Correspondence and Completeness for Generalized Quantifiers. [1]

Natasha Alechina   Michiel van Lambalgen
natasha@fwi.uva.nl   michiell@fwi.uva.nl

Department of Mathematics and Computer Science
University of Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam

1 Introduction

A generalized quantifier $\mathbf{Q}$ as defined by Mostowski (1957) is a class of subsets of the universe, so that a model M satisfies $Qx\varphi(x, \bar{d})$ if the set of elements $\{e : M \models \varphi[e, \bar{d}]\}$ is in $\mathbf{Q}$. Examples are: the ordinary existential quantifier (interpreted as the set of all non-empty subsets of the universe); the quantifier "there are exactly 2"; a filter quantifier (where the only requirement on $\mathbf{Q}$ is that it is a filter), "there are uncountably many" (where the domain is uncountable, and $\mathbf{Q}$ contains all uncountable subsets), etc. An obvious property of these quantifiers is extensionality: if two formulas φ and ψ are satisfied by the same sets of elements, $Qx\varphi$ is satisfied if and only if $Qx\psi$ is. Actually Mostowski also required that the generalized quantifier is invariant under permutations of the universe, thus restricting attention to quantifiers related to cardinality. Subsequently, other generalized quantifiers were considered which do not have the property of permutation invariance, such as topological quantifiers or measure quantifiers. For an overview of the subject, one may consult the collection "Model-Theoretic Logics", edited by Barwise and Feferman (1985).

The very title of this volume makes it plain that generalized quantifiers have been studied mostly from a model theoretic point of view. There are various sound reasons for this focus: many quantifiers are not axiomatizable, so proof theory wouldn't make much sense[2] and even if there exists an axiomatization as in the case of the quantifier "uncountably many", the failure of interpolation (see Keisler (1970)) would lead one to suspect that even if one could give a proof theory, it would not enjoy such pleasant properties as for instance cut elimination.

Nonetheless, in van Lambalgen (1991) an attempt was made to develop Gentzen style calculi for the filter quantifiers "almost all" (due to H. Friedman; cf. Steinhorn (1985)) and "co-countably many".

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[2]Although one can sometimes give a proof theory by altering the semantics; cf. M. Mostowski (1991).

The guiding intuition behind this attempt is the idea that one can set up a proof system once one can manage dependency relations between variables bound by quantifiers. This idea is of course not restricted to generalized quantifiers: it can also be found in Fine's natural deduction systems using arbitrary objects and a dependency relation (cf. his (1985)), or in various forms of the functional interpretation of the existential quantifier (cf. Gabbay and de Queiroz (1991)).

For example, if Q is a quantifier which determines a non-trivial free filter, we have $\forall y Qx(x \neq y)$ but not $Qx\forall y(x \neq y)$; the failure of commutation seems to indicate that x in some way depends on y, as it would if we were considering the quantifier combination $\forall y \exists x$.

So we think of axioms for quantifiers such as $QxQy\varphi \rightarrow QyQx\varphi$ (characteristic for "almost all") or $\forall yQx\varphi \wedge Qy\forall x\varphi \rightarrow Qx\forall y\varphi$ (characteristic for "co-countably many") as implicitely determining a dependency pattern between variables; one may now ask whether this pattern can be made explicit, e.g. whether the properties of the dependence relation are first order describable. For example, in the case of the existential quantifier, the dependence can be taken to be functional, and the required first order description is given by the axioms for Skolem functions. Indeed, as Fine shows, one can also give a graphical representation of these dependencies.

Somewhat surprisingly, the axioms mentioned do indeed determine a first order condition on dependence, even conditions which are true for a paradigmatic case of dependence, namely, linear dependence in vector spaces. It is the purpose of this paper to explore this phenomenon, both its extent and its limits, in greater detail.

As a first step, we introduce an analogue of the expansion of a language by Skolem functions. Consider a language $\mathcal{L}_{\forall\Box}$ which extends a first-order language with equality by introducing a unary generalized quantifier $\Box_x$. We use this notation to emphasize an analogy between generalized quantifiers and modal operators, that will become apparent below. The dual of $\Box_x$ is $\Diamond_x\varphi =_{df} \neg\Box_x\neg\varphi$. (In the examples above a filter quantifier would correspond to $\Box$.)

Definition 1 Let $R(x, \bar{y})$ be a relation of indefinite arity. The intuitive interpretation of $R(x, \bar{y})$ is "x depends on $\bar{y}$". The *standard translation* * : $\mathcal{L}_{\forall\Box} \longrightarrow \mathcal{L}(R)$ is defined inductively as follows:

$P(x_1, \ldots, x_n)^* = P(x_1, \ldots, x_n)$;

$(\neg\psi)^* = \neg\psi^*$;

$(\psi_1 \wedge \psi_2)^* = \psi_1^* \wedge \psi_2^*$;

$(\forall x\psi)^* = \forall x\psi^*$

$(\Box_x\psi(x, \bar{y}))^* = \forall x(R(x, \bar{y}) \rightarrow \psi^*(x, \bar{y}))$. $\Box$

In other words, in the formula $\Box_x\psi(x, \bar{y})$ the bound variable x depends on $\bar{y}$ in a way determined solely by $\Box_x$, not by ψ (this is different from Skolem functions).

