Syntax vs. Semantics on Finite Structures

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Abstract. Logic preservation theorems often have the form of a syntax/semantics correspondence. For example, the Łos-Tarski theorem asserts that a first-order sentence is preserved by extensions if and only if it is equivalent to an existential sentence. Many of these correspondences break when one restricts attention to finite models. In such a case, one may attempt to find a new semantical characterization of the old syntactical property or a new syntactical characterization of the old semantical property. The goal of this paper is to provoke such a study.

1 Introduction

It is well known that famous theorems about first-order logic fail in the case when only finite structures are allowed (see, for example, [8]). A more careful examination shows that it is wrong to lump all these failing theorems together. On one side we have theorems like completeness or compactness where the failure is really and truly hopeless. On the other side there are theorems like the Łos-Tarski theorem, which we prefer to formulate in the following form:

Theorem 1 (Łos and Tarski). *A first order formula is preserved by extensions iff it is equivalent to an existential formula.*

Theorems of this form are known as preservation theorems. Classical preservation theorems fail not only when the class of models is restricted to allow only finite models, but also when the language is modified (for example, only up to k many variables allowed in a formula). Recently a lot of interesting work was done on rescuing preservation theorems in non-classical contexts. Rosen and Weinstein in [12] start with analyzing the failure of the Łos-Tarski theorem on finite models and come up with a generalized notion of a preservation theorem: $L \cap EXT \subseteq L'$, where EXT is the set of formulas preserved by extensions on finite models, L is some first order quantifier prefix class, and L' is the existential fragment of $L^{\omega}_{\infty\omega}$ or positive Datalog. They also show that the analogue of the Łos - Tarski theorem fails for $L^{\omega}_{\infty\omega}$. Barwise and van Benthem in [4] rescue preservation theorems in $L^{\omega}_{\infty\omega}$ and its fragments (on arbitrary models) using a generalized notion of 'consequence along some model relation'.

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We view the Łos-Tarski theorem not so much as a preservation theorem, but rather as a theorem relating syntax and semantics. On the one hand, it can be seen as the semantical characterization of existential first-order formulas. It is natural to ask if there is an alternative characterization of such formulas in the case of finite structures[3]. It turns out that there is a natural characterization of that sort; see Theorem 5.

On the other hand, the Łos-Tarski theorem can be seen as a syntactical characterization of the semantical property 'being preserved by extensions' in the context of first-order logic. We say that a property $\mathcal{P}$ has a *syntactical characterization in the context of a logic* L if there is a recursive class F of formulas of L, such that

1. Every formula in F has the property $\mathcal{P}$.
2. Every L-formula which has the property $\mathcal{P}$ is L-equivalent to a formula in F.

A natural question arises whether such a characterization for the formulas preserved by extensions exists in the case of finite structures. We know that the classical characterization fails (Gurevich - Shelah, Tait), but this does not rule out the existence of another characterization.

The Łos-Tarski theorem is not the only syntax/semantics theorem.

Some other theorems of the same kind are:

Theorem 2 (Lyndon). *A first order formula is monotone in a predicate P iff it is equivalent to a formula positive in P[4].*

Theorem 3. *A first order formula is preserved by homomorphisms iff it is equivalent to a positive existential formula.*

Theorem 4. *A first order universal formula is preserved by finite direct products iff it is equivalent to a universal Horn formula.*

In this paper we concentrate on syntax-to-semantics characterization in the context of finite structures. We also make some preliminary remarks on the problem of semantics-to-syntax characterization. This is only the beginning; our investigation provides many questions and few answers.

The rest of the paper is organized as follows. In section 2, we give a semantical characterization of existential first order formulas on finite structures. In section 3 we use similar techniques to characterize positive existential, existential Horn and universal Horn formulas on finite structures. Section 4 deals with the problem of characterizing semantical properties in the context of first order logic. Here, we only have negative results which state that the class of formulas

[3] Indeed this question has been asked by Johan van Benthem (in September 1995) and by H. Jerome Keisler (in March 1996) when one of the authors lectured on finite model theory.

[4] A couple of years ago Jörg Flum asked explicitly if there is an alternative characterization of monotonicity over finite structures.

preserved under a given construction cannot be itself the desired characterization, since it is not recursive. In section 5 some syntactical characterizations are proposed in the context of extensions of first order logic. We conclude with stating some open problems.

Conventions. We assume that all structures are over a finite vocabulary which does not contain functional symbols of positive arity, unless explicitly stated otherwise. We assume that classes of structures are closed under isomorphisms, and 'there is a unique structure' or 'there are finitely many structures' means 'up to isomorphism'.

The notation is usually fairly standard. We use $M \preceq N$, where M, N are structures, for 'M is a substructure of N', that is: the universe of M is a subset of the universe of N, and the interpretation of all predicates and constants in M and N is the same on the universe of M. $Mod(T)$, where T is a theory, denotes the class of finite models of T.

For the sake of brevity, we speak about sentences and classes of structures rather than formulas and global relations (for the definition of global relations, see [8]).

2 Existential Formulas on Finite Structures

As we have already mentioned, the Łos-Tarski's characterization of existential first order formulas fails for finite structures (see Theorem 10). A question arises whether there is any natural characterization of existential formulas on finite structures. It turns out that such a characterization exists.

Define a *minimal* structure in a class K to be a structure in K with no proper substructures in K.

Let $\min(K) = \{M \in K : M \text{ is minimal in } K\}$.

Theorem 5. *Let K be an arbitrary class of finite structures over the same finite vocabulary which does not contain functional symbols of positive arity.*

The following are equivalent:

1. *K is closed under extensions and $min(K)$ is finite.*
2. *There is an existential first-order sentence φ such that K is the collection of finite models of φ.*

Proof. $1 \longrightarrow 2$. Let $A_1, .., A_j$ be the minimal models of K of cardinalities $n_1, .., n_j$ respectively. The desired φ has the form

$$\varphi_1 \vee \ldots \vee \varphi_j,$$

where φ_i states that there are elements $x_1, \ldots, x_{n_i}$ which form a structure isomorphic to A_i.

$2 \longrightarrow 1$. Fix an appropriate φ and let n be the number of quantifiers plus the number of constants in φ. By the classical theorem, K is closed under extensions.

