delta|$ .*

This makes it possible to translate a CGS into a  $K_n^\cap$  model, such that we can recover a CGS that is CL-equivalent to the former from the latter. Before formally defining the translation in Section 4 we take a closer look at injective games; the reader mainly interested in the translation can skip directly to that section.

### 3.1 Effectivity Functions and Representation

Although coalition logic cannot discern between injective games and non-injective games, there is still another pertinent question if we want to restrict our attention to injective games: the question of representation using effectivity functions. Which truly playable effectivity functions correspond to injective games? The answer is not necessarily “all”: this is similar to the relationship between playable and truly playable effectivity functions [14]; the latter is a proper subset of the former while coalition logic still cannot discern between the two. Indeed, not all truly playable effectivity functions are the  $\alpha$ -effectivity functions of injective games:

*Example 1.* Let  $N = \{1, 2\}$  and  $E$  be defined as follows:

$$E(\emptyset) = E(1) = E(2) = \{\{s, t\}\} \quad E(\{1, 2\}) = \{\{s\}, \{t\}, \{s, t\}\}$$

(where  $s \neq t$ ). The reader can check that  $E$  is truly playable. However, it is not the  $\alpha$ -effectivity function of an injective game. For assume it is, that  $E = E_G$  for some  $G$ . Because of Safety, the game has exactly two states  $s$  and  $t$ . Together with the fact that  $\{s\}, \{t\} \in E(\{1, 2\})$ , that means that one of the agents must have exactly one strategy, and the other exactly two: all other combinations violate injectivity

of a two-state game. Without loss of generality assume that  $\Sigma_1 = \{\sigma_1\}$  and  $\Sigma_2 = \{\sigma'_1, \sigma'_2\}$ .  $\{s\} \in E_G(\{1, 2\})$  implies that  $o(\sigma_1, \sigma'_1) = s$  or  $o(\sigma_1, \sigma'_2) = s$ ; without loss of generality assume the former. Then  $\{t\} \in E_G(\{1, 2\})$  implies that  $o(\sigma_1, \sigma'_2) = t$ . But that means that  $\{s\}, \{t\} \in E_G(2)$ , which is not the case.

We now state and prove a representation theorem (Theorem 2) for injective games. An effectivity function is *injectively playable* iff it is playable and (for all  $C, i, j, X, Y$ ):

$$E(C) \text{ has a complete nonmonotonic core} \quad (1)$$

$$E^{nc}(C) = \left\{ \bigcap_{i \in C} X_i : X_i \in E^{nc}(i) \right\} \quad C \neq \emptyset \quad (2)$$

$$X, Y \in E^{nc}(i) \text{ and } X \neq Y \Rightarrow X \cap Y = \emptyset \quad (3)$$

$$X \in E^{nc}(j) \text{ and } x \in X \Rightarrow \exists Y \in E^{nc}(i), x \in Y \quad (4)$$

Injective playability extends the true playability requirement of a complete nonmonotonic core [14] from the empty coalition to *all* coalitions. As a result,  $E(C)$  is completely determined by its nonmonotonic core (stated formally in the following Lemma). In addition, there are some restrictions on the structure of the core. None of the additional properties of injective playability, (1)–(4), hold in general for truly playable effectivity functions (in particular, true playability does not imply complete nonmonotonic core for non-empty coalitions).

**Lemma 1.** *Let  $E$  be an outcome monotonic effectivity function.  $E(C)$  has a complete nonmonotonic core iff  $E(C) = \{X : \exists Y \subseteq X, Y \in E^{nc}(C)\}$ .*

*Proof.* For the implication to the right, assume that  $E(C)$  has a complete nonmonotonic core.  $E(C) \subseteq \{X : Y \subseteq X, Y \in E^{nc}(C)\}$  is immediate. If  $Y \subseteq X$  and  $Y \in E^{nc}(C)$ , then  $X \in E(C)$  by outcome monotonicity. The implication to the left is immediate.  $\square$

Two derived properties are the following.

**Lemma 2.** *If  $E$  is injectively playable, then:*

$$(\forall_{i \in N} X_i \in E^{nc}(i)) \Rightarrow \left| \bigcap_{i \in N} X_i \right| = 1 \quad (5)$$

$$E^{nc}(\emptyset) = \{Z\} \text{ where } Z = \bigcup E^{nc}(N) \quad (6)$$

*Proof.* (5) Let  $X_i \in E^{nc}(i)$  for each  $i \in N$ . By (2),  $\bigcap_{i \in N} X_i \in E^{nc}(N)$ , and by true playability and [14, Proposition 5] there is an  $x \in \bigcap_{i \in N} X_i$  such that  $\{x\} \in E(N)$ . That means that  $\{x\} = \bigcap_{i \in N} X_i$ ; because  $\bigcap_{i \in N} X_i \neq \{x\}$  contradicts the facts that  $\bigcap_{i \in N} X_i \in E^{nc}(N)$ ,  $\{x\} \in E(N)$  and  $x \in \bigcap_{i \in N} X_i$ .

(6) Let  $Z = \bigcup E^{nc}(N)$ . Since injective playability implies true playability, we know that  $E^{nc}(\emptyset) = \{Z'\}$  for some  $Z'$  [14, Proposition 5]. We show that  $Z' = Z$ . We have that  $S \setminus Z \notin E(N)$ ; otherwise there would be a  $X \subseteq S \setminus Z$  such that  $X \in E^{nc}(N)$  (by (1)) and thus  $X \subseteq Z$  by definition of  $Z$ , which together with the fact that  $X \neq \emptyset$

(Liveness) is a contradiction. By  $N$ -maximality,  $S \setminus (S \setminus Z) = Z \in E(\emptyset)$ . Thus,  $Z' \subseteq Z$  by (1). Assume, towards a contradiction, that  $Z \not\subseteq Z'$ , i.e., that there is an  $x \in Z$  such that  $x \notin Z'$ . That  $x \in Z$  means that there is an  $X \in E^{nc}(N)$  with  $x \in X$ . Let  $X' = X \setminus \{x\}$ . That  $Z' \in E(\emptyset)$  and  $X \in E(N)$  implies by superadditivity that  $Z' \cap X \in E(N)$ , and by the fact that  $x \notin Z'$  we have that  $Z' \cap X \subseteq X'$ . By outcome monotonicity,  $X' \in E(N)$ . But that contradicts the fact that  $X \in E^{nc}(N)$ . Thus,  $Z = Z'$ .  $\square$

Before proving the main result (Theorem 2) we need the following lemma.

**Lemma 3.** *If  $E_G$  is the  $\alpha$ -effectivity function of an injective game  $G = (N, \{\Sigma_i : i \in N\}, o, S)$ , then for all  $C \subseteq N$ :*

$$E_G^{nc}(C) = \{ \{o(\sigma_C, \sigma_{\bar{C}}) : \sigma_{\bar{C}} \in \Sigma_{\bar{C}}\} : \sigma_C \in \Sigma_C \}$$

*Proof.*  $X \in E_G^{nc}(C)$  iff  $\exists \sigma_C \forall \sigma_{\bar{C}} o(\sigma_C, \sigma_{\bar{C}}) \in X$  and there is no  $Y \in E_G(C)$  such that  $Y \subset X$ . Let  $P = \{ \{o(\sigma_C, \sigma_{\bar{C}}) : \sigma_{\bar{C}} \in \Sigma_{\bar{C}}\} : \sigma_C \in \Sigma_C \}$ .