As in the case of Skolem functions, we would like to prove that every theory T has a conservative extension to a theory T' such that

$$(\flat) \quad T' \vdash \Box_x\varphi(x, y_1, \ldots, y_n) \equiv \forall x(R(x, y_1, \ldots, y_n) \rightarrow \varphi(x, y_1, \ldots, y_n)).$$

A moment's reflection shows that this can be true only for theories which are consistent with the *minimal logic*

Definition 2 *The minimal logic L_{min}* for $\mathcal{L}_{\forall\Box}$ is the smallest class of formulas closed with respect to classical first-order logic and the following axiom schemata:

A1 $\vdash \Box_x \top$;

A2 $\vdash \forall x(\varphi \to \psi) \wedge \Box_x \varphi \to \Box_x \psi$, *provided $\Box_x \psi$ and $\Box_x \varphi$ have the same free variables*;

A3 $\vdash \Box_x \varphi \wedge \Box_x \psi \to \Box_x(\varphi \wedge \psi)$, *provided $\Box_x \psi$ and $\Box_x \varphi$ have the same free variables*;

A4 $\vdash \Box_x \varphi \to \Box_y \varphi$, *provided y is free for x in φ.* $\Box$

In order to talk about *axioms* (that is, formulas) and not schemata, we introduce the substitution rule

$$\frac{\vdash \Phi(P(x_1, \ldots, x_n))}{\vdash \Phi(\varphi(x_1, \ldots, x_n))},$$

provided $P(\bar{x})$ and $\varphi(\bar{x})$ have precisely the same free variables. This restriction is necessary due to the fact that A2 entails only a restricted form of extensionality.

For theories which are consistent with the minimal model we can indeed prove that a conservative extension T' with ($\flat$) exists. First we define the required model expansion.

Definition 3 *A relational model* for $\mathcal{L}_{\forall\Box}$ is a triple $M = <D, R, V>$, where D is a non-empty domain, V interprets predicate symbols, and R is a relation of indefinite arity on D, called the dependence relation. The notion of a formula being satisfied in a model under a variable assignment is standard; the clause for $\Box_x$ reads as follows:

$$M \models^{\alpha} \Box_x \varphi(x, \bar{y}) \Leftrightarrow \forall d \in D(R(d, \alpha(\bar{y})) \to M \models^{\alpha} \varphi(d, \bar{y}))$$

A relational canonical model is a model where the accessibility relation is defined as follows:

$$R(x, \bar{y}) = \bigwedge_{\varphi(x,\bar{y}) \in \mathcal{L}_{\forall\Box}} \Box_x \varphi(x, \bar{y}) \to \varphi(x, \bar{y}).$$

$\Box$

The existence of T' satisfying ($\flat$) now follows from

Theorem 1 *Every L_{min}-consistent set of $\mathcal{L}_{\forall\Box}$ formulas has a Henkin canonical relational model.*

Proof can be found in van Benthem & Alechina (1993). $\Box$

Clearly, then, this way of making dependency patterns explicit will not work for all unary quantifiers; for quantifiers which are not (the dual of) a filter quantifier a translation more complicated than * might be necessary (see, however, the discussion in section 6.2). But it will be seen below that this simple case already has a rich theory.

Given the translation *, a quantifier axiom corresponds to a schema in the language $\mathcal{L}(R)$; the main question then becomes: when can this schema be replaced by a first-order condition on R?

The reader will have observed that both the set-up and the main problem are very much analogous to familiar themes in modal logic: R plays the role of the accessibility relation, and what we ask for is a Sahlqvist theorem, i.e. a characterization of a class of formulas for which there is a first order correspondent. To be more specific we need a definition:

Definition 4 If A is a quantifier axiom, a *correspondent in the sense of completeness* is a first order condition $A^\dagger$ on R with the following two properties:

i any set of sentences consistent with A has a relational model where $A^\dagger$ holds;

ii A is satisfied on any relational model where $A^\dagger$ holds. □

Recall that a modal logic L is called *first order complete* if there is a set Δ of first order sentences in the language $\{R, =\}$ (where R is the accessibility relation) such that

$$\vdash_L \varphi \text{ if and only if for every Kripke model } M\text{: if } M \models \Delta \text{ then } M \models \varphi;$$

equivalently, $\vdash_L \varphi$ if and only if φ is true on any frame which satisfies Δ. This notion can be reformulated for the generalized quantifiers as follows

Definition 5 If L is a generalized quantifier logic, then L is *first order complete* if there is a set Δ of first order sentences in the language $\mathcal{L}(R)$ so that for every quantifier formula φ, $\vdash_L \varphi$ if and only if for every relational model M: if $M \models \Delta$ then $M \models \varphi$. □

Note if L is finitely axiomatizable then by compactness Δ can be taken to be finite.

