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# A general method for proving decidability of intuitionistic modal logics ☆

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**Abstract**

We generalise the result of [H. Ganzinger, C. Meyer, M. Veanes, The two-variable guarded fragment with transitive relations, in: Proc. 14th IEEE Symposium on Logic in Computer Science, IEEE Computer Society Press, 1999, pp. 24–34] on decidability of the two variable monadic guarded fragment of first order logic with constraints on the guard relations expressible in monadic second order logic. In [H. Ganzinger, C. Meyer, M. Veanes, The two-variable guarded fragment with transitive relations, in: Proc. 14th IEEE Symposium on Logic in Computer Science, IEEE Computer Society Press, 1999, pp. 24–34], such constraints apply to one relation at a time. We modify their proof to obtain decidability for constraints involving several relations. Now we can use this result to prove decidability of multi-modal modal logics where conditions on accessibility relations involve more than one relation. Our main application is intuitionistic modal logic, where the intuitionistic and modal accessibility relations usually interact in a non-trivial way.


*Keywords:* Intuitionistic modal logics; Decidibility

## 1. Introduction

In this paper, we present a new general way of proving decidability of multi-modal modal logics. This method relies on the result of Ganzinger, Meyer and Veanes [14], that a

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monadic two-variable guarded fragment $GF^2_{\text{mon}}$ of classical first-order logic, where guard relations satisfy conditions that can be expressed as monadic second-order definable closure constraints, is decidable. Our contribution is a slight generalisation of this result to account for conditions which involve more than one guard relation. We believe that this method is particularly promising for intuitionistic modal logic, where there exists a variety of systems, most of them semantically defined, with various conditions connecting the intuitionistic and modal accessibility relations. General results on decidability and finite modal property of intuitionistic modal logic have been proved in [33–35] using an embedding of intuitionistic modal logics with $n$ modalities in classical modal logics with $n+1$ modalities. However, their results can only be used to prove decidability of those intuitionistic modal logics, for which the corresponding classical logic is known to be decidable.

The decidability proof presented in this paper does not give a good decision procedure, since it proceeds by reduction to satisfiability of formulas of *SkS* (monadic second-order theory of trees with constant branching factor $k$, [28]) which is non-elementary.[1] It does however provide a rather simple way to establish decidability, before looking for a decision procedure tailored for a particular logic.

## 2. Two-variable monadic guarded fragment

We start by defining $GF^2_{\text{mon}}$ as introduced in [14]. In the following definitions, $FV(\varphi)$ stands for the set of free variables of $\varphi$, and $\bar{x}$ stands for a sequence of variables. We assume a first order language which contains predicate letters of arbitrary arity, including equality $=$, and no constants or functional symbols.

**Definition 1.** The guarded fragment *GF* of first-order logic is the smallest set that contains all first-order atoms and is closed under boolean connectives and the following rule: if $\rho$ is an atom, $\varphi \in GF$, and $\bar{x} \subseteq FV(\varphi) \subseteq FV(\rho)$, then $\exists\bar{x}(\rho \wedge \varphi)$ and $\forall\bar{x}(\rho \to \varphi) \in GF$ (in such a case $\rho$ is called a guard).

The monadic two-variable guarded fragment $GF^2_{\text{mon}}$ is the subset of *GF* containing formulas $\varphi$ such that (i) $\varphi$ has no more than two variables (free or bound), and (ii) all non-unary predicate letters of $\varphi$ occur in guards.

## 3. Closure conditions

In this section we define the form of conditions on guards in $GF^2_{\text{mon}}$ which yield decidable fragments. We generalise the notion of mso-definable (monadic second order definable) closure conditions from [14] so that they can apply to more than one relation.

**Definition 2.** Let $W$ be a non-empty set. A unary function $C$ on $W$ is a *simple closure operator* if, for all $\mathcal{P}, \mathcal{P}' \subseteq W$,

[1] Better complexity bounds for the guarded fragment with transitive guards were obtained in [17] and [31], however their results apply only to transitivity, and it is not clear whether they could be extended to arbitrary closure conditions, which we need for intuitionistic modal logics.

(1) $\mathcal{P} \subseteq C(\mathcal{P})$ ($C$ is increasing),
(2) $\mathcal{P} \subseteq \mathcal{P}'$ implies $C(\mathcal{P}) \subseteq C(\mathcal{P}')$ ($C$ is monotone),
(3) $C(\mathcal{P}) = C(C(\mathcal{P}))$ ($C$ is idempotent).

An $(n+1)$-ary function $C$ on the powerset of $W$ is a *parametrised closure operator* if $C(\mathcal{P}_1, \ldots, \mathcal{P}_n, -)$ for any $\mathcal{P}_1, \ldots, \mathcal{P}_n \subseteq W$ is a simple closure operator. We use the notation $C^{\mathcal{P}_1,\ldots,\mathcal{P}_n}$ for a closure operator parametrised by $\mathcal{P}_1, \ldots, \mathcal{P}_n$.