First, let  $X \in E_G^{nc}(C)$  and let  $\sigma_C$  be as above (a witness for  $X$ ). Let  $Y = \{o(\sigma_C, \sigma_{\bar{C}}) : \sigma_{\bar{C}} \in \Sigma_{\bar{C}}\}$ .  $Y \in P$ .  $Y \subseteq X$ . We have that  $Y \in E_G(C)$  (by definition of  $\alpha$ -effectivity), so by the fact that  $X \in E_G^{nc}(C)$  it follows that  $Y \not\subseteq X$  and thus that  $Y = X$ .

Second, let  $X = \{o(\sigma_C, \sigma_{\bar{C}}) : \sigma_{\bar{C}} \in \Sigma_{\bar{C}}\} \in P$  for some  $\sigma_C$ .  $X \in E_G(C)$ . Assume towards a contradiction that there is a  $Y \in E_G(C)$  such that  $Y \subset X$ . Thus there is a  $\sigma'_C \in \Sigma_C$  such that for all  $\sigma'_{\bar{C}} \in \Sigma_{\bar{C}}$ ,  $o(\sigma'_C, \sigma'_{\bar{C}}) \in Y$ .  $\sigma_C \neq \sigma'_C$ ; otherwise  $X \subseteq Y$ , a contradiction.  $Y \neq \emptyset$ , so there is a  $y \in Y \cap X$ . In other words, there are  $\sigma_{\bar{C}}$  and  $\sigma'_{\bar{C}}$  such that  $o(\sigma_C, \sigma_{\bar{C}}) = o(\sigma'_C, \sigma'_{\bar{C}}) = y$ . But this contradicts the fact that  $G$  is injective, since  $\sigma_C \neq \sigma'_C$ . Thus, there is no such  $Y$ , and  $X \in E_G^{nc}(C)$ .  $\square$

**Theorem 2.** *An effectivity function  $E$  is injectively playable iff it is the  $\alpha$ -effectivity function of some injective game  $G$ .*

*Proof.* First, let  $E_G$  be the  $\alpha$ -effectivity function of some injective game  $G$ . We show that  $E_G$  is injectively playable. It is immediate from [24] that  $E_G$  is playable.

In order to show (1), let  $X \in E_G(C)$ , i.e., there is a  $\sigma_C$  such that for all  $\sigma_{\bar{C}}$   $o(\sigma_C, \sigma_{\bar{C}}) \in X$ . Let  $Y = \{o(\sigma_C, \sigma_{\bar{C}}) : \sigma_{\bar{C}} \in \Sigma_{\bar{C}}\}$ .  $Y \subseteq X$ , and  $Y \in E_G^{nc}(C)$  by Lemma 3.

In order to show (2), assume that  $|C| \geq 2$ , since (2) holds trivially for  $|C| = 1$ . For one direction, let  $X \in E_G^{nc}(C)$ . By Lemma 3,  $X = \{o(\sigma_C, \sigma_{\bar{C}}) : \sigma_{\bar{C}} \in \Sigma_{\bar{C}}\}$  for some  $\sigma_C$ . Let, for each  $i \in C$ ,  $\sigma_i = (\sigma_C)_i$  and  $X_i = \{o(\sigma_i, \sigma_i) : \sigma_i \in \Sigma_i\}$ .  $X_i \in E_G^{nc}(i)$  by Lemma 3.  $X \subseteq \bigcap_{i \in C} X_i$ . We must show that  $\bigcap_{i \in C} X_i \subseteq X$ . Let  $x \in \bigcap_{i \in C} X_i$ . For each  $i \in C$ , there exists some  $\sigma_i$  such that  $x = o(\sigma_i, \sigma_i)$ . For any arbitrary  $i, j \in C$ ,  $i \neq j$ , from  $o(\sigma_j, \sigma_j) = x = o(\sigma_i, \sigma_i)$  we get that  $(\sigma_j)_i = \sigma_i$  by injectivity. Thus,  $o(\sigma_C, \sigma_{\bar{C}}) = o(\sigma_j, \sigma_j)$ , for all  $j \in C$  and some  $\sigma_{\bar{C}}$ , and thus  $x \in X$ .

For the other direction of (2), let  $X = \bigcap_{i \in C} X_i$  with  $X_i \in E_G^{nc}(i)$ . Again, for each  $i \in C$ ,  $X_i = \{o(\sigma_i, \sigma_i) : \sigma_i \in \Sigma_i\}$  for some  $\sigma_i$ . Let  $\sigma_C$  be defined by  $(\sigma_C)_i = \sigma_i$ . Let  $Y = \{o(\sigma_C, \sigma_{\bar{C}}) : \sigma_{\bar{C}} \in \Sigma_{\bar{C}}\}$ .  $Y \in E_G^{nc}(C)$  by Lemma 3. We show that  $Y = X$ . First, let  $\sigma_{\bar{C}} \in \Sigma_{\bar{C}}$  be arbitrary. Since  $(\sigma_C)_i = \sigma_i$ ,  $o(\sigma_C, \sigma_{\bar{C}}) \in X_i$  for all  $i \in C$ , and  $o(\sigma_C, \sigma_{\bar{C}}) \in$

$X$ . Thus,  $Y \subseteq X$ . Let  $x \in X$ . For each  $i \in C$ , there is some  $\sigma_{\bar{i}}$  such that  $o(\sigma_i, \sigma_{\bar{i}}) = x$ . We can now reason as above. Let  $i, j \in C$ ,  $i \neq j$ . From  $o(\sigma_j, \sigma_{\bar{j}}) = x = o(\sigma_i, \sigma_{\bar{i}})$ , we get that  $(\sigma_{\bar{j}})_i = \sigma_i$  by injectivity. Thus,  $o(\sigma_j, \sigma_{\bar{j}}) = o(\sigma_C, \sigma_{\bar{C}})$ , for some arbitrary  $j \in C$  and some  $\sigma_{\bar{C}}$ , and thus  $x \in Y$ . Thus,  $X \subseteq Y$ .

In order to show that (3) holds, let  $X \neq Y \in E_G^{nc}(i)$ . By Lemma 3, there are  $\sigma_i, \sigma'_i$  such that  $X = \{o(\sigma_i, \sigma_{\bar{i}}) : \sigma_{\bar{i}} \in \Sigma_{\bar{i}}\}$  and  $Y = \{o(\sigma'_i, \sigma_{\bar{i}}) : \sigma_{\bar{i}} \in \Sigma_{\bar{i}}\}$ . Assume that  $x \in X \cap Y$ , i.e., that  $o(\sigma_i, \sigma_{\bar{i}}) = o(\sigma'_i, \sigma_{\bar{i}})$  for some  $\sigma_{\bar{i}}$  and  $\sigma'_i$ . Since the game is injective that means that  $\sigma_i = \sigma'_i$ , but that contradicts the fact that  $X \neq Y$ . Thus,  $X \cap Y = \emptyset$ .

In order to show that (4) holds, let  $X \in E_G^{nc}(j)$  and  $x \in X$ . By Lemma 3, there is a  $\sigma_j$  such that  $X = \{o(\sigma_j, \sigma_{\bar{j}}) : \sigma_{\bar{j}} \in \Sigma_{\bar{j}}\}$ . In particular,  $x = o(\sigma_j, \sigma_{\bar{j}})$  for some  $\sigma_{\bar{j}}$ . Let  $\sigma_i = (\sigma_{\bar{j}})_i$ , and let  $Y = \{o(\sigma_i, \sigma_{\bar{i}}) : \sigma_{\bar{i}} \in \Sigma_{\bar{i}}\}$ .  $x \in Y$ , and  $Y \in E_G^{nc}(i)$  by Lemma 3.