The definition 4 can be reformulated as follows: A has a correspondent for completeness $A^\dagger$ if $L_{min} + A$ is first order complete with respect to the class of models satisfying $A^\dagger$.

In modal logic one can even define a stronger notion of completeness. A modal logic L is canonical if it is first order complete and the corresponding set Δ of first order sentences holds in the canonical model for L. For generalized quantifiers, there is no unique canonical model. However, we can approximate the modal concept as follows:

Definition 6 If L is a generalized quantifier logic, then L is *canonical* if L is first order complete and the corresponding set Δ of first order sentences holds in every ω-saturated canonical relational model of L. □

This definition is motivated by the fact that in modal logic canonical models are ω-saturated.

The correspondence theory for generalized quantifiers developed in van Benthem and Alechina (1993) is not quite adequate for our purposes, since it is based on the notion of frame correspondence:

Definition 7 If A is a quantifier axiom, a *frame correspondent* is a first order condition $A^\star$ on R such that $<D, R> \models A^\star$ if and only if for *any* interpretation V, $<D, R, V> \models A$. □

It will be seen below that the two notions of having a correspondent are different; for example, there are formulas which have a frame correspondent, but do not have a correspondent for completeness. However, the following holds:

Proposition 1 *If A has a correspondent for completeness $A^\dagger$ and a frame correspondent $A^\star$, then $\vdash_{FOL} A^\dagger \rightarrow A^\star$.*

Proof. Let $< D, R > \models A^\dagger$. Then by the Definition 4 (ii), for every interpretation V $<D, R, V> \models A$. By the Definition 7, $<D, R> \models A^\star$. □

We restate the result on frame correspondence of van Benthem and Alechina (1993) here, and give an outline of its proof, because we shall need some concepts occurring in the proof in the sequel.

Theorem 2 *(Correspondence part of the Sahlqvist theorem). All formulas of the "Sahlqvist form" $\varphi \to \psi$, where*

1. *φ is constructed from*
   - *atomic formulas, possibly prefixed by $\Box_x$, $\forall$;*
   - *formulas in which predicate letters occur only negatively*

   using $\wedge, \vee, \Diamond_x, \exists$

2. *in ψ all predicate letters occur only positively*

have a frame correspondent. [3]

We shall sketch the proof of the theorem here.

A formula χ is valid in a frame if the following second-order formula is: $\forall P_1 \ldots \forall P_n \chi^*$, where $P_1, \ldots, P_n$ are all predicate letters in χ. If this formula is equivalent to a first-order formula without $P_1, \ldots, P_n$, then χ defines a first-order condition on frames. The proof of the theorem is based on the method of minimal substitutions, due to van Benthem (1983). After performing some syntactical transformations on χ the task can be reduced to proving that the following formula has a first-order equivalent:

$$\forall P_1 \ldots \forall P_n \forall \bar{u}(\bigwedge_j \mathcal{R}_j \wedge \bigwedge_i \forall \bar{x}(\mathcal{R}_i \to P_i) \wedge \bigwedge_k P_k \to \Psi)$$

where $\bigwedge_j \mathcal{R}_j$ corresponds to the translation of the truth conditions for $\Diamond$-quantifiers in the antecedent, $\bigwedge_i \forall \bar{x}(\mathcal{R}_i \to P_i)$ corresponds to the translations of occurrences of predicate symbols preceded by $\Box$-quantifiers, $\bigwedge_k P_k$ corresponds to occurrences of predicate symbols not preceded by $\Box$'s, and Ψ is a positive formula.

For example, assume that $\chi = \Diamond_x(\Box_y P(x,y) \wedge S(x)) \to \Box_z S(z)$. Then its translation reads

$$\forall P \forall S \forall x(R(x) \wedge \forall y(R(y,x) \to P(x,y)) \wedge S(x) \to \forall z(R(z) \to S(z))).$$

Here $\bigwedge_j \mathcal{R}_i = R(x)$, $\bigwedge_j \forall \bar{x}(\mathcal{R}_i \to P_i) = \forall y(R(y,x) \to P(x,y))$, and $\bigwedge_k P_k = S(x)$.