**Example 3.** A reflexive, transitive closure operator for binary relations $TC(\mathcal{P})$ is a simple closure operator.

**Example 4.** A function $Incl^{\mathcal{P}'}(\mathcal{P}) = \mathcal{P}' \cup \mathcal{P}$ is a closure operator parametrised by $\mathcal{P}'$.

**Definition 5.** A condition on relation $\mathcal{P}$ is a *simple closure condition* if it can be expressed in the form $C(\mathcal{P}) = \mathcal{P}$, where $C$ is a simple closure operator.

A condition on relation $\mathcal{P}$ is a *parametrised closure condition* if it can be expressed in the form $C^{\mathcal{P}_1,\ldots,\mathcal{P}_n}(\mathcal{P}) = \mathcal{P}$, where $C^{\mathcal{P}_1,\ldots,\mathcal{P}_n}$ is a parametrised closure operator.

**Example 6.** Reflexivity-and-transitivity is a simple closure condition, since it can be expressed in the form $TC(\mathcal{P}) = \mathcal{P}$.

**Example 7.** Condition $\mathcal{P}' \subseteq \mathcal{P}$ is a closure condition on $\mathcal{P}$ parametrised by $\mathcal{P}'$, since it can be stated as $Incl^{\mathcal{P}'}(\mathcal{P}) = \mathcal{P}$.

Given a set of closure conditions on a set of relations $S$, we want to preclude circularity while closing off relations in $S$.

**Definition 8.** Let $S$ be a finite set of relations, $\mathbf{C}$ a set of closure conditions on those relations, and $\mathbf{C}(\mathcal{P})$ be all the closure conditions on the relation $\mathcal{P}$ from $\mathbf{C}$. $\mathbf{C}$ is *acyclic* if there is an ordering $\mathcal{P}_1, \ldots, \mathcal{P}_n$ of $S$ such that all parameters in $\mathbf{C}(\mathcal{P}_{i+1})$ come from $\mathcal{P}_1, \ldots, \mathcal{P}_i$.

Furthermore, we are not interested in arbitrary closure operators, but only in those definable in monadic second-order logic. Let $\|\varphi(x_1, \ldots, x_n)\|^{\mathcal{M}}$ stand for the set of $n$-tuples satisfying $\varphi$ in model $\mathcal{M}$.

**Definition 9.** A closure operator $C^{\mathcal{P}_1,\ldots,\mathcal{P}_m}$ on $n$-ary relations is mso (-definable), if there exists a monadic second-order formula $\overline{C_P^{P_1,\ldots,P_m}}$ with predicate parameters $P_1, \ldots, P_m$ and $P$, such that, for any model $\mathcal{M}$ and any $n$-ary formula $\varphi$,

$$C^{\mathcal{P}_1,\ldots,\mathcal{P}_m}\big(\|\varphi\|^{\mathcal{M}}\big) = \big\|\overline{C_P^{P_1,\ldots,P_m}}(\varphi/P)\big\|^{\mathcal{M}}$$

**Example 10.** The closure operator $TC$ is definable by the mso formula

$$\overline{TC}_P(z_1, z_2) = \forall X\big(X(z_1) \wedge \forall x, y\big(X(x) \wedge P(x, y) \to X(y)\big) \to X(z_2)\big)$$

To see that $\overline{TC_P}$ defines the reflexive, transitive closure of $P$, assume that there is a $P$-chain $a_1 \xrightarrow{P} a_2 \ldots a_{n-1} \xrightarrow{P} a_n$, connecting $a_1$ and $a_n$, and that $X(a_1)$ and $\forall x, y(X(x) \wedge P(x, y) \to X(y))$ hold. Then $X(a_1)$ implies $X(a_2)$, $X(a_2)$ implies $X(a_3)$, etc., $X(a_n)$ is true, so $\overline{TC_P}(a_1, a_n)$ is true. Conversely, suppose there is no $P$-chain connecting $a_1$ and $a_n$. We can assign to $X$ the set containing $a_1$ and all the elements $P$-reachable from $a_1$, which makes $X(a_n)$ and $\overline{TC_P}(a_1, a_n)$ false.

**Example 11.** The closure operator $Incl^{\mathcal{P}'}$ is definable by the mso (in fact, first-order) formula

$$\overline{Incl_P^{P'}}(z_1, z_2) = P'(z_1, z_2) \vee P(z_1, z_2)$$

**Theorem 12.** *Let $\phi \in GF^2_{\text{mon}}$ and* **C** *be an acyclic set of mso closure conditions on relations in $\phi$ so that at most one closure condition is associated with each relation. It is decidable whether $\phi$ is satisfiable in a model satisfying* **C**.