Since there are only finitely many structures in K of cardinality $\leq n$, it suffices to prove that every A in K has a substructure of cardinality $\leq n$ satisfying φ. But this is obvious. The desired substructure is formed by a set of at most n elements witnessing that A satisfies φ. $\square$

Remark. The same holds for any global relation K (not only for a class of structures, i.e. a 0-ary global relation).

In one aspect, the semantical characterization of existential formulas in Theorem 5 is even preferable to the classical Łos-Tarski characterization: K is not supposed to be finitely axiomatizable in first-order logic. It may seem that Theorem 5 and its proof survive in the case of arbitrary structures. Flum pointed out that this is wrong ([7]). Counterexamples include (i) the class of extensions of a given (up to isomorphism) infinite structure, and (ii) the class of non-well-founded orders. In both cases, the class in question satisfies 1 (in case (ii) this happens because there are no minimal models in the class) but is not definable by an existential sentence. To adapt Theorem 5 to the case of infinite structures, the condition that $\min(K)$ is finite may be replaced by the following stronger condition: There exists a finite class $K_0 \subseteq K$ of finite structures such that every K-structure extends some K_0-structure.

Obviously, the Łos-Tarski theorem cannot be generalized as 'A class of arbitrary structures is closed under extensions if, and only if, it is axiomatizable by an existential formula'. Here is an example (building on the Gurevich-Shelah counter-example, cf. Theorem 10) of a class which is closed under extensions on arbitrary structures but is not finitely axiomatizable. Consider the class K_0 of finite linear orders with the successor relation, a minimal element a and a maximal element b, and close this class under extensions. Let us call the resulting class K. Assume by contradiction that K is definable by a first order formula. Then, by the Łos-Tarski's theorem, K is definable by an existential sentence. By the analog of Theorem 5 for arbitrary structures, $\min(K)$ is finite. But it is easy to see that $\min(K) = K_0$; in particular, if you remove the successor of some element c in a structure $M \in K_0$, then the remaining structure does not belong to K_0 because the element c does not have a successor there. Thus $\min(K)$ is infinite, which gives the desired contradiction.

We say that Theorem 5 *survives the restriction to a theory* T if for every class K of models of T, the following are equivalent:

1. K is closed under T-extensions (that is, if $M \in K$, $N \models T$ and $M \preceq N$ then $N \in K$) and $\min(K)$ is finite.
2. There is an existential sentence φ such that $K = \{M \in Mod(T) : M \models \varphi\}$.

Proposition 6. *If T is axiomatized by an $\exists^*\forall^*$ sentence, then Theorem 5 survives the restriction to T.*

Proof. $1 \longrightarrow 2$: as before.
$2 \longrightarrow 1$. Suppose that T is given by a sentence $\alpha = \exists x_1 \ldots \exists x_k \forall \bar{y} \psi$ and $K = \{M \in Mod(T) : M \models \exists z_1 \ldots \exists z_n \chi\}$, where ψ and χ are quantifier-free.

It is obvious that K is closed under T-extensions. Let $M \in K$. There are k elements satisfying $\forall \bar{y} \psi(x_1, \ldots, x_k)$ and n elements satisfying χ in M. The substructure generated by these $k+n$ elements still satisfies α and φ. Therefore the minimal structures in K are of size less or equal to $k+n+$ the number of constants. The rest of the proof is as above. $\square$

Observe that the same proof works in case T is a universal theory. However,

Proposition 7. *Theorem 5 may not survive the restriction to a theory given by an infinite set of existential axioms.*

Proof. Let φ be $\exists x(x = x)$. The following existential theory T has infinitely many minimal models:

$$T = \exists x_1 x_2 (x_1 R x_2), \exists x_1 x_2 x_3 (x_1 R x_2 R x_3), \exists x_1 x_2 x_3 x_4 (x_1 R x_2 R x_3 R x_4), \ldots$$

Notice that every cycle $a_1 R a_2 R \ldots R a_1$ (without any additional edges) is a model of T. It is also a minimal model, for, if one of the elements is deleted, and the longest chain of the form $a_1 R a_2 \ldots R a_n$ has length n, then

$$\exists x_1 \ldots x_{n+1} (x_1 R x_2 \ldots R x_{n+1})$$

is no longer satisfied. $\square$

Proposition 8. *Theorem 5 may not survive the restriction to a theory given by a $\forall\exists$ sentence α.*

Proof. Let α be $\forall x \exists y R(x, y)$ and φ be $\exists x(x = x)$. Every cycle $a_1 R a_2 R \ldots R a_1$ is a minimal model of $\alpha \wedge \varphi$. $\square$

Proposition 9. *Theorem 5 fails if functions are allowed.*

Proof. Let φ be $\exists x(f(x) = f(x))$. It has infinitely many minimal models, for example with $f(a_1) = a_2, f(a_2) = a_3, \ldots, f(a_n) = a_1$. (If any element is deleted, the resulting substructure is not closed under functional application.) $\square$

Theorem 5 allows us to simplify the counterexample to the Łos-Tarski theorem on finite structures given by Gurevich and Shelah in [8].

Theorem 10 (Tait 1959, Gurevich - Shelah 1984). *There exists a first order formula which is preserved by extensions on finite structures but is not equivalent to any existential formula on finite structures.*

Proof. Consider the first order language with equality containing two constants a and b, and two binary predicates $<$ and S (which is going to denote the successor relation) in addition to equality.

Let χ be $\chi_1 \wedge \chi_2 \wedge \chi_3 \rightarrow \chi_4$, where:

1. χ_1 says that $<$ is a linear order, that is, χ_1 is the conjunction of

$\chi_{11} = \forall x \forall y (x < y \vee y < x \vee x = y)$,
$\chi_{12} = \forall x \neg (x < x)$,
$\chi_{13} = \forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$;

2. χ_2 says that S is consistent with the successor relation in the following sense:

$$\chi_2 = \forall x \forall y (S(x, y) \rightarrow x < y \wedge \neg \exists z (x < z < y));$$

3. χ_3 says that a is the least element and b the greatest element, and a is not equal to b, that is, $\chi_3 = \chi_{31} \wedge \chi_{32} \wedge a < b$, where $\chi_{31} = \forall x (a \leq x)$ and $\chi_{32} = \forall x (x \leq b)$ (where $\leq$ is defined as usual);
4. χ_4 says $\forall x (x < b \rightarrow \exists y S(x, y))$ (every element except for b has a successor). Together with $\chi_1 - \chi_3$, χ_4 implies that S is in fact the successor relation.