Note that we quantify over all possible assignments to the predicate symbols, so a second order quantifier $\forall P_i^n$ can be instantiated using any suitable set of n-tuples. That is what we are going to do. If P_i occurs in the antecedent in a subformula of the form $\forall \bar{x}(\mathcal{R}_i \to P_i(\bar{x}))$, the *minimal substitution* for this occurrence of P_i is precisely the set of those tuples for which $\mathcal{R}_i$ holds. (In the running example, $P(u_1, u_2) := R(u_2, u_1) \wedge u_1 = x$.) If P_i occurs as a member of conjunction $\bigwedge_k P_k$, then the minimal substitution for this occurrence is a singleton set: in the example, $S(u) := u = x$. Finally, the minimal substitution for P_i is the disjunction of minimal substitutions for its occurrences. (Precise definitions and details are given in van Benthem and Alechina (1993)). Denote the result of the substitution χ'. (In the example, χ' is

$$\forall x(R(x) \wedge \forall y(R(y,x) \to R(y,x) \wedge x = x) \wedge x = x \to \forall z(R(z) \to z = x)),$$

which is equivalent to $\forall x(R(x) \to \forall z(R(z) \to z = x))$.)

[3] This theorem can be made slightly more general by allowing an arbitrary long prefix of $\Box$-quantifiers before the formula, constant formulas in the antecedent(formulas which contain only $=$, $\top$ and $\bot$ as predicate symbols), and also stating that a conjunction of Sahlqvist formulas is a Sahlqvist formula.

χ' follows from $\forall \bar{P} \chi^*$ as a substitutional instance. The converse also holds, as follows.

Denote the substitutions for P_i as p_i. Each p_i is a first-order formula. The p_i's were chosen so that the antecedent of χ' becomes trivially true except for the first-order part $\bigwedge_j \mathcal{R}_j$,

$$\chi' = \forall \bar{u}(\bigwedge_j \mathcal{R}_j \wedge \bigwedge_i \forall \bar{x}(\mathcal{R}_i \to \mathcal{R}_i) \wedge \bigwedge (w = w) \to \Psi(p_1, \ldots, p_n)),$$

that is $\chi' = \forall \bar{u}(\bigwedge_j \mathcal{R}_j \to \Psi(p_1, \ldots, p_n))$. The consequent $\Psi(p_1, \ldots, p_n)$ is a first-order formula. Assume that χ' is true and there is a valuation (an assignment to predicate symbols) V which makes the antecedent of χ^* true:

$$<D, R, V> \models \bigwedge_j \mathcal{R}_j \wedge \bigwedge_i \forall \bar{x}(\mathcal{R}_i \to P_i) \wedge \bigwedge_k P_k$$

$$<D, R> \models \bigwedge_j \mathcal{R}_j \to \Psi(p_1, \ldots, p_n)).$$

We want to show that under this valuation also the consequent of χ^*, $\Psi(P_1, \ldots, P_n)$, is true. First of all, observe that $\Psi(p_1, \ldots, p_n)$ is true. The substitutions p_i which we used are minimal in the sense that always when the antecedent of χ^* is true under some assignment of values to predicate letters, then this assignment should contain the tuples satisfying p_i:

$$\{< d_1, \ldots, d_n >: p_i(d_1, \ldots, d_n)\} \subseteq V(P_i),$$

that is, $< D, R, V > \models \forall \bar{d}(p_i(\bar{d}) \to P_i(\bar{d}))$. $\Psi(P_1, \ldots, P_n)$ follows from $\Psi(p_1, \ldots, p_n)$ by monotonicity of Ψ. $\square$

A disadvantage of the notion of frame correspondence is that there may be consistent quantifier logics which are frame incomplete, i.e. logics L such that for no D, R: for all V, $<D, V, R> \models A$. In particular this holds for logics where the quantifier determines a free filter, such as 'almost all' or 'co-countably many'. It will be seen below that this is mainly due to the property of upwards monotonicity for the filter quantifier; unlike the modal case, where monotonicity is automatic given Kripke semantics, this property is highly non-trivial here. Actually, inspection of the proofs below shows that this result can be strengthened: the troublesome property is extensionality, that is, $\forall x(\varphi \equiv \psi) \to (\Box_x \varphi \equiv \Box_x \psi)$. This is interesting in view of the fact that extensionality is generally taken to be the sine qua non for logicality of a quantifier.

2 Statement of the result and idea of the proof

We now formulate the completeness part of the Sahlqvist theorem, which describes a class of formulas φ defining first-order conditions on R so that for any logic L in the language of $\mathcal{L}_{\forall\Box}$ which has a canonical relational model, L with φ as an axiom is complete for the class of models where R has the first order property corresponding to φ. This class is strictly smaller than the class of formulas described above. We shall call those formulas *weak* Sahlqvist formulas.

Theorem 3 *(Completeness part of the Sahlqvist theorem) All "weak Sahlqvist formulas" χ of the form $\bigwedge Qz_1 \ldots Qz_n(A \to B)$, where $n \geq 0$, each Q is either $\forall$ or $\Box$, and*

1. *A is constructed from*

a. atomic formulas, possibly with a quantifier prefix $Qx_1 \dots Qx_k$, where each Q is a $\Box$- or $\forall$-quantifier;

b. formulas in which atomic formulas occur only negatively,

c. constant formulas (where the only predicate letters are $\top, \bot$ and $=$),

using $\wedge$ and $\vee$,

2. in B all predicate letters occur only positively

3. every occurrence of a predicate letter has the same free variables [4]

have a correspondent in the sense of completeness; moreover, this correspondent holds in every ω-saturated canonical relational model of a logic in which χ is provable.