Second, let  $E$  be an injectively playable effectivity function over  $N$  and  $S$ . We construct a game  $G = (N, \{\Sigma_i : i \in N\}, o, S)$  as follows:

$$\Sigma_i = E^{nc}(i) \quad o(X_1, \dots, X_g) = x \quad \text{where} \quad \{x\} = \bigcap_{i \in N} X_i$$

The property (5) (Lemma 2) ensures that the game is well defined. To see that  $G$  is injective, assume that  $o(X_1, \dots, X_g) = o(X'_1, \dots, X'_g) = x$ . That means that, for each  $i$ ,  $x \in X_i \cap X'_i$ , and by (3) it follows that  $X_i = X'_i$ . Thus,  $G$  is injective.

We must show that  $E_G = E$ . By (1), outcome monotonicity and Lemma 1 it suffices to show that  $E_G^{nc}(C) = E^{nc}(C)$  for all  $C \subseteq N$ .

First assume that  $C \neq \emptyset$ . For any  $X$ ,  $X \in E_G^{nc}(C)$  iff (by Lemma 3)  $\exists \sigma_C$  such that  $X = \{o(\sigma_C, \sigma_{\bar{C}}) : \sigma_{\bar{C}} \in \Sigma_{\bar{C}}\}$  iff  $\exists \{X_i \in E^{nc}(i) : i \in C\}$  such that  $X = \{x : \{x\} = \bigcap_{i \in N} X_i, X_j \in E^{nc}(j), j \in N \setminus C\}$ . On the other hand,  $X \in E^{nc}(C)$  iff  $\exists \{X_i \in E^{nc}(i) : i \in C\}$  such that  $X = \bigcap_{i \in C} X_i$ , by (2). Thus, let, for each  $i \in C$ ,  $X_i \in E^{nc}(i)$ . It suffices to show that  $\{x : \{x\} = \bigcap_{i \in N} X_i, X_j \in E^{nc}(j), j \in N \setminus C\} = \bigcap_{i \in C} X_i$ . For inclusion towards the left, assume that  $x \in \bigcap_{i \in C} X_i$ . If  $C = N$  we are done. Otherwise, from (4) it follows that there is a  $X_j \in E^{nc}(j)$  such that  $x \in X_j$ , for every  $j \in N \setminus C$ . Thus,  $x \in \bigcap_{i \in N} X_i$ . For inclusion towards the right, let  $\{x\} = \bigcap_{i \in N} X_i$  for some  $\{X_i \in E^{nc}(i) : i \in N \setminus C\}$ . It immediately follows that  $x \in \bigcap_{i \in C} X_i$ .

Second, consider the case that  $C = \emptyset$ .  $X \in E_G^{nc}(\emptyset)$  iff (by Lemma 3)  $X = \{o(\sigma_N) : \sigma_N \in \Sigma_N\}$  iff  $X = \{x : \{x\} = \bigcap_{i \in N} X_i, X_i \in E^{nc}(i)\}$  iff (by (2))  $X = \bigcup E^{nc}(N)$  iff (by Lemma 2)  $X \in E^{nc}(\emptyset)$ .  $\square$

## 4 Multi-modal $K$ with Intersection for Games

We now show how  $K_n^\cap$  formulae can be interpreted in game structures, by identifying a class of  $K_n^\cap$  models corresponding to (injective) game structures.

### 4.1 Joint Action Models

Let  $Act$  be a finite set of actions and  $N$  a set of  $g$  agents. Define a set of atomic modalities as follows:

$$\Pi_{ActN}^0 = N \times Act$$

Intuitively, an atomic modality is a pair  $(i, a)$  intended to represent an action and the agent that executes that action. We will call an atomic modality in  $\Pi_{ActN}^0$  an *individual action*, and a composite modality  $\pi = \pi_1 \cap \pi_2$  a *joint action*. Since the intersection operation is associative, we can write any joint action  $\pi$  as an intersection of a set of individual actions:  $\pi = (i_1, a_1) \cap \dots \cap (i_k, a_k)$ . Joint actions of the form  $(1, a_1) \cap \dots \cap (g, a_g)$  with one individual action for *every* agent in  $N$  will be called *complete (joint) actions*.

The following are some properties of  $K_n^\cap$  models over  $\Pi_{ActN}^0$  that will be of particular interest. We say that an action  $a \in Act$  is *enabled* for agent  $i$  in a state  $s$  iff there is a state  $s'$  such that  $(s, s') \in R_{(i,a)}$ .

**Seriality (SER)** For any state  $s$  and agent  $i$ , at least one action is enabled in  $s$  for  $i$ .

**Independent Choice (IC)** For any state  $s$ , agents  $C = \{i_1, \dots, i_k\}$  and actions  $a_1, \dots, a_k \in Act$ , if for every  $j$   $a_j$  is enabled for  $i_j$  in  $s$ , then there is a state  $s'$  such that  $(s, s') \in R_{(i_1, a_1) \cap \dots \cap (i_k, a_k)}$ .

**Deterministic Joint Actions (DJA)** For any complete joint action  $\alpha$  and states  $s, s_1, s_2$ ,  $(s, s_1), (s, s_2) \in R_\alpha$  implies that  $s_1 = s_2$ .

**Unique Joint Actions (UJA)** For any complete joint actions  $\alpha$  and  $\beta$  and states  $s, t$ , if  $(s, t) \in R_\alpha \cap R_\beta$  then  $\alpha = \beta$ .

A  $K_n^\cap$  model over  $\Pi_{ActN}^0$  (where  $Act$  is finite) is a *joint action model* if it satisfies the properties SER, IC, DJA, and UJA.

Given (finite)  $Act$  and  $N$ , we now translate any CGS over  $Act$  and  $N$  into a joint action model. We start with injective CGSS.

**Definition 1.** Given an injective CGS  $M = \langle S, V, Act, d, \delta \rangle \in \mathcal{M}_{cgs}(N, \Theta, Act)$  where  $Act$  is finite, the corresponding joint action model  $\hat{M} = \langle S, V, \{R_\pi : \pi \in \Pi\} \rangle$  over  $\Theta$  and  $\Pi_{ActN}^0$  is defined as follows:

- $R_{(i,a)} = \{(s, s') : \exists \mathbf{a} \in D(s) \text{ such that } a_i = a \text{ and } s' = \delta(s, \mathbf{a})\}$ , when  $(i, a) \in \Pi_{ActN}^0$
- $R_{\pi_1 \cap \pi_2} = R_{\pi_1} \cap R_{\pi_2}$

As discussed in Section 3, this translation views an action by a single agent as a non-deterministic action where the non-determinism is due to the possible choices by the other agents (the existential quantification over full joint actions in the definition above). Full joint actions are recovered by using intersection.

We use the following property to show that  $\hat{M}$  is indeed a joint action model.

**Lemma 4.** Let  $M = \langle S, V, Act, d, \delta \rangle$  be injective and  $\hat{M} = \langle S, V, \{R_\pi : \pi \in \Pi\} \rangle$  be the corresponding joint action model, and let  $\pi = (i_1, a_1) \cap \dots \cap (i_k, a_k)$ . Then  $(s, t) \in R_\pi$  iff there is an  $\mathbf{a}' \in D(s)$  such that  $a'_j = a_j$  for all  $1 \leq j \leq k$  and  $\delta(s, \mathbf{a}') = t$ .