Notice that χ is preserved by extensions. For, let $M \models \chi$ and N be a proper extension of M. Suppose $M \not\models \chi_1 \wedge \chi_2 \wedge \chi_3$. Since $\neg(\chi_1 \wedge \chi_2 \wedge \chi_3)$ is existential, it is preserved by extensions. Therefore $N \models \chi$. Suppose $M \models \chi_1 \wedge \chi_2 \wedge \chi_3$. It suffices to check that N fails to satisfy $\chi_1 \wedge \chi_2 \wedge \chi_3$. Consider an element c of N which does not belong to M. If it does not fit in the linear order between a and b, χ_1 or χ_3 are false in N. Since M is finite, c has to fit between a and the successor of a, or between the successor of a and its successor, et cetera, or between the predecessor of b and b. In all these cases χ_2 is violated.

This formula has infinitely many minimal models, because every initial segment of natural numbers with a interpreted as 0, b as the greatest element and the predicates having their standard interpretation is a minimal model of χ. Indeed, let M be such a model (an initial segment of natural numbers), and N a proper substructure of M. Since χ_1, χ_2, χ_3 are universal sentences satisfied in M, they are satisfied in N, but χ_4 necessarily fails in N.

It follows from Theorem 5 that χ is not equivalent to an existential formula. □

Observe that χ is not preserved by extensions on infinite structures. Let ω^R be the order type of the reversed ω: $\ldots, 3, 2, 1, 0$. Consider a structure of order type $\omega + \omega^R$:

$$a = 0, 1, \ldots, \ldots, -2, -1 = b$$

with the standard successor relation. It is a model of χ. If we extend this structure by putting an element in the middle, then χ is not true any more.

If a formula with only unary predicates, constants and equality is preserved by extensions on finite models, it is equivalent to an existential formula. This is easy to check. However, one binary relation is sufficient for a counter-example.

Theorem 11. *Let L be a language with only one binary relation and without constants or equality. There is an L-sentence preserved by extensions that is not equivalent to any existential sentence on finite structures.*

We only sketch a proof of Theorem 11. The proof consists in step-by-step replacing the formula χ above with a formula which does the same job but contains only one binary predicate.

As a first step, we get rid of the individual constants and equality. The first auxiliary language contains $<$, S and two more unary predicates, F ('first') and L ('last'). Let $x \equiv y =_{df} \forall z([(x < z) \leftrightarrow (y < z)] \wedge [(z < x) \leftrightarrow (z < y)] \wedge [S(x,z) \leftrightarrow S(y,z)] \wedge [S(z,y) \leftrightarrow S(z,y)] \wedge [F(x) \leftrightarrow F(y)] \wedge [L(x) \leftrightarrow L(y)])$.

Our first modification of χ is

$\chi' = \chi'_1 \wedge \chi'_2 \wedge \chi'_3 \to \chi'_4$, where χ'_1 is like in the proof of Theorem 10, but with $=$ replaced by $\equiv$, χ'_2 is like χ_2, χ'_3 is the conjunction of

$$\forall x \forall y (F(x) \to x < y \vee x \equiv y);$$

$$\forall x \forall y (L(x) \to y < x \vee x \equiv y);$$

$$\forall x \forall y (F(x) \wedge L(y) \to x < y),$$

and χ'_4 is the conjunction of $\forall x(\neg L(x) \to \exists y S(x,y))$ and $\exists x F(x) \wedge \exists x L(x)$.

We leave it for the reader to check that this formula is preserved by extensions on finite structures and has infinitely many minimal models. (Notice that if $M \models \chi'$ and N is a proper extension of M, N may satisfy the antecedent of χ' provided that every new element fits into some $\equiv$-equivalence class in M. But then every element in N still has a successor.)

Now we translate χ' into the language containing just one binary predicate R. We are going to code old elements with clusters consisting of three points, and code the relations $<$, S, F and L between the clusters in terms of relation R among their components.

We use the following abreviations: $M(x)$ ('x is a main element') for $R(x,x)$; $A(x)$ ('x is an auxiliary element') for $\neg R(x,x)$; and $Reg(x, x_1, x_2)$ ('x, x_1, x_2 is a regular triple') for the conjunction of

- $M(x)$;
- $A(x_1) \wedge A(x_2)$;
- $R(x_1, x) \wedge R(x_2, x)$ ('x_1, x_2 are auxiliary to x');
- $R(x, x_1) \wedge \neg R(x, x_2)$ ('x_1 is the first auxiliary, x_2 is the second auxiliary').

Let $Reg(x, x_1, x_2)$, $Reg(y, y_1, y_2)$. The relations between the main elements are coded as follows:

- $x < y \Leftrightarrow x_1 R y_1$;
- $S(x,y) \Leftrightarrow x_2 R y_2$;
- $F(x) \Leftrightarrow x_1 R x_2$;
- $L(x) \Leftrightarrow x_2 R x_1$.

Abbreviate $\forall z(R(x,z) \leftrightarrow R(y,z)) \wedge (R(z,x) \leftrightarrow R(z,y))$ as $x \equiv y$.

Translate $\chi'_1 - \chi'_4$ introduced above using this coding. For example, χ'_{11} becomes

$$\forall x x_1 x_2 y y_1 y_2 (Reg(x, x_1, x_2) \wedge Reg(y, y_1, y_2) \to x_1 R y_1 \vee y_1 R x_1 \vee x \equiv y).$$

Denote the resulting translations $\chi_1'' - \chi_4''$.

To guarantee that the coding is well defined, we need the following conditions. Let χ_0'' be the conjunction of the universal closure of

$M(x) \wedge M(y) \wedge R(x,y) \rightarrow x \equiv y$ (main elements are not connected or they are indistinguishable);

$A(y) \wedge A(z) \wedge R(x,y) \wedge R(y,x) \wedge R(x,z) \wedge R(z,x) \rightarrow y \equiv z$ (every main element has only one first auxiliary element up to $\equiv$);

$M(x) \wedge A(y) \wedge A(z) \wedge R(y,x) \wedge \neg R(x,y) \wedge R(z,x) \wedge \neg R(x,z) \rightarrow y \equiv z$ (every main element has only one second auxiliary);

$A(x) \wedge M(y) \wedge M(z) \wedge R(x,y) \wedge R(x,z) \rightarrow y \equiv z)$ (every auxiliary element has only one 'master').