**Proof.** The proof is very similar to the proof given in [14] for non-parametrised closure conditions. In fact, it is slightly simpler, because in the original proof all relations are assumed to be closed under equivalence (to show decidability of the fragment with equality). However, closure under equivalence is a special case of a parametrised closure condition, so we do not need to treat it separately.

Let $\phi \in GF^2_{\text{mon}}$ and let **C** be an acyclic set of mso closure conditions on relations in $\phi$. $\phi$ is satisfiable in a model satisfying **C** iff $N$, the Skolemised form of $\phi$, is satisfiable in a Herbrand model in which all conditions from **C** hold. The idea of the decidability proof is to reduce the latter problem to satisfiability of formulas of *SkS* (the mso theory of trees with constant branching factor $k$), where $k$ is the number of Skolem function symbols in $N$. We construct an mso formula $MSO_N$, in the vocabulary of *SkS* (an mso formula containing only unary relation variables, unary functions and equality), such that $MSO_N$ is satisfiable in a tree model iff $N$ has a Herbrand model satisfying closure conditions from **C**. The construction proceeds in three stages: defining counterparts for predicate letters, for clauses in $N$ and finally for $N$ itself.

*Stage* 1. For each predicate $P$ in $N$, construct a formula $\varphi_P$ in the vocabulary of *SkS*.

Let $P(\overline{t_1}), \ldots, P(\overline{t_m})$ be all positive literals of $N$ containing $P$. Note that since $\phi \in GF^2_{\text{mon}}$, each $P$ is either a unary or a binary predicate; each positive literal will contain at most one free variable. For each $P(\overline{t_i})$ above, a new unary second-order variable $X_{P(\overline{t_i})}$ is introduced. Let $\bar{t}[z]$ be the result of substituting a variable $z$ for the free variable of $\bar{t}$. Then, if $P$ is a unary predicate,

$$\varphi_P(z_1) = \bigvee_{i=1}^{m} \exists z \big(X_{P(t_i)}(z) \wedge z_1 = t_i[z]\big)$$

and if $P$ is a binary predicate,

$$\varphi_P(z_1, z_2) = \bigvee_{i=1}^{m} \exists z \big(X_{P(t_{i1}, t_{i2})}(z) \wedge z_1 = t_{i1}[z] \wedge z_2 = t_{i2}[z]\big)$$

Intuitively, the relation defined by $\varphi_P$ is the minimal extension of $P$.

Next, for each predicate that has a closure condition imposed on it, we define the closure $\psi_P$ of $\varphi_P$ with respect to the closure condition on $P$. For each such $P$ we have a single closure condition $C_P$, which may be parametrised by other predicates. For simplicity, assume that $C_P$ is parametrised by a single predicate $P'$ that, in its own turn, has a simple closure condition $C_{P'}$. We know, then, that $C_{P'}$ is definable by an *MSO* formula $\overline{C_{P'}}(z_1, z_2)$ containing $P'$, and $C_P$ is definable by an *MSO* formula $\overline{C_P^{P'}}(z_1, z_2)$, containing $P'$ and $P$. First, we define the closure of $P'$ with respect to its simple closure condition:

$$\psi_{P'}(z_1, z_2) = \overline{C_{P'}}(z_1, z_2)[\varphi_{P'}/P']$$

that is, we replace every occurrence of $P'$ in $\overline{C_{P'}}(z_1, z_2)$ with $\varphi_{P'}$.

Next, we define the closure of $P$ with respect to its parametrised condition:

$$\psi_P(z_1, z_2) = \overline{C_P^{P'}}(z_1, z_2)[\psi_{P'}/P', \varphi_P/P]$$

In general, for any acyclic set **C** of conditions on the collection of relations $S$, we first define the simple closures, then the closures parametrised by relations with simple closure conditions, etc. The acyclicity of **C** ensures that this procedure can be carried out.

*Stage* 2. For each clause $\chi = \{\rho_1, \ldots, \rho_l\}$ in $N$, construct a formula $MSO_\chi$ in the vocabulary of *SkS*.

For every literal $\rho$ in $\chi$, a formula $MSO_\rho$ is defined according to the following rule:

$$MSO_\rho = \begin{cases} X_\rho(x), & \text{if } \rho \text{ is a non-ground atom containing } x \\ \exists z X_\rho(z), & \text{if } \rho \text{ is a ground atom} \\ \neg\psi_P(\bar{t}), & \text{if } \rho \text{ is } \neg P(\bar{t}) \end{cases}$$

where $\psi_P$ is the formula constructed at stage 1. Now $MSO_\chi$ is defined as $MSO_\chi = \bigvee_{\rho\in\chi} MSO_\rho$.

*Stage* 3. Finally, $MSO_N = \exists\overline{X}\forall\bar{x} \bigwedge_{\chi\in N} MSO_\chi$, where $\overline{X}$ are all the free second order variables and $\bar{x}$ are all the first order variables in $\bigwedge_{\chi\in N} MSO_\chi$.