*Proof.*  $(s, t) \in R_\pi$  iff there are  $\mathbf{a}^1, \dots, \mathbf{a}^k \in D(s)$  such that for all  $1 \leq j \leq k$ :  $a_{i_j}^j = a_j$  and  $t = \delta(s, \mathbf{a}^j)$ . Since  $M$  is injective, it must be the case that  $\mathbf{a}^1 = \dots = \mathbf{a}^k$ .  $\square$

**Lemma 5.**  $\hat{M}$  is a joint action model.

*Proof.*  $\hat{M}$  is a proper  $\mathcal{L}_K^\cap(\Pi_{ActN}^0, \Theta)$  model by definition. SER holds because  $d_i(s)$  is always non-empty. IC holds because  $D(s)$  is defined as the cross product of  $d_i(s)$  for all  $i$ . For DJA,  $(s, s_1), (s, s_2) \in R_\alpha$  implies that  $\delta(s, \mathbf{a}) = s_1 = s_2$  by Lemma 4. For UJA,  $(s, t) \in R_\alpha \cap R_\beta$  implies that  $\delta(s, \mathbf{a}) = t$  and  $\delta(s, \mathbf{b}) = t$  by Lemma 4, which by injectivity implies that  $\mathbf{a} = \mathbf{b}$  which again implies that  $\alpha = \beta$ .  $\square$

Let us call a formula  $\phi \in \mathcal{L}_K^\cap(\Pi_{ActN}^0, \Theta)$  a *complete action formula* if the modal operators in  $\phi$  only contain complete actions. The following lemma shows that as far as complete action formulae are concerned, all joint action models correspond to CGSS, which is going to be useful in showing the embedding.

**Lemma 6.** Given a finite set of actions  $Act$ , for every joint action model  $M'$  over  $\Pi_{ActN}^0$  with states  $S$ , there is an injective CGS  $M \in \mathcal{M}_{cgs}(N, \Theta, Act)$  with corresponding joint action model  $\hat{M}$  such that for all complete action formulas  $\phi \in \mathcal{L}_K^\cap(\Pi_{ActN}^0, \Theta)$  and  $s \in S$ ,  $M', s \models \phi$  iff  $\hat{M}, s \models \phi$ .

*Proof.* Given a joint action model  $M' = \langle S, V, \{R_\pi : \pi \in \Pi\} \rangle$  over  $\Theta$  and  $\Pi_{ActN}^0$ , we construct  $M = \langle S, V, Act, d, \delta \rangle$  as follows:

- $Act = \{(i, a) : (i, a) \in \Pi_{ActN}^0\}$
- $d_i(s) = \{(i, a) : \exists s' R_{(i,a)}(s, s')\}$
- $\delta(s, ((1, a_1), \dots, (g, a_g))) = s'$  iff  $R_{(1,a_1) \cap \dots \cap (g,a_g)}(s, s')$ .

Note that  $M$  is a well defined CGS, in particular,  $d_i(s) \neq \emptyset$  and  $\delta$  is a function because of SER, IC and DJA.  $M$  is injective because of UJA.

Now consider  $\hat{M}$  which is the joint action model corresponding to  $M$ .  $\hat{M}$  may be different from  $M'$  in that  $R_{(i,a)}(s, s')$  holds in  $\hat{M}$  only if  $s$  and  $s'$  are connected by a complete joint action. However, it is easy to show that for every complete joint action  $\pi$ ,  $R_\pi(s, s')$  holds in  $M'$  iff  $R_\pi(s, s')$  holds in  $\hat{M}$ . Given this fact, it is easy to show by induction that for every complete action formula  $\phi$ , and every state  $s \in S$ ,  $M', s \models \phi$  iff  $\hat{M}, s \models \phi$ .  $\square$

## 4.2 Embedding of CL

Given a coalition logic formula  $\phi \in \mathcal{L}_{CL}(N, \Theta)$  and a finite set of actions  $Act$ , we define the translation  $T^{Act}(\phi) \in \mathcal{L}_K^\cap(\Pi_{ActN}^0, \Theta)$  as follows:

$$\begin{aligned}
T^{Act}(p) &\equiv p \\
T^{Act}(\neg\psi) &\equiv \neg T^{Act}(\psi) \\
T^{Act}(\psi_1 \wedge \psi_2) &\equiv T^{Act}(\psi_1) \wedge T^{Act}(\psi_2) \\
T^{Act}([\{i_1, \dots, i_k\}]\psi) &\equiv \bigvee_{a_1, \dots, a_k \in Act} ( \\
&\bigwedge_{a_{k+1}, \dots, a_g \in Act} \langle (i_1, a_1) \cap \dots \cap (i_k, a_k) \cap (i_{k+1}, a_{k+1}) \cap \dots \cap (i_g, a_g) \rangle \top \wedge \\
&\bigwedge_{a_{k+1}, \dots, a_g \in Act} [(i_1, a_1) \cap \dots \cap (i_k, a_k) \cap (i_{k+1}, a_{k+1}) \cap \dots \cap (i_g, a_g)] T^{Act}(\psi)
\end{aligned}$$

The translation of the CL formula  $[C]\psi$  says that there is an action for each agent in  $C$ , such that (i) the actions are enabled and (ii) for all possible states resulting from executing the actions at the same time together with any other actions by  $N \setminus C$ , the translation of  $\psi$  holds. The translation assumes that the set of possible actions  $Act$  is given and that it is finite.

The following theorem shows that the model and formula translations preserve satisfaction.

**Theorem 3.** *Let  $\Theta$ ,  $N$  and  $Act$  (finite) be given. For any  $\phi \in \mathcal{L}_{CL}(N, \Theta)$  and any injective CGS  $M \in \mathcal{M}_{CGS}(N, \Theta, Act)$  and any state  $s$  in  $M$ :*

$$M, s \models \phi \text{ iff } \hat{M}, s \models T^{Act}(\phi)$$

*Proof.* Let  $\Theta$ ,  $N$  and  $Act$  (finite) be given,  $\phi \in \mathcal{L}_{CL}(N, \Theta)$  and  $M \in \mathcal{M}_{CGS}(N, \Theta, Act)$  be injective and  $s$  in  $M$ . The proof is by induction on the structure of  $\phi$ . The only interesting case is when  $\phi = [C]\psi$  where  $C = \{1, \dots, k\}$ .  $M, s \models [C]\psi$  iff there is a  $\langle a_1, \dots, a_k \rangle \in D_C(s)$  such that for all  $\mathbf{a} \in D(s)$  with  $\mathbf{a} = (a_1, \dots, a_k, a_{k+1}, \dots, a_g)$ ,  $M, \delta(s, \mathbf{a}) \models \psi$  iff by the induction hypothesis and the construction of  $\hat{M}$ , there exists at least one  $t$  such that  $R_{(1, a_1) \cap \dots \cap (g, a_g)}(s, t)$  for some  $a_{k+1}, \dots, a_g$ , and for all  $t$  with  $R_{(1, a_1) \cap \dots \cap (g, a_g)}(s, t)$ ,  $\hat{M}, t \models T^{Act}(\psi)$  iff  $\hat{M}, s \models T^{Act}([C]\psi)$ .  $\square$

The following follows immediately from Theorems 3 and 1.

**Corollary 1.** *Let  $\Theta$ ,  $N$  and  $Act$  (finite) be fixed. For any  $\phi \in \mathcal{L}_{CL}(N, \Theta)$  and any CGS  $M \in \mathcal{M}_{CGS}(N, \Theta, Act)$  there is a joint action model  $\bar{M}$  over  $\Pi_{ActN}^0$  such that for any state  $s$  in  $\bar{M}$ :*

$$M, s \models \phi \text{ iff } \bar{M}, s \models T^{Act}(\phi)$$

The translation assumes that the (finite) set of actions  $Act$  is given. In model checking this can be obtained directly from the model. Thus, the model checking problem for CL can be reduced to the model checking problem for  $K_n^\cap$ .