Again we leave it to the reader as an exercise to verify that $\chi_0'' \wedge \ldots \wedge \chi_3'' \rightarrow \chi_4''$ is preserved by extensions on finite structures and has infinitely many minimal models.

3 More Examples

In this section we look at some other well known preservation theorems from classical logic and give some characterizations for the finite case.

The following characterizations of universal Horn and existential Horn formulas are known in classical model theory (cf. [5], p.337, Propositions 6.2.8 and 6.2.9).

Fact 12. *A first order formula is preserved by substructures and finite direct products iff it is equivalent to a universal Horn formula.*

Fact 13. *A first order formula is preserved by extensions and finite direct products iff it is equivalent to an existential Horn formula.*

We do not know if the first statement fails on finite structures. The second one is false:

Proposition 14. *There exists a formula which is preserved by extensions and finite direct products on finite structures but is not equivalent to an existential Horn formula.*

Proof. Consider the formula $\exists x \exists y (x \neq y) \wedge \chi$, where χ is defined as in the proof of Theorem 10. This formula is preserved by extensions on finite structures (since both conjuncts are) and is not equivalent to an existential formula (since it has infinitely many minimal models). It is easy to show that it is preserved by direct products. Let M_1 and M_2 satisfy the formula. Obviously if M_1 and M_2 contain at least two elements, so does their product. With respect to χ, we have two cases. First, assume that for all elements e of M_1 and M_2 $e < e$ holds. Then all elements of $M_1 \times M_2$ are reflexive as well, and the antecedent of χ which states that the order is strict, fails. Therefore χ is true on $M_1 \times M_2$. Now suppose that one of the structures (let it be M_2) has a non-reflexive element e. Since M_1 has at least two elements e_1 and e_2, $M_1 \times M_2$ contains (e_1, e) and (e_2, e) which are incomparable. Hence the antecedent of χ which states the existence of a linear order, fails, and χ itself is true on $M_1 \times M_2$. □

A characterization of universal Horn and existential Horn formulas on finite structures can be easily obtained from Theorem 5.

Theorem 15. *Let K be an arbitrary class of finite structures over the same finite vocabulary which does not contain functional symbols of positive arity.*
The following are equivalent:

1. *K is closed under extensions and direct products, and min(K) is finite.*
2. *K is definable by an existential Horn formula.*

Proof. $1 \longrightarrow 2$. By Theorem 5, K is definable by an existential formula φ. We also know that φ is preserved by finite direct products of finite structures. It remains to show that φ is preserved by finite direct products of arbitrary structures.

By contradiction, assume that this is not the case. Then there are possibly infinite structures $M_1, \dots, M_k$, each satisfying φ, such that their product does not satisfy φ. Since φ is existential, there are finite substructures of structures M_i satisfying φ. Then φ is true on the product of these substructures, since it is preserved by products of finite structures. But this product is a substructure of the big product, so φ is true in the big product, which contradicts the assumption.

The direction from 2 to 1 is easy. □

Analogously it is possible to characterize universal Horn sentences.

We say that a structure M is a *minimal counterexample* to the class K if M is not in K and every proper substructure of M is in K.

Theorem 16. *Let K be an arbitrary class of finite structures over the same finite vocabulary which does not contain functional symbols of positive arity.*
The following are equivalent:

1. *K is closed under substructures and direct products, and K has finitely many (up to isomorphism) minimal counterexamples.*
2. *K is definable by a universal Horn sentence.*

Proof. $1 \longrightarrow 2$. By Theorem 5, K is definable by a universal statement. Classical argument (see [5], Proposition 6.2.8) shows how to construct a finite counterexample to preservation by direct products for any universal sentence which is not equivalent to a Horn sentence. Since K is closed under direct products, K is definable by a Horn sentence.

$2 \longrightarrow 1$. Let φ be a universal Horn sentence. Then the set of models of φ is closed wrt direct products and substructures. Since φ is a universal formula, $\neg\varphi$ is existential and has finitely many minimal models. This means that φ has finitely many minimal counterexamples. □

Remark. Observe that the same reasoning again applies to any global relation (not only to a class of structures).

Observe that on arbitrary structures the restriction on minimal models or minimal counterexamples cannot be thrown away. For example, take a class K of (finite) graphs which do not contain cycles. It is closed under substructures and direct products but is not first-order definable. The class of (finite) graphs which do contain cycles is preserved by extensions and direct products as well, but it is not first order definable hence not definable by an existential Horn formula.

Given two structures M and N in the same relational vocabulary, a mapping $h : M \longrightarrow N$ is a *homomorphism* (or a *homomorphism into*) if for every n-ary relation R in the vocabulary of M and N, and for all n-tuples of elements $x_1, \dots, x_n$ in M, if $R_M(x_1, \dots, x_n)$, then $R_N(h(x_1), \dots, h(x_n))$. Observe that if a class K of structures K is closed under homomorphisms, then K is closed under extensions and under adding more tuples in the relations. The latter means that if $M \in K$ and M' is like M except for $R_M \subset R_{M'}$, then $M' \in K$. Let M and N be two structures in the same vocabulary. Call N a *weak substructure* of M if N can be obtained from M by deleting elements or decreasing relations in M. Let $\min_h(K)$ be a collection of $M \in K$ such that no proper weak substructure of M belongs to K.

It is a known fact that if a first order formula is preserved by homomorphisms on arbitrary structures, then it is equivalent to a positive existential formula. It is unknown whether this result remains true on finite structures. We have the following characterization of positive existential first order sentences on finite structures:

Theorem 17. *Let K be a class of finite structures of a given finite vocabulary. The following are equivalent:*

1. *K is closed under homomorphisms and $min_h(K)$ is finite.*
2. *K is definable by a positive existential sentence.*

Proof. The direction from 2 to 1 is easy. Assume 1. Then $\min_h(K)$ has finitely many elements, $M_1, \dots, M_n$. As in the proof of Theorem 5, we describe each of them by an existential formula, but now we describe only the positive diagram of each M_i, $1 \leq i \leq n$. More precisely, let $\Delta_i(\bar{a})$ be the positive diagram of M_i, and let φ_i be $\exists \bar{x} \Delta_i(\bar{x})$. Let φ be the disjunction of φ_i. Obviously, φ is a positive existential formula. It remains to show that φ defines K.

Suppose $M \in K$. Then for some M_i there is a homomorphism $h : M_i \longrightarrow M$. By the classical result, $M \models \varphi_i$, hence $M \models \varphi$.