It remains to show that $N$ has a Herbrand model satisfying the closure conditions in **C** iff $MSO_N$ is satisfiable in a tree. Let $\mathcal{T}$ be the tree corresponding to the term algebra of the Herbrand universe of $N$.

($\Leftarrow$) Assume that $N$ has a Herbrand model $\mathcal{A}$ satisfying the closure conditions in **C**. We want to show that $\mathcal{T}$ satisfies $MSO_N$. Fix witnesses for second-order variables $X_\rho$ of $MSO_N$ as follows:

(i) If $\bar{t_i}$ is non-ground, then $X_{P(\bar{t_i})} = \{a\colon \mathcal{A} \models P(\bar{t_i}[a])\}$.
(ii) If $\bar{t_i}$ is ground, then $X_{P(\bar{t_i})}$ is a non-empty set.

We know that for each clause $\chi$ of $N$, and each tuple $\bar{a}$, $\mathcal{A} \models \chi(\bar{a})$. This means that for each $\bar{a}$, there is a literal $\rho$ in $\chi$ such that $\mathcal{A} \models \rho(\bar{a})$. We show that for any $\bar{a}$ and $\rho$, if $\mathcal{A} \models \rho(\bar{a})$, then $\mathcal{T} \models MSO_\rho(\bar{a})$. Hence $\mathcal{A} \models \chi(\bar{a})$ implies $\mathcal{T} \models MSO_\chi(\bar{a})$.

There are three cases to consider, depending on the form of $\rho$. The first two (non-ground atom $P(\bar{t_i})$ and ground atom) are exactly the same as in [14]. If $\rho$ is a negative literal $\neg P(\bar{t_i})$, we need to show that $\mathcal{T} \models \neg\psi_P(\bar{t})(\bar{a})$. It suffices to show that $\|\psi_P\|^{\mathcal{A}} \subseteq P^{\mathcal{A}}$. Indeed, this, together with our assumption that $\mathcal{A} \models \neg P(\bar{t})[\bar{a}]$, implies $\mathcal{T} \models \neg\psi_P(\bar{a})$.

First, the definition of $\mathcal{T}$ guarantees that $\|\varphi_P\|^{\mathcal{A}} \subseteq P^{\mathcal{A}}$. Hence, by monotonicity of closure operators, $C_P^{P_1^{\mathcal{A}}}(\|\varphi_P\|^{\mathcal{A}}) \subseteq C_P^{P_1^{\mathcal{A}}}(P^{\mathcal{A}})$. By definition of $\psi_P$, $C_P^{P_1^{\mathcal{A}}}(\|\varphi_P\|^{\mathcal{A}}) = \|\psi_P\|^{\mathcal{A}}$; furthermore, since $\mathcal{A}$ satisfies conditions in **C**, $C_P^{P_1^{\mathcal{A}}}(P^{\mathcal{A}}) = P^{\mathcal{A}}$; hence, $\|\psi_P\|^{\mathcal{A}} \subseteq P^{\mathcal{A}}$.

($\Rightarrow$) Assume that $MSO_N$ is true in $\mathcal{T}$. Define a Herbrand model $\mathcal{A}$ as follows. The universe of $\mathcal{A}$ is the set of nodes of $\mathcal{T}$, and $P^{\mathcal{A}} = \|\psi_P\|$. First, we prove that $\mathcal{A}$ satisfies closure conditions **C**. To this end, we have to show that $C_P^{\mathcal{P}_1}(P^{\mathcal{A}}) = P^{\mathcal{A}}$. Indeed, $C_P^{P_1^{\mathcal{A}}}(P^{\mathcal{A}}) = C_P^{P_1^{\mathcal{A}}}(\|\psi_P\|) = C_P^{\|\psi_{P_1}\|}(C_P^{\|\psi_{P_1}\|}(\|\varphi_P\|)) = C_P^{\|\psi_{P_1}\|}(\|\varphi_P\|) = \|\psi_P\| = P^{\mathcal{A}}$.

Finally, we need to show that $\mathcal{A}$ satisfies all clauses in $N$. This part of the proof is exactly the same as in [14]. $\square$

## 4. Intuitionistic modal logics

One of the most promising applications of the result above is propositional intuitionistic modal logic. Intuitionistic modal logic is simply a modal logic with intuitionistic, rather than classical, base. The work on intuitionistic modal logic has several motivations: mathematical interest; preference for intuitionistic rather than classical logic; desire to give intuitionistic account of the notions studied in modal logic; and suitability of intuitionistic modal logic for modelling certain computational phenomena. There exists an extensive literature on intuitionistic modal logics, for example [5–7,10,12,13,15,20,22,23,26,27,32–35]. A comprehensive survey can be found in [29]; for later references, see [36] and [24].