All conditions of the theorem can be shown to be necessary, namely, if a formula does not satisfy one of them, then it need not have a correspondent for completeness. The conditions which are common with the Theorem 2, are shown to be necessary in van Benthem & Alechina (1993). The additional conditions are: no existential quantifiers in the antecedent and all occurrences of a predicate symbol have the same free variables. The necessity of those is shown in section 6.

The idea of the proof of the Theorem 3 (very similar to the one used in Sambin & Vaccaro (1989)) can be illustrated by the following example.

Example 1 Let $\mathcal{C}$ be a canonical[5] model. We show that if for every P and S

$$\Box_x P(x, \bar{y}) \rightarrow \Box_x (P(x, \bar{y}) \vee S(x, \bar{z}))$$

is valid in $\mathcal{C}$, then the accessibility relation in $\mathcal{C}$ has the property $R(x, \bar{y}\bar{z}) \rightarrow R(x, \bar{y})$.

Proof. It is easy to see (since the consequent is monotone in S) that the axiom above is equivalent to $\Box_x P(x, \bar{y}) \rightarrow \Box_x (P(x, \bar{y}) \vee \bot(x, \bar{z}))$. We translate the validity conditions using second-order quantifiers which range only over *definable* relations of $\mathcal{C}$ (this is the difference with the case of the frame correspondence). To emphasize this difference we use quantifiers $\forall\varphi$. Note that due to the restricted Substitution Rule, if P is an n-place predicate symbol, then formulas which can be substituted for P must have precisely n variable places. Using this notation the validity condition of the axiom reads as follows:

$$\forall\varphi[\forall x(R(x, \bar{y}) \rightarrow \varphi(x, \bar{y})) \rightarrow \forall x(R(x, \bar{y}\bar{z}) \rightarrow \varphi(x, \bar{y}) \vee \bot(x, \bar{z}))]$$

This is equivalent to $\forall\varphi[\forall x(R(x, \bar{y}) \rightarrow \varphi(x, \bar{y})) \rightarrow \forall x(R(x, \bar{y}\bar{z}) \rightarrow \varphi(x, \bar{y}))]$, and, in turn, to $\bigwedge_{\varphi(x,\bar{y})}\{\forall x(R(x, \bar{y}\bar{z}) \rightarrow \varphi(x, \bar{y})) : \forall x(R(x, \bar{y}) \rightarrow \varphi(x, \bar{y}))\}$. Moving the conjunction inside (the proof that this can be done for any positive logical function of φ, is given in the Intersection Lemma; in the given case the proof is obvious), we obtain

$$\forall x(R(x, \bar{y}\bar{z}) \rightarrow \bigwedge_{\varphi(x,\bar{y})} \{\varphi(x, \bar{y})) : \forall x(R(x, \bar{y}) \rightarrow \varphi(x, \bar{y}))\})$$

[4] Actually, the sufficient condition is that all positive occurrences of a predicate letter in the antecedent have the same free variables, say, $\bar{y}$, all negative occurrences of a predicate letter in the antecedent and all occurrences in the consequent have the same free variables, say, $\bar{z}$, and $\bar{y} \subseteq \bar{z}$. It can be shown that in this case the condition *3* can be forced to hold. But we assume that this holds from the very beginning to make the formulation of the theorem readable.

[5] Henceforth, "canonical model" will always mean "canonical relational model".

But in $\mathcal{C}$ $\bigwedge_{\varphi(x,\bar{y})}\{\varphi(x,\bar{y})) : \forall x(R(x,\bar{y}) \to \varphi(x,\bar{y}))\}) \equiv R(x,\bar{y})$. Substituting $R(x,\bar{y})$ instead of the infinite conjunction yields the first-order equivalent $\forall x(R(x,\bar{y}\bar{z}) \to R(x,\bar{y}))$.

It is easy to check that in every model where $\forall x(R(x,\bar{y}\bar{z}) \to R(x,\bar{y}))$ holds, the axiom is valid. □

The general case is slightly more complicated because to obtain a correspondent we must sometimes move to a different canonical model, namely, to an ω-saturated canonical model. The existence of such model for every L_{min}-consistent set of sentences is proved in section 3. In a canonical ω-saturated model the following lemma holds:

Intersection Lemma If B is a positive formula and X is a set of formulas with the same free variables, closed with respect to $\wedge$, then in an ω-saturated model

$$\bigwedge\{B(\varphi) : \varphi \in X\} \equiv B(\bigwedge\{\varphi : \varphi \in X\}).$$

The proof of the Theorem 3 consists of the same three ingredients as those in the example: translation of the validity conditions of an axiom (eventually accompanied by some syntactic transformations), application of the Intersection Lemma, and making use of the fact that for some first-order expression $\mathcal{R}$ with R as the only predicate symbol

$$\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge_{\varphi(\bar{x},\bar{y})} \{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathcal{R}(\bar{x},\bar{y}) \to \varphi(\bar{x},\bar{y}))\}.$$

Of course, $\mathcal{R}$ is nothing else than a "minimal substitution", but of a special kind, which will be formally defined below.