Suppose $M \models \varphi$. Then for some i, $M \models \varphi_i$. This gives a homomorphism $h : M_i \longrightarrow M$, hence $M \in K$. □

We do not have a semantic characterization of the property 'being positive in a given predicate P' on finite structures (cf. Lyndon's theorem stated in the Introduction). It is straightforward to characterize *existential* sentences positive (negative) in a given predicate P analogously to the theorem above. Given the characterization of existential sentences positive (negative) in a given predicate, the characterization of universal sentences negative (positive) in a given predicate follows. We leave the following theorem as an exercise for the reader.

Theorem 18. *Let φ be an existential or a universal sentence. If φ is increasing (resp. descreasing) in P on finite structures, then it is equivalent, on finite structures, to an existential sentence φ' positive (resp. negative in P).*

4 Undecidability of Semantical Properties

In this section we show that the easiest answer to the question whether a given semantical property has a syntactic characterization (in the sense defined in the Introduction), namely, taking the class of formulas having the property itself as a characterization, often does not work. We prove undecidability results, both for finite and arbitrary models, for a large class of semantical properties. We do it by using Turing machine encodings and recursive inseparability. There are much easier ways of proving undecidability for particular properties. For example, Johan van Benthem pointed to us that monotonicity is undecidable since a formula φ is valid iff $\varphi \vee \neg q$, for a fresh propositional variable q, is upward monotone in q.

For a formal definition of a Turing machine, we refer the reader to e.g. [6]. Here we give a brief and informal description of the kind of Turing machines appropriate to our purposes.

A Turing machine T has a 'head' and a linear tape, divided into cells, which is bounded on the left and unbounded on the right. The head can move one cell at a time to the left or to the right, read and write in a cell a symbol from some finite alphabet $\mathcal{A}$, or erase the current symbol in the cell; let us say that an empty cell contains the letter '*blank*'.

For technical reasons, we introduce an additional symbol α which is not in $\mathcal{A}$ and which occurs in the leftmost cell of the tape. T has finitely many states $q_1, \ldots, q_n$. The instructions of T are of the form: 'If the state is q and you are reading a, replace a with b, move the head one cell to the right (or to the left, or stay in the same cell) and change the state to q''.

Without loss of generality we restrict attention to Turing machines which make $\geq max(3, s_T)$ steps, where s_T is the number of states of T.

An *n-th configuration* of T is the description of T after n steps of its computation.

It is well known that the problem whether a Turing machine with a given set of instructions stops after a finite number of steps (the halting problem) is undecidable (see, for example, [11]).

To formalize Turing machine computations we use several predicates.

Denote the configurations of a given Turing machine T by C_0 (initial configuration), $C_1, \ldots, C_t, \ldots$. Let

$Q(q,t)$ mean that in configuration C_t the state of the machine is q;
$H(i,t)$ mean that in configuration C_t the head is reading cell i;

$L_a(i,t)$ mean that in configuration C_t the letter a is in cell i.

We will use the following formulas in the proofs to follow:

ψ_1 is $\chi_1 \wedge \ldots \wedge \chi_4$ from the proof of Theorem 10 (with a replaced by 0),
ψ_2 says that there are at least $max(3, s_T)$ elements, where s_T is the number of states of T; that for each t, there is precisely one state q and cell i with $Q(q,t)$ and $H(i,t)$; for every pair (i,t), precisely one of $\{L_a(i,t) : a \in \mathcal{A} \cup \{\alpha, blank\}\}$ holds.
ψ_3 is $Q(0,0) \wedge H(0,0) \wedge L_\alpha(0,0) \wedge \forall x(x \neq 0 \rightarrow L_{blank}(x,0))$ (this is the description of C_0);
ψ_4 is conjunction of formulas describing instructions of T, e.g.:

$$\forall y \forall t(Q(q,t) \wedge H(y,t) \wedge L_a(y,t) \rightarrow \exists y' \exists t'(S(t,t') \wedge S(y,y') \wedge Q(q',t') \wedge$$
$$\wedge H(y',t') \wedge L_b(y,t') \wedge \forall v(v \neq y \rightarrow \bigwedge_{a \in \mathcal{A} \cup \{\alpha, blank\}} (L_a(v,t) \rightarrow L_a(v,t'))))$$

Similarly for the case when the head has to move left (replace $S(y,y')$ by $S(y',y)$) or not move (replace $S(y,y')$ by $y = y'$). It is assumed that the head does not move to the left when it reads α.

Recall that two sets X and Y, say of strings in a given alphabet, are called *recursively inseparable* (in short, r.i.) if there is no recursive set R with $X \subseteq R$ and $R \cap Y = \emptyset$ (see, for example, [11]).

We will use the following well known fact:

Lemma 19 (Reduction Lemma). *Suppose (X_1, X_2) are r.i., and there is a recursive function f such that*

$$\forall x(x \in X_1 \rightarrow f(x) \in Y_1)$$
$$\forall x(x \in X_2 \rightarrow f(x) \in Y_2)$$

Then (Y_1, Y_2) are r.i.

This gives us a method of proving that two sets are recursively inseparable by reducing to them some sets which are already known to be recursively inseparable.

Fix two members q_1, q_2 of the alphabet of states. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are r.i., where

$\mathcal{M}_1$ is the set of Turing machines which halt in state q_1.
$\mathcal{M}_2$ is the set of Turing machines which halt in state q_2.

Recursive inseparability of $\mathcal{M}_1$ and $\mathcal{M}_2$ follows from the existence of two r.e. sets of natural numbers which are recursively inseparable (Theorem XII in [11]). Let X_1 and X_2 be two such sets. Then for any natural number n, let T_n be a Turing machine which is computing X_1 and X_2 and halting in q_1 if $n \in X_1$ and in q_2 if $n \in X_2$. The function $f(n) = T_n$ is the required reduction.

Assume that we have proved that sets X and Y are r.i.; then any $X' \supseteq X$ and $Y' \supseteq Y$ are recursively inseparable (otherwise the set separating X' and Y' would also separate X and Y).

Theorem 20. *The following are r.i.:*

The set F_1 of logically false first order formulas.
The set F_2 of first order formulas which have only one finite model of size > 1.