One of the motivations for intuitionistic modal logic is modelling computational phenomena. A considerable strand of work in this area is based on the work by Moggi [21] who extended a typed λ-calculus style semantics for functional programming languages with an additional construct—a monad—to model effects in functional programming languages (such as the raising of exceptions etc.). The correspondence between simply-typed λ-calculus and intuitionistic propositional logic is well known; it turns out that monads correspond to S4-type modalities. This created a considerable interest in intuitionistic S4 modal logic, its proof theory and categorical and Kripke semantics [1,3,4,8,9,16,18,24,25]. Other applications of intuitionistic modal logic to modelling computational phenomena included modelling incomplete information [32], communicating systems [30], hardware verification [11,19], etc.

Intuitionistic modal languages are obtained by adding either or both of the unary connectives $\Box$ (necessity) and $\Diamond$ (possibility) to the language of propositional intuitionistic logic, which contains a set of propositional parameters $\mathtt{Par} = \{p_1, p_2, \ldots\}$, a unary connective $\sim$, and binary connectives $\wedge$, $\vee$, and $\Rightarrow$. Analogously to $\forall$ and $\exists$, in intuitionistic logic $\Box$ and $\Diamond$ are not required to be dual. It is also to be expected that for example $\Box(\varphi \vee \sim\varphi)$ is not valid. In some intuitionistic modal logics, $\Diamond(\varphi \vee \psi) \equiv (\Diamond\varphi \vee \Diamond\psi)$ is not valid either; see for example [32].

Kripke semantics of intuitionistic modal logics extends Kripke semantics for intuitionistic propositional logic. An intuitionistic Kripke model is a structure $\mathcal{M} = (W, \mathcal{R}, V)$ such that (i) $W \neq \emptyset$, (ii) $\mathcal{R}$ is a reflexive and transitive binary relation on $W$, and (iii) $V$ is a function from $\mathtt{Par}$ into the powerset of $W$ such that, for all $w \in W$ and $p \in \mathtt{Par}$, if $w \in V(p)$ and $w\mathcal{R}v$, then $v \in V(p)$ (condition we will refer to as upward persistence for propositional variables). Elements of $W$ are called nodes. Truth at a node is defined as follows ($\to$ and $\neg$ stand for classical implication and negation, respectively):

$$\begin{array}{lll}
\mathcal{M}, w \models p & \text{iff} & w \in V(p); \\
\mathcal{M}, w \models \sim\varphi & \text{iff} & \forall v\big(\mathcal{R}(w,v) \to \neg(\mathcal{M}, v \models \varphi)\big); \\
\mathcal{M}, w \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi; \\
\mathcal{M}, w \models \varphi \vee \psi & \text{iff} & \mathcal{M}, w \models \varphi \text{ or } \mathcal{M}, w \models \psi; \\
\mathcal{M}, w \models \varphi \Rightarrow \psi & \text{iff} & \forall v\big(\mathcal{R}(w,v) \to \neg(\mathcal{M}, v \models \varphi) \text{ or } \mathcal{M}, v \models \psi\big);
\end{array}$$

To accommodate formulas of the form $\Box\varphi$ and $\Diamond\varphi$, intuitionistic Kripke models are augmented with binary relations $\mathcal{R}_\Box$ and $\mathcal{R}_\Diamond$. There is no single accepted way of defining the meaning of $\Box$ and $\Diamond$ in intuitionistic logic. The following clauses are encountered in the literature (see Chapter 3 of [29] for a comprehensive survey):

$$\begin{array}{lll}
(\Box_1)\ \mathcal{M}, w \models \Box\varphi & \text{iff} & \forall v(w\mathcal{R}_\Box v \to \mathcal{M}, v \models \varphi) \\
(\Box_2)\ \mathcal{M}, w \models \Box\varphi & \text{iff} & \forall v\big(w\mathcal{R}v \to \forall u(v\mathcal{R}_\Box u \to \mathcal{M}, u \models \varphi)\big) \\
(\Diamond_1)\ \mathcal{M}, w \models \Diamond\varphi & \text{iff} & \exists v(w\mathcal{R}_\Diamond v \wedge \mathcal{M}, v \models \varphi) \\
(\Diamond_2)\ \mathcal{M}, w \models \Diamond\varphi & \text{iff} & \forall v\big(w\mathcal{R}v \to \exists u(v\mathcal{R}_\Diamond u \wedge \mathcal{M}, u \models \varphi)\big)
\end{array}$$

Observe that definition $(\Diamond_2)$ gives rise to a modality which does not distribute over disjunction (hence to a non-normal modal logic).