Definition 8 Let M be a canonical model, and A a conjunction of atomic formulas which are prefixed by universal and □-quantifiers, so that all occurrences of a predicate symbol have the same free variables. Every occurrence of a predicate symbol P in A is therefore of the form $\bar{Q}_i\bar{x}P(\bar{x},\bar{y})$, where $\bar{Q}_i$ is the quantifier prefix of the ith occurrence. P has *a good minimal substitution* in A if $M \models \bigwedge\{\varphi(\bar{x},\bar{y}) : \bigwedge_i \bar{Q}_i\bar{x}\varphi(\bar{x},\bar{y})\} \equiv p$, where p is a first-order formula built using the predicates R, $=$, $\top$ and $\bot$ only. □

For example, we have seen that if the only occurrence of P in A is of the form $\Box_x P(x,\bar{y})$, then P has a good minimal substitution in A: for every canonical model M,

$$M \models \bigwedge\{\varphi(x,\bar{y}) : \Box_x\varphi(x,\bar{y})\} \equiv R(x,\bar{y}).$$

Before we formulate the Closure Lemma, we shall get rid of two degenerate cases.

(a) Let P be preceded by a vacuous □: e.g., the only occurrence of P in A is of the form $\Box_x P(y)$. Then every canonical model M satisfies

$$\bigwedge\{\varphi(y) : \Box_x\varphi(y)\} \equiv \bigwedge\{\varphi(y) : \forall x(R(x,y) \to \varphi(y))\} \equiv \bigwedge\{\varphi(y) : \exists x R(x,y) \to \varphi(y)\}.$$

If $M \models \exists x R(x,y)$, then $M \models \bigwedge\{\varphi(y) : \exists x R(x,y) \to \varphi(y)\} \equiv \top(y)$. If not, then $M \models \bigwedge\{\varphi(y) : \exists x R(x,y) \to \varphi(y)\} \equiv \bot(y)$. This means that axioms with vacuous □'s in the antecedent can have different correspondents in different models.

Henceforth we assume that *all quantifiers are non-vacuous.* Formally, this corresponds to assuming that R is always non-empty, or that the axiom $\Diamond_x\top(x,\bar{y})$ holds in all canonical models. This is an innocuous assumption, because if R is empty, every $\mathcal{L}_{\forall\Box}$-formula is equivalent to a first-order formula ($\Box_x\varphi$ becomes equivalent to $\top$ and $\Diamond_x\varphi$ becomes $\bot$).

We shall also assume that every quantifier prefix in A contains at least one $\Box$. That this is no loss of generality can be seen as follows.

Let P occur in A with a purely universal prefix, $\forall x_1 \ldots \forall x_n P(\bar{x}, \bar{y})$. Then this occurrence implies all other possible occurrences of P in A, and A is equivalent to a conjunction where $\forall x_1 \ldots \forall x_n P(\bar{x}, \bar{y})$ is the only occurrence of P.

$$M \models \bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall x_1 \ldots \forall x_n \varphi(\bar{x}, \bar{y})\} \equiv \top(\bar{x}, \bar{y}),$$

so P has a good minimal substitution in A.

Now we can state the Closure Lemma which is proved in section 4:

Closure Lemma. Let A be a conjunction of atomic formulas prefixed by $\forall$ and $\Box$-quantifiers, so that all occurrences of a predicate symbol in A have the same free variables. Then every atomic formula in A has a good minimal substitution, and this minimal substitution is the same as the one used to obtain the frame correspondent.

3 ω-saturated models and the Intersection Lemma.

Let $X = \{\varphi_1(x), \varphi_2(x), \ldots\}$ be a finitely realizable type in a model M, that is, for every n there is an element a_n in the domain of M such that $\varphi_1(a_n), \ldots, \varphi_n(a_n)$ is true in M. If M is just an ordinary Henkin model, there does not necessarily exist an element a such that for every φ in X $\varphi(a)$ is true in M. Among other things this implies that $\Diamond_x \bigwedge\{\varphi : \varphi \in X\}$ is not equivalent to $\bigwedge\{\Diamond_x \varphi : \varphi \in X\}$ in M. But in the proof we do need that

$$M \models\ \Diamond_x \bigwedge\{\varphi : \varphi \in X\} \equiv \bigwedge\{\Diamond_x \varphi : \varphi \in X\}$$

(an analogue of Esakia's lemma). We therefore move from the original Henkin model to a mildly saturated extension.