Proof. Given a Turing machine T, we are going to write a formula β_T, such that β_T is logically false if T halts in q_1 and has exactly one finite model if it halts in q_2. In other words,

$$\forall T(T \in \mathcal{M}_1 \to \beta_T \in F_1)$$
$$\forall T(T \in \mathcal{M}_2 \to \beta_T \in F_2)$$

First we give an informal description of β_T. It describes the 'standard model' of the computation of T, that is, it says that the model is linearly ordered, the first element corresponds to the initial configuration, every element has a successor satisfying the requirements imposed by instructions, the last element corresponds to a configuration where the state is q_2 and there is only one such configuration, and there is no configuration where the state is q_1.

More formally (we use the notation introduced above)

$$\beta_T = \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4 \wedge Q(q_2, b) \wedge \forall t(t < b \to \neg Q(q_2, t) \wedge \neg Q(q_1, t))$$

If T halts in q_1, then β_T has neither finite nor infinite model. Indeed, assume that every non-halting configuration has a successor, that is, the linear order is long enough to correspond to the whole computation of T. Then there is a q_1-configuration, or one of the instructions has been violated. If the order is not long enough, again one of the instructions is violated, since the instructions imply the existence of successor.

If T halts in q_2, then β_T has precisely one model, which is finite and has the size equal to the length of the computation of T. □

Observe that this result cannot be strengthened by restricting the cardinality of the model in F_2 by some n. In that case the recursive set of formulas not having a model of cardinality less or equal to n would separate F_1 and F_2.

This gives us a whole range of recursive inseparability / undecidability results for semantical preservation properties. Namely, logically false formulas are usually preserved by semantical constructions. For semantical constructions of interest for us in this paper, it holds that if a formula has precisely one finite model of size > 1, then it is not preserved by this construction.

Corollary 21. *Each of the following pairs are r.i.:*

1. *The set of formulas preserved by extensions on arbitrary models (EPA) and the set of formulas* not *preserved by extensions on finite models (not-EPF);*
2. – *The set of formulas preserved by direct products on arbitrary structures.*
   – *The set of formulas which are not preserved by direct products on finite structures.*
3. – *The set of formulas preserved by direct products and substructures on arbitrary structures.*

- *The set of formulas which are not preserved by direct products and substructures on finite structures.*

4. - *The set of formulas preserved by direct products and extensions on arbitrary structures.*
   - *The set of formulas which are not on preserved by direct products and substructures on finite structures.*
5. - *The set of formulas preserved by homomorphisms on arbitrary structures.*
   - *The set of formulas which are not preserved by homomorphisms on finite struc tures.*
6. - *The set of formulas monotone in a given predicate P on arbitrary structures.*
   - *The set of formulas which are not monotone in P on finite structures.*

Proof. For (1), notice that $F_1 \subseteq EPA$, $F_2 \subseteq \overline{EPF}$. The proofs of (2), (3), (4) and (5) are analogous.

For (6), replace β_T from the previous theorem by

$$\beta'_T = \beta_T \wedge \forall t \neg P(t).$$

If T halts in q_1, β_T has no model, hence β'_T does not. Therefore β'_T is trivially monotone in P. If T halts in q_2, β'_T has exactly one model where P is empty. If P is extended, β'_T fails. Hence in this case β'_T is not monotone in P. □

Corollary 22. *The following sets of formulas are not recursive:*

- *The set of formulas preserved by substructures.*
- *The set of formulas preserved by extensions.*
- *The set of formulas preserved by direct products.*
- *The set of formulas preserved by direct products and substructures.*
- *The set of formulas preserved by direct products and extensions.*
- *The set of formulas preserved by homomorphisms.*
- *The set of formulas monotone in a given predicate P.*

Proof. If any of those classes were recursive, it would separate a pair which we proved to be recursively inseparable. □

Corollary 23. *Corollary 22 remains true if we restrict attention to finite structures.*

5 From Semantics to Syntax

5.1 Łos-Tarski as a Normal Form Theorem

So far, we do not have any syntactic characterization of semantical properties in the context of first-order logic restricted to finite models. Here we give characterizations of EPF and monotonicity in the context of extensions of first-order logic where the notion of a substructure is definable (monadic second order logic, fixed point logic).

But before going into that, we would like to reformulate the Łos-Tarski Theorem as a *normal form theorem* for formulas preserved by extensions.

Proposition 24. *There is a partial recursive function which reduces every EPA formula to an equivalent existential formula.*

Proof. Define f as follows. Take a Turing machine which derives all logically true formulas in the vocabulary of φ; if φ is EPA, then after finitely many steps a tautology of the form $\varphi \leftrightarrow \psi$, where ψ is existential, will appear. Take ψ be $f(\varphi)$. □

Given the results above, there is no total recursive function f which assigns every formula an equivalent formula so that $f(\varphi)$ is existential iff φ is EPA - since that would make EPA decidable. Still the question remains whether there exists a total recursive function f, such that $f(\varphi)$ is an existential formula equivalent to φ in case φ is EPA (and arbitrary otherwise). We are going to show that this is not the case.

We use this opportunity to answer negatively a question of Andréka, van Benthem and Németi in [3]: is there a recursive function f which gives an upper bound on the number of variables needed to write an existential equivalent of an EPA formula, given the number of variables of this formula? In other words, is there a recursive bound on the number of quantifiers in the existential equivalent?[5]

We use the technique from [8], where an analogous result for finite structures is proved. Let $\sharp(\varphi)$ denote the number of quantifiers in φ.

Theorem 25. *There is no total recursive function f such that for every first order EPA formula φ, $f(\varphi)$ is greater or equal to the number of quantifiers needed to write the shortest existential equivalent of φ.*

Proof. By contradiction, assume that there is such a function f.

If ψ is an existential sentence, then every minimal model of ψ has at most $\sharp(\psi)$ elements. So f would also give a bound on the size of the minimal models of EPA sentences.

For every Turing machine T, we show how to write a formula φ_T which corresponds to a computation of T in the following way. If T halts, then φ_T is EPA and has a minimal model which has as many elements as there are steps in the computation of T. If φ_T is EPA, then the size of any minimal model of φ_T is less or equal to $f(\varphi_T)$. Therefore f can be used to solve the halting problem by letting T run for $f(\varphi_T)$ steps: we know that if T halts at all, it halts after at most $f(\varphi_T)$ steps. This shows that such a function f cannot exist.