On top of the requirement that $\mathcal{R}$ is reflexive and transitive, some additional conditions are usually imposed on $\mathcal{R}$, $\mathcal{R}_\Box$, and $\mathcal{R}_\Diamond$. As a rule, these conditions specify the way $\mathcal{R}$, $\mathcal{R}_\Box$, and $\mathcal{R}_\Diamond$ interact. For example, the following conditions usually accompany truth clauses $(\Box_1)$ and $(\Diamond_1)$ (see [34]):

$$\mathcal{R} \circ \mathcal{R}_\Box \circ \mathcal{R} = \mathcal{R}_\Box \tag{1}$$

$$\mathcal{R} \circ \mathcal{R}_\Diamond^{-1} \circ \mathcal{R} = \mathcal{R}_\Diamond^{-1} \tag{2}$$

In the conditions above, $\circ$ stands for relational composition:

$$\mathcal{R} \circ \mathcal{R}' = \big\{\langle a, b\rangle\colon\ \exists c\ (\langle a, c\rangle \in \mathcal{R}\ \&\ \langle c, b\rangle \in \mathcal{R}')\big\}$$

Another condition occurring in the literature (see for example [11]) stipulates that

$$\mathcal{R}_\Diamond \subseteq \mathcal{R} \tag{3}$$

It turns out that many of the conditions on $\mathcal{R}$, $\mathcal{R}_\Box$ and $\mathcal{R}_\Diamond$, including conditions (1)–(3) above, are mso-definable closure conditions as introduced in Section 3. For condition (3), see Examples 4 and 7. Below is a proof for (1) and (2).

**Proposition 13.** *Condition of the form $\mathcal{P} = \mathcal{P}' \circ \mathcal{P} \circ \mathcal{P}'$ is an mso-definable closure condition, provided that $\mathcal{P}'$ is reflexive and transitive.*

**Proof.** Consider a function $Comp^{\mathcal{P}'}(\mathcal{P}) = \mathcal{P}' \circ \mathcal{P} \circ \mathcal{P}'$. If $\mathcal{P}'$ is reflexive and transitive, then $\mathcal{P} \subseteq \mathcal{P}' \circ \mathcal{P} \circ \mathcal{P}'$ by the reflexivity of $\mathcal{P}'$. $\mathcal{P}' \circ \mathcal{P} \circ \mathcal{P}'$ is obviously monotone in $\mathcal{P}$;

and $Comp^{\mathcal{P}'}$ is idempotent because of the transitivity of $\mathcal{P}'$. This proves that $Comp^{\mathcal{P}'}$ is a closure operator provided that $\mathcal{P}'$ is reflexive and transitive. Conditions of the form $\mathcal{P}' \circ \mathcal{P} \circ \mathcal{P}' = \mathcal{P}$ can be expressed as closure conditions: $Comp^{\mathcal{P}'}(\mathcal{P}) = \mathcal{P}$. This condition is mso-definable; in fact, it is definable by a first order formula:

$$\overline{Comp^{\mathcal{P}'}_{\mathcal{P}}}(z_1, z_2) = \exists x \exists y \big(P'(z_1, x) \wedge P(x, y) \wedge P'(y, z_2)\big) \qquad \square$$

## 5. Embedding into the two-variable monadic fragment

In this section, we show that every intuitionistic modal logic $L$ defined semantically with any of the truth clauses $(\square_1)$–$(\diamond_2)$ can be translated into $GF^2_{\mathrm{mon}}$.

We define, by mutual recursion, two translations, $\tau_x$ and $\tau_y$, so that a first-order formula $\tau_v(\varphi)$ $(v \in \{x, y\})$ contains a sole free variable $v$, which intuitively stands for the world at which $\varphi$ is being evaluated in the Kripke model. $\tau_x$ is defined by

$$\begin{aligned}
&\tau_x(p) := P(x)\\
&\tau_x(\sim\varphi) := \forall y\big(R(x, y) \rightarrow \neg\tau_y(\varphi)\big)\\
&\tau_x(\varphi \wedge \psi) := \tau_x(\varphi) \wedge \tau_x(\psi)\\
&\tau_x(\varphi \vee \psi) := \tau_x(\varphi) \vee \tau_x(\psi)\\
&\tau_x(\varphi \Rightarrow \psi) := \forall y\big(R(x, y) \rightarrow \big(\neg\tau_y(\varphi) \vee \tau_y(\psi)\big)\big)\\
&\tau_x(\square\varphi) := \forall y\big(R(x, y) \rightarrow \forall x\big(R_\square(y, x) \rightarrow \tau_x(\varphi)\big)\big)\\
&\tau_x(\diamond\varphi) := \forall y\big(R(x, y) \rightarrow \exists x\big(R_\diamond(y, x) \wedge \tau_x(\varphi)\big)\big)
\end{aligned}$$