Theorem 4 *Every consistent set of $\mathcal{L}_{\forall\Box}$ formulas has a model $\mathcal{A}$ which is ω-saturated and canonical, that is*

i *every finitely realizable type which contains finitely many parameters is realized;*

ii $R_{\mathcal{A}}(d, \bar{d}) =_{df} \bigwedge_{\varphi(x,\bar{d}) \in \mathcal{L}_{\forall\Box}} \Box_x \varphi(x, \bar{d}) \to \varphi(d, \bar{d})$

Proof From the completeness proof for the minimal logic we know that every consistent set of formulas has a canonical model $\mathcal{C}$ where

$$R_{\mathcal{C}}(x, \bar{y}) \equiv \bigwedge_{\varphi(x,\bar{y}) \in \mathcal{L}_{\forall\Box}} \Box_x \varphi(x, \bar{y}) \to \varphi(x, \bar{y})$$

By the truth definition

$$\mathcal{C} \models \Box_x \varphi(x, \bar{y}) \Leftrightarrow \forall x(R_{\mathcal{C}}(x, \bar{y}) \Rightarrow \mathcal{C} \models \varphi(x, \bar{y}))$$

Therefore there is a first-order model $\mathcal{C}^*$ (with $R = R_{\mathcal{C}}$ just an ordinary predicate) such that if $\psi \in \mathcal{L}_{\forall\Box}$

$$\mathcal{C} \models \psi \Leftrightarrow \mathcal{C}^* \models \psi^*$$

where ψ^* is the standard translation of ψ.

We shall use this fact to build the saturated model which we need, because one can apply the standard procedure of constructing an ω-saturated extension of $\mathcal{C}^*$. (While extending a model for a generalized quantifier is much more difficult, see for example Hodges (1985).)

Take an ω-saturated elementary extension $\mathcal{A}^*$ of $\mathcal{C}^*$. It is clear that

$$\mathcal{C} \models \psi \Leftrightarrow \mathcal{C}^* \models \psi^* \Leftrightarrow \mathcal{A}^* \models \psi^*,$$

for every sentence ψ of $\mathcal{L}_{\forall\Box}$.

Every type finitely realizable in $\mathcal{C}$ is finitely realizable in $\mathcal{C}^*$ and is therefore realized in $\mathcal{A}^*$. But $\mathcal{A}^*$ is still a first-order model; to make an $\mathcal{L}_{\forall\Box}$ model $\mathcal{A}$ out of it, we could take the interpretation of R in $\mathcal{A}^*$ to be the accessibility relation in $\mathcal{A}$, i.e. stipulate

$$\mathcal{A} \models \Box_x \varphi(x, \bar{y}) \Leftrightarrow \ \forall x(R(x, \bar{y}) \Rightarrow \mathcal{A} \models \varphi(x, \bar{y})).$$

However, it is not obvious that $\mathcal{A}$ is still canonical.

Instead we define the accessibility relation anew in $\mathcal{A}$. $\mathcal{A}$ will be the expansion $< \mathcal{A}^*, R_{\mathcal{A}} >$ of $\mathcal{A}^*$, where $R_{\mathcal{A}}$ is defined on $\mathcal{A}^*$ as

$$R_{\mathcal{A}}(x, \bar{y}) = \bigwedge_{\varphi^*(x,\bar{y}):\varphi(x,\bar{y})\in\mathcal{L}_{\forall\Box}} \forall x(R(x, \bar{y}) \to \varphi^*(x, \bar{y})) \to \varphi^*(x, \bar{y}).$$

Note that the intersection is only over the formulas $\varphi^*(x, \bar{y})$ such that $\varphi(x, \bar{y}) \in \mathcal{L}_{\forall\Box}$.

We are done if we can show that

Lemma 1 $\mathcal{A} \models \varphi \Leftrightarrow \mathcal{A}^* \models \varphi^*$ *for all formulas* $\varphi \in \mathcal{L}_{\forall\Box}$.

Proof By induction on the complexity of φ. The only non-trivial case is $\varphi = \Box_x \psi(x, \bar{y})$.

To prove the direction from right to left, assume that $\mathcal{A}^* \models (\Box_x \psi(x, \bar{y}))^*$, that is, $\mathcal{A}^* \models \forall x(R(x, \bar{y}) \to \psi^*(x, \bar{y}))$. We want to prove $\mathcal{A} \models \Box_x \psi(x, \bar{y})$, that is $\mathcal{A} \models \forall x(R_{\mathcal{A}}(x, \bar{y}) \to \psi(x, \bar{y}))$.