But what about the satisfiability problem, can that be translated? Here we are only given a CL formula, which does not mention actions explicitly. We can here make use of a result from [13]: that every ATL-consistent formula  $\phi$  has a model where the number of actions of each agent equals  $1 + |\{\theta \in ecl(\phi) : \theta = [C]\psi \text{ or } \theta = [C]\psi\}|^1$ , where  $ecl(\phi)$  is the set of subformulas of  $\phi$  closed under single negations and the condition that if  $[C]\psi \in ecl(\phi)$ , then  $[C']\psi \in ecl(\phi)$  for all  $C' \subseteq N$ . The same result immediately applies to CL. For a CL formula  $\phi$ , we will write  $Act^\phi$  for the number of

<sup>1</sup> We are using CL rather than ATL syntax here for clarity.

actions ‘required’ to construct a satisfying model for  $\phi$ . We use this fact (‘bounded action property’ of coalition logic) to provide a satisfiability-preserving embedding of CL into a normal modal logic.

**Theorem 4.** *For any  $\phi \in \mathcal{L}_{CL}(N, \Theta)$ :  $\phi$  is satisfiable in some CGS (over an arbitrary set of actions  $Act$ ) iff  $T^{Act^\phi}(\phi)$  is satisfiable in some joint action model over  $\Pi_{Act^\phi N}^0$ .*

*Proof.* Let  $\phi \in \mathcal{L}_{CL}(N, \Theta)$ . If  $\phi$  is satisfiable in CL, it is satisfiable in a CGS over  $Act^\phi$  [13]. By Corollary 1  $T^{Act^\phi}(\phi)$  is satisfiable in some joint action model over  $\Pi_{Act^\phi N}^0$ . The other way around follows from Cor. 1 and Lemma 6, since  $T^{Act^\phi}(\phi)$  is a complete action formula.  $\square$

Thus, the satisfiability problem for CL can be reduced to a satisfiability problem for  $K_n^\cap$ .

## 5 Axiomatisation of joint action models

We now give an axiomatisation for the language  $\mathcal{L}_K^\cap(\Pi_{Act N}^0, \Theta)$  and prove that it is sound and complete with respect to the class of all joint action models over  $\Pi_{Act N}^0$  and  $\Theta$ . The axiom system  $\mathcal{S}$  is defined as follows:

- K**  $[\pi](\phi \rightarrow \psi) \rightarrow ([\pi]\phi \rightarrow [\pi]\psi)$
- A1**  $\bigvee_{a \in Act} \langle (i, a) \rangle \top$
- A2**  $\langle \pi \rangle \phi \rightarrow \bigvee_{a \in Act} \langle \pi \cap (i, a) \rangle \phi$
- A3**  $\bigwedge_{i \in N} \langle (i, a_i) \rangle \top \rightarrow \langle (1, a_1) \cap \dots \cap (g, a_g) \rangle \top$
- A4**  $\langle (1, a_1) \cap \dots \cap (g, a_g) \rangle \phi \rightarrow [(1, a_1) \cap \dots \cap (g, a_g)] \phi$
- A5**  $[\pi]\phi \rightarrow [\pi \cap \pi'] \phi$
- A6**  $[(i, a) \cap (i, b)] \perp$  when  $a \neq b$
- MP** From  $\phi \rightarrow \psi$  and  $\phi$  infer  $\psi$
- G** From  $\phi$  infer  $[\pi]\phi$

**K**, **MP** and **G** says that the  $[\pi]$  modalities are normal. **A1** says that at least one action is enabled for each agent in every state, **A2** says that if there is a joint action for some agents that can ensure  $\phi$ , then any agent can do some action at the same time such that  $\phi$  is still ensured, **A3** says that all joint actions composed of enabled individual actions are enabled, **A4** says that complete joint actions are deterministic, **A5** is the standard axiom for intersection, and **A6** says that an agent cannot do two actions simultaneously.

**Theorem 5.** *The axiom system  $\mathcal{S}$  is sound and complete wrt. all joint action models.*

*Proof.* Soundness is straightforward. To prove completeness, we introduce some conventions and auxiliary concepts and show some intermediate properties. When  $\pi = (i_1, a_1) \cap \dots \cap (i_k, a_k)$  and  $\pi' = (j_1, b_1) \cap \dots \cap (j_l, b_l)$  are joint actions, we write

$\pi \leq \pi'$  to denote that for every  $1 \leq u \leq k$  there is a  $1 \leq v \leq l$  such that  $i_u = j_v$  and  $a_u = b_v$ . Recall that a complete joint action is an expression of the form  $(1, a_1) \cap \dots \cap (g, a_g)$  where  $a_i \in Act$ , giving one action for each agent in the system. Let  $JA$  denote the (finite) set of complete joint actions. We use  $\alpha, \beta, \dots$  to denote complete joint actions.

We will make use of a notion of *pseudomodels*, which only have transition relations for complete joint actions. Formally, a pseudomodel is a tuple  $(S, \{R_\alpha : \alpha \in JA\}, V)$  where:  $S$  is a set of states,  $R_\alpha \subseteq S \times S$  for each  $\alpha \in JA$  and  $V : \Theta \rightarrow 2^S$ .

First, we construct the *canonical* pseudomodel  $M^c = (S^c, \{R_\alpha^c : \alpha \in JA\}, V^c)$  as follows:

- $S^c$  is the set of  $\mathcal{L}_K^{\cap}(\Pi_{ActN}^0, \Theta)$ -maximal  $\mathcal{S}$ -consistent sets  $\Gamma$
- $V^c(p) = \{\Gamma : p \in \Gamma\}$
- $R_\alpha^c \Gamma \Gamma'$  iff for any  $\psi$ , if  $\psi \in \Gamma'$  then  $\langle \alpha \rangle \psi \in \Gamma$

**Lemma 7 (Existence Lemma).** *For any  $s \in S^c$ , if  $\langle \alpha \rangle \gamma \in s$  for some  $\alpha \in JA$ , then there is an  $s'$  such that  $(s, s') \in R_\alpha^c$  and  $\gamma \in s'$ .*

The proof of the existence lemma is as in standard normal modal logic.

Now let  $\phi$  be a consistent formula; we show that it is satisfied in some joint action model. Let  $x \in S^c$  be such that  $\phi \in x$ . We now take the *unravelling* of the canonical pseudomodel around  $x$ . The pseudomodel  $M^x = (S^x, \{R_\alpha^x : \alpha \in JA\}, V^x)$  is defined as follows:

- $S^x$  is the set of all finite sequences  $(s_0, R_{\alpha_0}^c, s_1, \dots, R_{\alpha_{k-1}}^c, s_k)$  such that  $s_0, \dots, s_k \in S^c$ ,  $s_0 = x$ , and  $(s_i, s_{i+1}) \in R_{\alpha_i}^c$  for all  $0 \leq i \leq k-1$ .
- $(s, u) \in R_\alpha^x$  iff  $s = (x, R_{\alpha_0}^c, s_1, \dots, R_{\alpha_{k-1}}^c, s_k)$  and  $u = (x, R_{\alpha_0}^c, s_1, \dots, R_{\alpha_{k-1}}^c, s_k, R_\alpha^c, s_{k+1})$  for some  $s_{k+1} \in S^c$ .
- $V^x(p) = \{(x, R_{\alpha_0}^c, s_1, \dots, R_{\alpha_{k-1}}^c, s_k) : s_k \in V^c(p)\}$