Now we write down the formula in question. Given a Turing machine T with the halting state 1, let φ_T be as follows:

$$\varphi_T =_{df} \chi_1 \wedge \chi_2 \wedge \chi_3 \to \chi_4 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4 \wedge Q(1,b) \wedge \forall t(t < b \to \neg Q(1,t)).$$

Claim 26. *If T halts, then φ_T is preserved by extensions on arbitrary structures. Moreover, φ_T has a minimal model of the size equal to the number of steps in the computation of T.*

[5] In the meantime, we have found out that this question is also answered in [12].

Proof of the claim. Let $M \models \varphi_T$. If the universal antecedent is false in M, then it will remain false in any extension of M, therefore φ_T is true on all extensions of M.

Assume that $M \models \chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \chi_4 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4 \wedge Q(1,b) \wedge \forall t(t < b \rightarrow \neg Q(1,t))$. Then M is a linear order where the initial point corresponds to the initial configuration of T, and each next point corresponds to the next step in the computation of T. Since T halts, on a finite distance from 0 there will be a point t with $Q(1,t)$. But the last conjunct says that this point has to be b. This gives us a finite rigid linear order as in Theorem 10, but on arbitrary structures. In this case, any extension satisfies the formula because in any proper extension one of $\chi_1 - \chi_3$ is violated. No proper submodel of this model satisfies the formula, because χ_4 is violated, so it is a minimal model.

Now the theorem follows. □

Corollary 27. *There is no total recursive function f such that, for every first order EPA formula φ, $f(\varphi)$ is an existential first order formula equivalent to φ.*

Proof. Assume that such function existed; then it would give a bound on the number of quantifiers of an existential equivalent. But this is impossible. □

5.2 Extensions of First-Order Logic

So far, we have no syntactic characterization of EPF and monotonicity in the context of first order logic. In this section, we show that such a characterization exists in the context of logics where the notion of a substructure is definable, namely (extensions of) monadic second order logic. On ordered finite structures, a characterization can be given in $FO^< + PFP$ (partial fixed point). The following weaker semantical property may be of interest on ordered finite structures: preservation by end extensions. We show that the problem of syntactical characterization of this weaker property has a positive solution in the context of e.g. $FO^< + IFP$ (inflationary fixed point). What is more, the reduction to normal form in all these cases is effective, unlike for first-order logic.

Theorem 28. *Let L be monadic second order logic. There is a linear time algorithm A such that for every formula φ of L, (i) $A(\varphi)$ is EPF and (ii) $A(\varphi)$ is equivalent to φ iff φ is EPF.*

Proof. Consider a formula φ of L. $A(\varphi)$ will be a formula saying 'there is a substructure satisfying φ'. Let X be a monadic second order variable not occurring in φ, and φ_X be the formula obtained from φ by restricting all quantifiers in φ to X, that is, replacing $\forall y$ by $\forall y \in X$, $\exists y$ by $\exists y \in X$, and $\forall Y$ and $\exists Y$ by $\forall Y \subseteq X$ and $\exists Y \subseteq X$, respectively. Then $A(\varphi)$ is $\exists X(\varphi_X \wedge X \neq \emptyset)$.

Obviously, $A(\varphi)$ is EPF. We check that if φ is EPF, then φ is equivalent to $A(\varphi)$. Let φ be EPF. Obviously, any formula φ implies $A(\varphi)$, just take X to be the whole universe. Assume that a structure M satisfies $A(\varphi)$. This means that there is a substructure of M which satisfies φ. Since φ is EPF, $M \models \varphi$.

Finally, if φ is not EPF, then φ is not equivalent to $A(\varphi)$ (which is always EPF). □

Remark. Theorem 28 generalizes to extensions of monadic second order logic.

We denote by $FO^<$ the restriction of first-order logic to the case of structures where $<$ is linear order. In the rest of this section we consider extensions of $FO^<$.

The next normal form uses a partial fixed point operator, i.e. it is about the language $FO^< + PFP$.

Recall that given any $\varphi(X,y)$, the partial fixed point operator generates a sequence $F_0, F_1, \ldots$, such that

$$F_0 = \emptyset \text{ and}$$
$$F_n = \{y : \varphi(F_{n-1}, y)\}.$$

Set $F_\infty = F_i$, if $F_i = F_{i+1}$ for some i, and $\emptyset$ otherwise. The partial fixed point $\mathrm{PFP}_{X,y}\varphi(X,y)$ of $\varphi(X,y)$ with respect to X, y denotes F_∞.

The following theorem is of no surprise.

Theorem 29. *Monadic second order quantification on ordered structures is expressible in $FO^< + PFP$.*

Proof. First we define a formula $\alpha(X,y)$, such that the fixed point operator applied to this formula with respect to X, y generates the list of all substructures of a given structure in a lexicographic order, repeated cyclically infinitely many times.

Let M be a structure with n elements, $1, \ldots, n$. There is a natural ordering on the substructures of M since they correspond to sequences of 0's and 1's of length n:

$$\begin{array}{cccc} 1\,2 & \ldots & n-1 & n \\ 0\,0 & \ldots & 0 & 0 \\ 0\,0 & \ldots & 0 & 1 \\ 0\,0 & \ldots & 1 & 0 \\ 0\,0 & \ldots & 1 & 1 \\ & \ldots & & \\ 1\,1 & \ldots & 1 & 1 \end{array}$$

Let us call the subset of M corresponding to the ith number in this ordering F_i, with $F_0 = \emptyset$. Let us call F_{i+1} the successor of F_i, and F_0 the successor of F_{2^n-1}. Observe that given F_i for any $0 \le i < 2^n$, we can describe the elements in its successor by a first order formula $\alpha(F_i, y)$. Namely, the maximal element of the complement of F_i is in F_{i+1}, and all elements such that they are in F_i and they are less than some element which is not in F_i, should be in F_{i+1}:

$$\alpha(F_i, y) =_{df} (\neg F_i(y) \wedge \forall z(\neg F_i(z) \to z \le y)) \vee (F_i(y) \wedge \exists z(y \le z \wedge \neg F_i(z)))$$

Now it is easy to check that $\mathrm{PFP}_{X,y}\alpha(X,y)$ generates the sequence described above.