$\tau_y$ is defined analogously, switching the roles of $x$ and $y$. This translation assumes modal truth clauses $(\square_2)$ and $(\diamond_2)$. Clauses for $(\square_1)$ and $(\diamond_1)$ are even simpler (and familiar from classical modal logic):

$$\begin{aligned}
&\tau'_x(\square\varphi) := \forall y\big(R_\square(x, y) \rightarrow \tau'_y(\varphi)\big)\\
&\tau'_x(\diamond\varphi) := \exists y\big(R_\diamond(x, y) \wedge \tau'_y(\varphi)\big)
\end{aligned}$$

Not surprisingly, since $\tau_x$ is a natural generalisation of the standard translation of modal logic into classical predicate logic, the following theorem holds:

**Theorem 14.** *Let $\phi$ be an intuitionistic modal formula and $\mathsf{M}$ be a class of models of intuitionistic modal logic. Let $\mathcal{M} \in \mathsf{M}$. Then, $\mathcal{M}, w \models \phi$ iff $\mathcal{M} \models \tau_x(\phi)[w]$ (where $\mathcal{M}$ is taken as a model of first order logic with $\mathcal{R}$, $\mathcal{R}_\square$, $\mathcal{R}_\diamond$ interpreting $R$, $R_\square$, $R_\diamond$).*

From the theorem it follows that if the satisfiability problem of $GF^2_{\mathrm{mon}}$ over $\mathsf{M}$ is decidable, then the satisfiability problem of intuitionistic modal logic over $\mathsf{M}$ is decidable.

It is well known that the guarded fragment is decidable over the class of all first order models [2]. Decidability of $GF^2_{\mathrm{mon}}$ over models with reflexive, transitive guards is proved in [14]. From this and from the fact that upward persistence for propositional variables occurring in $\phi$ is expressible in $GF^2_{\mathrm{mon}}$ it follows immediately that basic intuitionistic modal logic (with no conditions connecting $\mathcal{R}$, $\mathcal{R}_\Box$, and $\mathcal{R}_\Diamond$) is decidable. The purpose of this paper is to generalise the result of [14] to include classes of models defined using conditions involving interaction between $\mathcal{R}$, $\mathcal{R}_\Box$ and $\mathcal{R}_\Diamond$.

Theorems 14 and 12 give us our main theorem:

**Theorem 15.** *Let* $\mathsf{M}$ *be a class of intuitionistic modal models defined by an acyclic set of mso closure conditions on* $\mathcal{R}$, $\mathcal{R}_\Box$, *and* $\mathcal{R}_\Diamond$ *so that at most one closure condition is associated with each relation, and let* $\phi$ *be an intuitionistic modal formula. Then, it is decidable whether* $\phi$ *is satisfiable in* $\mathsf{M}$.

## 6. Examples

In this section, we state several decidability results just to illustrate our approach.

The first example is by no means a surprise, although we doubt if anyone has proved this for all possible combinations of truth definitions for modalities. Essentially this is decidability of several flavours of basic intuitionistic modal logic (no conditions on the modal accessibility relation).

**Proposition 16.** *An intuitionistic modal logic* $L$ *with two modalities* $\Box$ *and* $\Diamond$, *defined by a class of models where*

$$\mathcal{R} \circ \mathcal{R}_\Diamond^{-1} \circ \mathcal{R} = \mathcal{R}_\Diamond^{-1}$$
$$\mathcal{R} \circ \mathcal{R}_\Box \circ \mathcal{R} = \mathcal{R}_\Box$$

*and employing any of the truth definitions for modalities* $(\Box_1)$, $(\Box_2)$, $(\Diamond_1)$, $(\Diamond_2)$ (*in any combination, e.g.* $(\Box_1)$ *with* $(\Diamond_2)$; *possibly with more modalities, provided that all truth definitions can be translated in* $GF^2_{\mathrm{mon}}$), *is decidable.*

**Proof.** The class of models of $L$ is defined by the following closure conditions on $\mathcal{R}_\Box$, $\mathcal{R}_\Diamond$ and $\mathcal{R}$:

(1) $\mathcal{R}$ is reflexive and transitive;
(2) $\mathcal{R} \circ \mathcal{R}_\Diamond^{-1} \circ \mathcal{R} = \mathcal{R}_\Diamond^{-1}$;
(3) $\mathcal{R} \circ \mathcal{R}_\Box \circ \mathcal{R} = \mathcal{R}_\Box$.

There is clearly at most one condition for each of the relations $\mathcal{R}$, $\mathcal{R}_\Diamond$ and $\mathcal{R}_\Box$, and the set of conditions is acyclic. We have shown in Examples 3 and 6 that the condition on $\mathcal{R}$ is a closure condition and in Example 10 that it is mso-definable. By Proposition 13, conditions on $\mathcal{R}_\Box$ and $\mathcal{R}_\Diamond$ are also mso-definable closure conditions.