We now transform the pseudomodel  $M^x$  into a (proper) model  $M = (S^x, \{R_\pi : \pi \in \Pi\}, V^x)$  as follows:

- $R_{(i,a)} = \bigcup_{\alpha \in JA, (i,a) \leq \alpha} R_\alpha^x$
- $R_{\pi_1 \cap \pi_2} = R_{\pi_1} \cap R_{\pi_2}$

**Lemma 8.**  $R_\alpha = R_\alpha^x$ , for any complete joint action  $\alpha \in JA$ .

*Proof.* Let  $\alpha = (1, a_1) \cap \dots \cap (g, a_g)$ , and observe that

$$R_\alpha = \left( \bigcup_{(1, a_1) \leq \alpha'} R_{\alpha'}^x \right) \cap \dots \cap \left( \bigcup_{(g, a_g) \leq \alpha'} R_{\alpha'}^x \right).$$

First, assume that  $(s, t) \in R_\alpha^x$ . It follows immediately that  $(s, t) \in \bigcup_{(i, a_i) \leq \alpha'} R_{\alpha'}^x$  for all  $i$  by taking  $\alpha' = \alpha$ . Second, assume that  $(s, t) \in R_\alpha$ , i.e., that there are  $\alpha_1, \dots, \alpha_g$  such that  $(i, a_i) \leq \alpha_i$  for all  $i$  and  $(s, t) \in R_{\alpha_1}^x \cap \dots \cap R_{\alpha_g}^x$ . By construction of  $M^x$ , that implies that  $\alpha_1 = \dots = \alpha_g$ . So  $\alpha_1 = \dots = \alpha_g = \alpha$ .  $\square$

Let  $last(s)$  denote the last element  $s_k \in S^c$  in a sequence  $s \in S^x$ .

**Lemma 9 (Truth Lemma).** *For any  $s$  and  $\psi$ ,  $M, s \models \psi$  iff  $\psi \in \text{last}(s)$ .*

*Proof.* The proof is by induction on the structure of  $\psi$ . The cases for propositional atoms and Boolean connectives are straightforward, so let  $\psi = \langle \pi \rangle \gamma$ . Let  $\pi = (i_1, a_1) \cap \dots \cap (i_k, a_k)$  for some  $1 \leq k \leq n$ .

First assume that  $i_l = i_m = i$  and  $a_l \neq a_m$  for some  $l \neq m$ . Observe that  $R_\pi = (\bigcup_{(i_1, a_1) \leq \alpha'} R_{\alpha'}^x) \cap \dots \cap (\bigcup_{(i_k, a_k) \leq \alpha'} R_{\alpha'}^x)$ . Let  $\alpha_1$  and  $\alpha_2$  be arbitrary complete joint actions such that  $(i_l, a_l) \leq \alpha_1$  and  $(i_m, a_m) \leq \alpha_2$ .  $\alpha_1 \neq \alpha_2$  since  $i_l = i_m$  and  $a_l \neq a_m$ . By construction of  $M^x$ ,  $R_{\alpha_1}^x \cap R_{\alpha_2}^x = \emptyset$ . Thus,  $R_\pi = \emptyset$  (since  $\alpha_1$  and  $\alpha_2$  were arbitrary), and  $M, s \not\models \langle \pi \rangle \gamma$  for any  $\gamma$ . On the other hand, by **A6**  $\neg \langle (i, a_l) \cap (i, a_m) \rangle \gamma \in \text{last}(s)$  for any  $\gamma$ , and it follows by **A5** that  $\neg \langle \pi \rangle \gamma \in \text{last}(s)$ . Thus, the Lemma holds in this case and we henceforth assume that  $a_l = a_m$  whenever  $i_l = i_m$ .

For the implication to the right, assume that  $M, s \models \langle \pi \rangle \gamma$ , i.e., that  $M, t \models \gamma$  for some  $(s, t) \in R_\pi = \bigcup_{(i_1, a_1) \leq \alpha'} R_{\alpha'}^x \cap \dots \cap \bigcup_{(i_k, a_k) \leq \alpha'} R_{\alpha'}^x$ . Thus, for each  $1 \leq j \leq k$ , there is an  $\alpha_j$  such that  $(s, t) \in R_{\alpha_j}^x$  and  $(i_j, a_j) \in \alpha_j$ . By construction of  $M^x$ ,  $\alpha_1 = \dots = \alpha_k = \alpha$  (the state  $t$  has only one ‘‘incoming’’ transition). Thus,  $(s, t) \in R_\alpha^x$  with  $\pi \leq \alpha$ . Assume that  $s = (x, R_{\alpha_0}^c, s_1, \dots, R_{\alpha_{k-1}}^c, s_k)$ . Then  $t = (x, R_{\alpha_0}^c, s_1, \dots, R_{\alpha_{k-1}}^c, s_k, R_{\alpha}^c, s_{k+1})$  for some  $s_{k+1}$  such that  $(s_k, s_{k+1}) \in R_\alpha^c$ . By the induction hypothesis  $\gamma \in \text{last}(t) = s_{k+1}$ . By construction of  $R_\alpha^c$ ,  $\langle \alpha \rangle \gamma \in s_k = \text{last}(s)$ . By (repeated applications of) **A5**,  $\langle \pi \rangle \gamma \in s_k = \text{last}(s)$ .

For the implication to the left, let  $s = (x, R_{\alpha_0}^c, s_1, \dots, R_{\alpha_{k-1}}^c, s_k)$  and let  $\langle \pi \rangle \gamma \in \text{last}(s) = s_k$ . Let  $X = \{l_0, \dots, l_m\}$  be the agents not mentioned in  $\pi$ , i.e.,  $X = \{l \in N : l \neq i_j, 1 \leq j \leq n\}$ . Let  $\pi_0 = \pi$ , and  $\pi_{j+1} = \pi_j \cap (l_j, a_j)$  for each  $0 \leq j \leq m$  where  $a_j \in \text{Act}$  is such that  $\langle \pi_j \cap (l_j, a_j) \rangle \gamma \in s_k$ . The existence of such  $a_j$ s are ensured by axiom **A2**. Finally, let  $\alpha = \pi_{m+1}$ . This construction together with the assumption that  $a_l = a_m$  whenever  $i_l = i_m$  ensures that  $\alpha$  is a complete joint action. By the fact that  $\langle \alpha \rangle \gamma \in s_k$  and the existence lemma, there is a state  $s_{k+1} \in S^c$  such that  $(s_k, s_{k+1}) \in R_\alpha^c$  and  $\gamma \in s_{k+1}$ . Let  $t = (x, R_{\alpha_0}^c, s_1, \dots, R_{\alpha_{k-1}}^c, s_k, R_\alpha^c, s_{k+1})$ ;  $t \in S^x$  and  $(s, t) \in R_\alpha^x$  by definition of  $M^x$ . By Lemma 8  $(s, t) \in R_\alpha$ , and from the fact that  $\pi \leq \alpha$  it is easy to see that  $R_\alpha \subseteq R_\pi$  by definition of  $R_\pi$ , so  $(s, t) \in R_\pi$ . Since  $\gamma \in \text{last}(t)$ , by the induction hypothesis  $M, t \models \gamma$ , and thus  $M, s \models \langle \pi \rangle \gamma$ .  $\square$

**Lemma 10.**  *$M$  is a joint action model.*

*Proof.* INT: Immediate from the definition.