Using the formula defined above, we can translate any expression of monadic second order logic into an $FO^< + PFP$ expression equivalent to it on ordered structures. Let $\exists X\varphi(X)$ be a monadic second order formula. Suppose we know how to translate $\varphi(X)$ into a $FO^{<} + PFP$ formula $\varphi'(X)$. Consider $\mathrm{PFP}_{X,y}\theta(X, y)$, where

$$\theta(X, y) = (\varphi'(X) \wedge X(y)) \vee (\neg\varphi'(X) \wedge \alpha(X, y)).$$

The operator starts with the empty X and checks whether $\varphi'(X)$ is true. If it is, then $F_\infty = \emptyset$. If not, the next substructure is checked. Eventually either a nonempty substructure X satisfying $\varphi(X)$ is found, or the fixed point is empty. The translation of $\exists X\varphi(X)$ is therefore $\varphi'(\emptyset) \vee \exists x(\mathrm{PFP}_{X,y}\theta(X, y))(x)$. □

Theorem 30. *Let L be $FO^{<} + PFP$. Then there is a linear time algorithm A, such that for every formula φ of L, (i) $A(\varphi)$ is EPF and (ii) $A(\varphi)$ is equivalent to φ iff φ is EPF.*

Proof. From the theorem above and the existence of the algorithm for monadic second order logic. □

It is impossible to check all substructures using the inflationary fixed point operator,[6] where the sequence generated by the operator is always increasing, i.e. $F_i \subseteq F_{i+1}$. If we restrict attention to the structures in the language of $<$, S and $=$ satisfying χ_1, χ_2, χ_{31} and χ_4, then the only legitimate substructures are those given by initial segments. We denote the first order part of such languages by $FO^{<,S}$. In this case, monadic second order quantification over substructures can be defined in a weaker fixed point logic, namely in $FO^{<,S} + IFP$. We leave this as an exercise to the reader.

Theorem 31. *Let L be $FO^{<,S} + IFP$. Then there is a linear time algorithm A, such that for every formula φ of L, (i) $A(\varphi)$ is EPF and (ii) $A(\varphi)$ is equivalent to φ iff φ is EPF.*

Proof. Analogously to Theorem 30. □

As before, the theorem also holds for extensions of $FO^{<,S} + IFP$.

It is known that Lyndon's theorem fails on finite structures; there is a first order sentence monotone in a given predicate P which is not equivalent to any formula positive in P ([1]). A question arises, is there any alternative characterization of monotonicity on finite structures?

We do not know the answer in the context of first-order logic; here we give a characterization in monadic second order logic and partial fixed point logic on ordered structures using the same trick as above.

Theorem 32. *Let L be monadic second order logic. There is a linear time algorithm A such that for every formula φ of L, (i) $A(\varphi)$ is positive in a predicate P and (ii) $A(\varphi)$ is equivalent to φ iff φ is monotone in P.*

[6] For a definition of inflationary fixed point operator, see [9].

Proof. Define $A(\varphi)$ to be $\exists X \subseteq P(X \neq \emptyset \wedge \varphi[P/X])$, where X is a new variable and $\varphi[P/X]$ denotes the result of replacing all occurrences of P in φ by X. The rest is as in Theorem 28. □

Theorem 33. *Let L be any extension of $FO^< + PFP$. Then there is a linear time algorithm A, such that for every formula φ of L, (i) $A(\varphi)$ is positive in a predicate P and (ii) $A(\varphi)$ is equivalent to φ iff φ is monotone in P.*

Proof. Follows from the theorem above and the fact that monadic second order quantification is definable in $FO^< + PFP$. □

6 Some Open Questions

Many open problems are implicit in the text above. Let us formulate some of them explicitly.

Problem 1. Is it true that, for every first order sentence φ, the following are equivalent:

1. φ is preserved by substructures and direct products on finite structures.
2. φ is equivalent, on finite structures, to a universal Horn formula?

Problem 2. Is it true that the following are equivalent:

1. φ is preserved by homomorphisms on finite structures.
2. φ is equivalent, on finite structures, to a positive existential first order sentence? [7]

Remark. The positive solution for this problem announced by Gurevich and Shelah in [10], has collapsed. Among the survivors is their construction, also announced in [10], of a series of formulas φ_n preserved by homomorphisms, such that the size of the smallest model of φ_n is a tower $2^{2^{2^{\cdot^{\cdot}}}}$ of n twos.

Problem 3. Does there exist a recursive class F of first order formulas such that

1. Every F-sentence has the property $\mathcal{P}$ on finite structures.
2. Every first order sentence having $\mathcal{P}$ is equivalent, on finite structures, to some F-sentence,

where $\mathcal{P}$ is one of: preserved by extensions, preserved by products, monotone in a given predicate.

Problem 4. Give semantical characterization, on finite structures, of the following classes of first order sentences: Horn sentences; sentences positive in a given predicate.

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[7] This question is due to Phokion Kolaitis.

References

1. M. Ajtai and Y. Gurevich. Monotone versus positive. *Journal of ACM*, 34:1004 – 1015, 1987.
2. M. Ajtai and Y. Gurevich. DATALOG vs. first-order logic. In *Proceedings 30th IEEE Symposium on Foundations of Computer Science*, pages 142 – 146, 1989.
3. H. Andréka, J. van Benthem, and I. Németi. Submodel preservation theorems in finite variable fragments. In M. de Rijke A. Ponse and Y. Venema, editors, *Modal Logic and Process Algebra.* Cambridge University Press, 1994.
4. J. Barwise and J. van Benthem. Interpolation, preservation, and pebble games. Technical Report ML-96-12, ILLC, University of Amsterdam, 1996.
5. C. C. Chang and H. J. Keisler. *Model theory.* North Holland, 1973.
6. H. - D. Ebbinghaus and J. Flum. *Finite model theory.* Springer, 1995.
7. J. Flum. Private communication.
8. Y. Gurevich. Toward logic tailored for computational complexity. In M. M. Richter et al., editor, *Computation and Proof Theory.* Springer lecture Notes in Mathematics 1104, 1984.
9. Y. Gurevich. Logic and the challenge of computer science. In E. Börger, editor, *Current Trends in Theoretical Computer Science*, pages 1 – 57. Computer Science Press, 1988.
10. Y. Gurevich. On finite model theory. In S. R. Buss and P. J. Scott, editor, *Feasible Mathematics*, pages 211 – 219. Birkhauser, 1990.
11. H. Rogers. *Theory of recursive functions and effective computability.* MIT Press, 1992.
12. E. Rosen and S. Weinstein. Preservation theorems in finite model theory. In *Logic and Computational Complexity,* volume 960 of *Lecture Notes in Computer Science*, pages 480–502, 1995.
13. W. Tait. A counterexample to a conjecture of Scott and Suppes. *Journal of Symbolic Logic*, 24:15 – 16, 1959.