We have shown that the class of models of $L$ conforms to the conditions of Theorem 15 which proves that $L$ is decidable. $\Box$

The next example is related to a known result (decidability of *PLL* [11]), but for a slightly different logic (without fallible worlds):

**Proposition 17.** *An intuitionistic modal logic $L$ with one modality $\Diamond$, defined by a class of models where*

$\mathcal{R}_\Diamond$ *is reflexive and transitive*

$\mathcal{R}_\Diamond \subseteq \mathcal{R}$

*and employing the truth definition* $(\Diamond_2)$ *for the modality, is decidable.*

**Proof.** The class of models of $L$ is defined by the following closure conditions:

(1) $TC(\mathcal{R}_\Diamond) = \mathcal{R}_\Diamond$;
(2) $TC(\mathcal{R}) = \mathcal{R}$;
(3) $Incl^{\mathcal{R}_\Diamond}(\mathcal{R}) = \mathcal{R}$ (see Examples 4 and 7).

This set of conditions is acyclic and each condition is mso definable. However there are two constraints associated with $\mathcal{R}$: it is required to be closed both with respect to $TC$ and to $Incl^{\mathcal{R}_\Diamond}$. To satisfy the conditions of Theorem 15 we need to combine them into one mso definable closure condition. Observe that $TC \circ Incl^{\mathcal{P}'}$ is a closure operator with the property that for any relation $\mathcal{P}$,

$$TC\big(Incl^{\mathcal{P}'}(\mathcal{P})\big) = \mathcal{P} \quad \Leftrightarrow \quad TC(\mathcal{P}) = \mathcal{P} \text{ and } Incl^{P'}(\mathcal{P}) = \mathcal{P}$$

First of all, $TC \circ Incl^{\mathcal{P}'}$ is monotone and increasing, since both $TC$ and $Incl^{\mathcal{P}'}$ are. It is also idempotent, because the result of applying $TC \circ Incl^{\mathcal{P}'}$ to any relation $\mathcal{P}$ is a transitive relation containing $\mathcal{P}'$, and any subsequent applications of $TC \circ Incl^{\mathcal{P}'}$ are not going to change it. So, $TC \circ Incl^{\mathcal{P}'}$ is a closure operator. To prove that closure with respect to this operator is equivalent to closure with respect to $TC$ and $Incl^{\mathcal{P}'}$ separately, observe that one direction is immediate: if $\mathcal{P}$ is closed with respect to $TC$ and $Incl^{\mathcal{P}'}$, then it is closed with respect to $TC \circ Incl^{\mathcal{P}'}$. For the other direction, assume first that

$$TC\big(Incl^{\mathcal{P}'}(\mathcal{P})\big) = \mathcal{P}$$

but $\mathcal{P}$ is not closed with respect to $Incl^{\mathcal{P}'}$, that is, it is a proper subset of $Incl^{\mathcal{P}'}(\mathcal{P})$. But since $TC$ is increasing, $\mathcal{P}$ is then a proper subset of $TC(Incl^{\mathcal{P}'}(\mathcal{P}))$, which contradicts the assumption. Now assume that $\mathcal{P}$ is not closed with respect to $TC$, so that it is a proper subset of $TC(\mathcal{P})$. However, since $\mathcal{P} \subseteq Incl^{\mathcal{P}'}(P)$, we have

$$TC(\mathcal{P}) \subseteq TC\big(Incl^{\mathcal{P}'}(P)\big)$$

so $\mathcal{P}$ is a proper subset of $TC(Incl^{\mathcal{P}'}(P))$, which again contradicts the assumption. This means that the conditions can be reformulated as

(1) $TC(\mathcal{R}_\Diamond) = \mathcal{R}_\Diamond$;
(2) $TC(Incl^{\mathcal{R}_\Diamond}(\mathcal{R})) = \mathcal{R}$;

and it is straightforward to show that the second condition is mso definable. □

Finally, two non-examples. We failed to reformulate the condition $\mathcal{R}_\Box \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{R}_\Box$ defining an intuitionistic modal logic in [1] as a closure condition. We also could not apply our method to the logic IS4 defined in [29], since the truth conditions for IS4 formulas are defined on pairs $(w, d)$ (where $w$ is a possible world and $d$ an element from its domain), so the image of IS4 under the standard translation is not in $GF^2_{\mathrm{mon}}$.

## 7. Conclusions

We have described a general method for proving decidability of an intuitionistic modal logic by translating it into monadic $GF^2$ and showing that conditions on the intuitionistic and modal accessibility relations can be expressed using mso definable closure operators. We illustrate this method by showing that it works for various truth definitions for modalities and various conditions on the intuitionistic and modal accessibility occurring in the literature. Most of the decidability results for particular logics obtained as illustrations of our proof are already known, but we believe that our method can easily yield new results, especially for logics with non-normal modalities defined using the truth definition $(\Diamond_2)$ which are less well studied. Obviously, the same method works for intuitionistic logic with more than two modalities, provided all truth definitions can be translated in $GF^2_{\mathrm{mon}}$.

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