SER: Let  $s$  be a state and  $i$  an agent. From **A1** and **A3** there is some  $\alpha = (1, a_1) \cap \dots \cap (g, a_g)$  such that  $\langle \alpha \rangle \top \in s$ . From the truth lemma and Lemma 8 there is a  $s'$  such that  $(s, s') \in R_\alpha = R_\alpha^x$ . From the definition of  $R_{(i, a_i)}$ ,  $R_\alpha^x \subseteq R_{(i, a_i)}$ , and thus  $(s, s') \in R_{(i, a_i)}$ .

IC: Let  $s$  be a state,  $C = \{i_1, \dots, i_k\}$  a coalition, and assume that for each  $j$ ,  $(i_j, a_{i_j})$  is enabled for  $i_j$  in  $s$ . Let  $\pi = (i_1, a_{i_1}) \cap \dots \cap (i_k, a_{i_k})$ . By the truth lemma,  $\bigwedge_{i_j \in C} \langle (i_j, a_{i_j}) \rangle \top \in \text{last}(s)$ . For each  $i_j \in N \setminus C$ , let  $a_{i_j}$  be such that  $\langle (i_j, a_{i_j}) \rangle \top \in \text{last}(s)$  – existing by **A1**. Let  $\alpha = (1, a_1) \cap \dots \cap (g, a_g)$ . By **A3**  $\langle \alpha \rangle \top \in \text{last}(s)$ , and by **A5**  $\langle \pi \rangle \top \in \text{last}(s)$ . By the truth lemma, there is an  $s'$  such that  $(s, s') \in R_\pi$ .

DJA: Assume that  $s = (x, R_{\alpha_0}^c, s_1, \dots, R_{\alpha_{k-1}}^c, s_k)$ . Let  $(s, s_1), (s, s_2) \in R_\alpha$ , where  $\alpha$  is complete. By Lemma 8, there are  $s_{k+1}^1, s_{k+1}^2 \in S^c$  such that

$s_1 = (x, R_{\alpha_0}^c, s_1, \dots, R_{\alpha_{k-1}}^c, s_k, R_{\alpha}^c, s_{k+1}^1)$ ,  $s_2 = (x, R_{\alpha_0}^c, s_1, \dots, R_{\alpha_{k-1}}^c, s_k, R_{\alpha}^c, s_{k+1}^2)$ , and  $(s_k, s_{k+1}^1), (s_k, s_{k+1}^2) \in R_{\alpha}^c$ . Assume that  $s_1 \neq s_2$ , i.e., since  $s_1$  and  $s_2$  are identical up to the last state, that  $s_{k+1}^1 \neq s_{k+1}^2$ . By the definition of  $S^c$ , there must be a formula  $\psi \in s_{k+1}^1$  such that  $\neg\psi \in s_{k+1}^2$ . By the truth lemma,  $M, s_1 \models \psi$  and  $M, s_2 \models \neg\psi$  and thus  $M, s \models \langle \alpha \rangle \psi \wedge \langle \alpha \rangle \neg\psi$ . By the truth lemma again,  $\langle \alpha \rangle \psi, \langle \alpha \rangle \neg\psi \in \text{last}(s)$ . By **A4**,  $[\alpha]\psi, [\alpha]\neg\psi \in \text{last}(s)$ . But  $\langle \alpha \rangle \psi, [\alpha]\neg\psi \in \text{last}(s)$  contradicts, via standard modal reasoning, the fact that  $\text{last}(s)$  is consistent. Thus,  $s_1 = s_2$ .

UJA: Immediate by Lemma 8 and construction of  $M^x$ .  $\square$

Since  $\phi \in x = \text{last}(x)$ ,  $\phi$  is satisfied in a joint action model by Lemma 9 and 10. This concludes the completeness proof.  $\square$

## 6 Complexity

We show that the complexity of deciding satisfiability in joint action models of  $\mathcal{L}_K^{\cap}(\Pi_{ActN}^0, \Theta)$  formulae is in PSPACE. The proof uses ideas from [20].

In what follows, we assume without loss of generality that formulas do not contain diamond modalities and disjunctions. Given a set of formulas  $X$ , we use  $Cl(X)$  to denote the smallest set containing all subformulas of formulas in  $X$  such that:

- (a) for each agent  $i$  and action  $a$ ,  $[(i, a)]\perp \in Cl(X)$
- (b) for every complete joint action  $\alpha \in JA$ ,  $[\alpha]\perp \in Cl(X)$
- (c) if  $\neg[(1, a_1) \cap \dots \cap (g, a_g)]\psi \in Cl(X)$ , then  $[(1, a_1) \cap \dots \cap (g, a_g)]\sim\psi \in Cl(X)$ , where  $\sim\psi = \neg\psi$  if  $\psi$  is not of the form  $\neg\chi$ , and  $\sim\psi = \chi$  otherwise
- (d) for each  $i$  and  $a \neq b$ ,  $[(i, a) \cap (i, b)]\perp \in Cl(X)$
- (e) if  $\psi \in Cl(X)$ , then  $\sim\psi \in Cl(X)$

The following procedure  $Tab$  is based on the  $K_{\omega}^{\cap \cup}$ -World procedure of [20]. For sets of formulas  $\Delta$  and  $S$  where  $S$  is closed as above,  $Tab(\Delta, S)$  returns true iff

- (A)  $\Delta$  is a maximally propositionally consistent subset of  $S$ , that is, for each  $\neg\psi \in S$ ,  $\psi \in \Delta$  iff  $\neg\psi \notin \Delta$  and for each  $\psi_1 \wedge \psi_2 \in S$ ,  $\psi_1 \wedge \psi_2 \in \Delta$  iff  $\psi_1 \in \Delta$  and  $\psi_2 \in \Delta$ .
- (B) There is a partition of the set  $\{\neg[\pi]\psi : \neg[\pi]\psi \in \Delta\}$  into sets  $W_{\alpha}$  (at most one for each  $\alpha \in JA$ ) such that if  $\neg[\pi]\psi \in W_{\alpha}$  then  $\pi \leq \alpha$  (where the definition of  $\leq$  is as in the proof of Theorem 5) and
  - (i)  $\neg\psi \in \Delta_{\alpha}$
  - (ii) for each  $\pi'$  and  $\psi'$ , if  $[\pi']\psi' \in \Delta$  and  $\pi' \leq \alpha$ , then  $\psi' \in \Delta_{\alpha}$
  - (iii)  $Tab(\Delta_{\alpha}, S')$  returns true, where  $S' = Cl(\{\psi' : [\pi']\psi' \in \Delta \text{ and } \pi' \leq \alpha\} \cup \{\neg\psi : \neg[\pi]\psi \in W_{\alpha}\})$
- (C) for each  $i \in N$ ,  $\neg[(i, a)]\perp \in \Delta$  for some  $a \in Act$
- (D) if  $\neg[(1, a_1)]\perp, \dots, \neg[(g, a_g)]\perp \in \Delta$ , then  $\neg[(1, a_1) \cap \dots \cap (g, a_g)]\perp \in \Delta$
- (E) if  $\neg[\alpha]\psi \in \Delta$ , then  $[\alpha]\sim\psi \in \Delta$
- (F) for every  $i \in N$  and  $a, b \in Act$  such that  $a \neq b$ ,  $[(i, a) \cap (i, b)]\perp \in \Delta$

We require  $Tab(\Delta, S)$  to terminate when the only modal formulas in  $S$  are those introduced by the clauses (a), (b) and (d) of the definition of  $Cl(X)$ . Note that otherwise formulas of the form  $\neg[\alpha]\perp$  will continue triggering new calls to  $Tab(\Delta, S)$ .

**Lemma 11.** *A formula  $\phi$  is satisfiable in a joint action model iff there exists  $\Delta \subseteq Cl(\phi)$  with  $\phi \in \Delta$  such that  $Tab(\Delta, Cl(\phi))$  returns true.*

*Proof.* One direction is easy. For the other direction, we will show how to construct a model for  $\phi$  if there exists  $\Delta \subseteq Cl(\phi)$  with  $\phi \in \Delta$  such that  $Tab(\Delta, Cl(\phi))$  returns true. Suppose for  $\phi$  such  $\Delta$  exists, and let us call it  $\Delta_0$ . The model  $M$  and the state  $s_0$  satisfying  $\phi$  are constructed as follows. Each  $\Delta$  in successive recursive calls of  $Tab(\Delta, S)$  corresponds to a (partial specification of a) state. The existence of a propositional assignment satisfying formulas in  $\Delta$  is ensured by clause (A). The initial state  $s_0$  corresponds to  $\Delta_0$ . In each  $\Delta$ , each formula of the form  $\neg[\pi]\psi$  by clause (B) belongs to a set  $W_\alpha$  and has a ‘witness’  $\Delta_\alpha$  for  $\neg\psi$  accessible by a complete joint action  $\alpha$  such that  $\pi \leq \alpha$  and  $\neg\psi \in \Delta_\alpha$ . In the model we stipulate  $R_\alpha(s_\Delta, s_{\Delta_\alpha})$  holds together with  $R_{\pi'}(s_\Delta, s_{\Delta_\alpha})$  for every  $\pi' \leq \alpha$ . The rest of clause (B) makes sure that  $\Delta_\alpha$  contains all formulas  $\psi'$  such that  $[\pi']\psi' \in \Delta$ , which makes sure both that the truth definition for  $[\pi]$  and the semantics of intersection work as expected. This part is almost identical to the proof for  $K_\emptyset^\cap$  (apart from requiring a unique  $\alpha$ -successor). All we need to prove is that in addition, the resulting model satisfies the properties of seriality, independent choice, determinism for complete joint actions, and uniqueness of joint actions.

SER is trivial by clause (C). When we terminate the procedure, to ensure seriality we add one more successor state for each  $\alpha$  with an  $\alpha$  loop to itself. This modification will not affect the truth of  $\phi$  in  $s_0$  because it is at a modal distance from  $s_0$  which is greater than the modal depth of  $\phi$ . IC is ensured by clause (D). DJA is ensured by (B); the existence of partition is enabled by (E) which makes the set of formulas  $\sim\psi$  for  $\neg[\alpha]\psi \in \Delta$  consistent. UJA is ensured by (F); namely there is no  $\Delta'$  accessible by  $\alpha \cap \alpha'$  from  $\Delta$ , where  $\alpha, \alpha' \in JA$  and  $\alpha \neq \alpha'$ ; otherwise by clause (F) for some agent  $i$  which performs a different action in  $\alpha$  and  $\alpha'$ ,  $[(i, a) \cap (i, b)]\perp \in \Delta$  and hence  $\perp \in \Delta'$ , but by the definition of the procedure then it cannot return true for  $\Delta'$ .  $\square$

**Theorem 6.** *The complexity of satisfiability problem of formulas in joint action models is PSPACE-complete.*

*Proof.* Satisfiability is decided by  $Tab(\Delta, Cl(\phi))$  by the previous lemma. To see that  $Tab(\Delta, Cl(\phi))$  requires polynomial space, consider the size of  $Cl(\phi)$ . The set of subformulas of  $\phi$  is clearly polynomial in  $|\phi|$ . The number of formulas added to  $Cl(\phi)$  by clause (a) is  $gm$ , the number of formulas added by clause (b) is  $m^g$ , the number of formulas added by clause (d) is  $gm^2$ , and (c) and (e) at most double the number of formulas in  $Cl(\phi)$ . Note that  $g$  and  $m$  are constant factors, hence the size of  $Cl(\phi)$  and  $\Delta$  is polynomial in  $|\phi|$ . PSPACE-hardness follows from  $K$  being PSPACE-complete.  $\square$

The following is an immediate consequence of the result for model checking complexity of PDL with intersection [18]:

**Theorem 7.** *Model checking the  $\mathcal{L}_K^\cap(\Pi_{ActN}^0, \Theta)$  language in joint action models is in PTIME.*

The complexity results above are encouraging from the point of view of using the logic of joint actions for verifying properties of game structures using standard theorem-proving and model-checking tools for normal modal logic. However, verification of properties involving coalitional ability comes at the price of performing a translation from CL to the language of  $K_n^\cap$ . The size of the translation may grow exponentially in the size of the input formula (nested coalition modalities give rise to nested disjunctions and conjunctions over all possible complete actions).

## 7 Discussion

In this paper we defined and studied a class of  $K_n^\cap$  models corresponding to concurrent game structures that are (1) injective and (2) parameterised by a fixed and finite set of actions, and showed that on this model class coalition modalities can be expressed in the  $K_n^\cap$  language. Along the way we proved a representation theorem for injective games (this result holds also for games over infinite sets of actions). Given the fact that any satisfiable CL formula  $\phi$  is satisfiable in an injective concurrent game structure with  $Act^\phi$  actions, we also have a satisfiability-preserving embedding from CL into a normal modal logic based on  $K_n^\cap$ .

As mentioned in the introduction, the idea of interpreting PDL-like languages in games is not new, and the idea of using intersection of strategies to capture coalitional ability is also not entirely new (see below). However, we are not aware of existing completeness or complexity results for  $K_n$  with intersection interpreted in game structures.

[5] uses propositional dynamic logic (PDL) interpreted directly in extensive-form games, and also suggests extending the language with a “forcing” operator  $\{G, i\}\phi$ , with the meaning that agent  $i$  has a strategy in game  $G$  which forces a set of outcomes that all will satisfy  $\phi$ . However, the forcing operator is not defined in terms of intersection, and the operator is only defined for singleton coalitions. [7] briefly mentions that coalition operators can be captured by intersection, but does not work out the details of resulting logics or relate them to standard PDL. Broersen et al. [8] have already shown that coalition logic can be embedded in a normal modal logic, namely in a variant of STIT (*seeing-to-it-that*) logic [4]. While this is a valuable result for several reasons, we argue that embedding in  $K_n^\cap$  is of additional interest because the latter is a more standard logic (see the introduction). A closely related work is [19], which sets out from a similar starting point as the current paper: defining a “minimalistic” logical framework based on PDL that is interpreted in models where agents perform joint actions. *Deterministic Dynamic Logic of Agency* ( $\mathcal{D}\mathcal{D}\mathcal{L}\mathcal{A}$ ) [19] has modalities of the form  $\langle i : a \rangle$  where  $i$  is an agent and  $a$  is an action, very similar to the modalities in the current paper in other words, and is shown to embed coalition logic. The interpretation of the modalities is slightly different:  $\langle i : a \rangle\phi$  informally means that “ $i$  performs action  $a$  and  $\phi$  holds afterwards”. The lan-

guage is not standard PDL; it also includes a modality  $\diamond$  that quantifies over actions. The language does not use intersection. In contrast, the current paper has focused on reasoning about joint action using only standard PDL modalities and operators, in particular intersection. Finally, the emphasis of [19] is somewhat different from the current paper, and does not contain most of the types of results (characterisation of effectivity, translation of the model checking problem, complexity, etc.) found in the current paper. We leave the precise relationship between the two logics to future work.

In this paper we studied a “minimal” language with intersection, sufficient to capture the coalition operators. For future work, extensions of the language with other PDL operators would be of interest, building on existing results on PDL with intersection such as [9].

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