Let $R_{\mathcal{A}}(x, \bar{y})$ hold in $\mathcal{A}$. By the definition of $R_{\mathcal{A}}$, $\mathcal{A}^* \models \forall x(R(x, \bar{y}) \to \psi^*(x, \bar{y})) \to \psi^*(x, \bar{y})$. We know that $\mathcal{A}^* \models \forall x(R(x, \bar{y}) \to \psi^*(x, \bar{y}))$. Therefore $\mathcal{A}^* \models \psi^*(x, \bar{y})$ and, by the inductive hypothesis, $\mathcal{A} \models \psi(x, \bar{y})$.

From left to right: let $\mathcal{A} \models \Box_x \psi(x, \bar{y})$, that is $\mathcal{A} \models \forall x(R_{\mathcal{A}}(x, \bar{y}) \to \psi(x, \bar{y}))$. Let $R(x, \bar{y})$ hold in $\mathcal{A}^*$. We want to show that $\mathcal{A}^* \models \psi^*(x, \bar{y})$. It is enough to show that $R(x, \bar{y})$ implies $R_{\mathcal{A}}(x, \bar{y})$. If this is so, we obtain $\psi(x, \bar{y})$ from $R(x, \bar{y})$ and the fact that $\mathcal{A} \models \forall x(R_{\mathcal{A}}(x, \bar{y}) \to \psi(x, \bar{y}))$, and hence applying the inductive hypothesis we also get $\psi^*(x, \bar{y})$.

Let $R(x, \bar{y})$. Take an arbitrary formula χ^* such that $\forall x(R(x, \bar{y}) \to \chi^*(x, \bar{y}))$. Then $\chi^*(x, \bar{y})$. This way we prove that for all χ^*, $R(x, \bar{y}) \to (\forall x(R(x, \bar{y}) \to \chi^*(x, \bar{y})) \to \chi^*(x, \bar{y}))$. Therefore $R(x, \bar{y}) \to \bigwedge_{\chi^*}(\forall x(R(x, \bar{y}) \to \chi^*(x, \bar{y})) \to \chi^*(x, \bar{y}))$, which means that $R(x, \bar{y})$ implies $R_{\mathcal{A}}(x, \bar{y})$. $\Box$

Comment. $\mathcal{C}$ is an elementary extension of $\mathcal{A}$ with respect to $\mathcal{L}_{\forall\Box}$ formulas, but not necessarily with respect to $\mathcal{L}(R)$ formulas if R is interpreted as $R_{\mathcal{A}}$.

Now we are ready to prove that in $\mathcal{A}$ the Intersection Lemma holds.

Lemma 2 *(Intersection Lemma)* *If B is a positive formula and X is a set of formulas with the same free variables, closed with respect to $\wedge$, then in an ω-saturated model*

$$\bigwedge\{B(\varphi) : \varphi \in X\} \equiv B(\bigwedge\{\varphi : \varphi \in X\})$$

Proof By induction on the complexity of B.

- $B(\varphi) = \varphi$: trivial, because $\bigwedge\{\varphi : \varphi \in X\} = \bigwedge\{\varphi : \varphi \in X\}$.

  B can be a formula with principal sign $\wedge, \vee, \forall, \exists, \Diamond_x$ or $\Box_x$ (since B is positive). Assume that for the components of B the claim holds.

- Let $B = B_1 \wedge B_2$;

  $$\bigwedge\{B_1(\varphi) \wedge B_2(\varphi) : \varphi \in X\} = \bigwedge\{B_1(\varphi) : \varphi \in X\} \wedge \bigwedge\{B_2(\varphi) : \varphi \in X\}$$

  (by the associativity of $\bigwedge$) and, by the inductive hypothesis, the latter is equivalent to $B_1(\bigwedge\{\varphi : \varphi \in X\}) \wedge B_2(\bigwedge\{\varphi : \varphi \in X\})$.

- Let $B = B_1 \vee B_2$.

  Assume $M, \bar{e} \models B_1(\bigwedge\{\varphi : \varphi \in X\}) \vee B_2(\bigwedge\{\varphi : \varphi \in X\})$. Then one of the disjuncts is true, for example $M, \bar{e} \models B_1(\bigwedge\{\varphi : \varphi \in X\})$. By the inductive hypothesis, $M, \bar{e} \models \bigwedge\{B_1(\varphi) : \varphi \in X\}$. Clearly, $M, \bar{e} \models \bigwedge\{B_1(\varphi) \vee B_2(\varphi) : \varphi \in X\}$.

  For the other direction, assume $M, \bar{e} \not\models B_1(\bigwedge\{\varphi : \varphi \in X\}) \vee B_2(\bigwedge\{\varphi : \varphi \in X\})$. This means that $M, \bar{e} \not\models B_1(\bigwedge\{\varphi : \varphi \in X\})$ and $M, \bar{e} \not\models B_2(\bigwedge\{\varphi : \varphi \in X\})$. By the inductive hypothesis, $M, \bar{e} \not\models \bigwedge\{B_1(\varphi) : \varphi \in X\}$ and $M, \bar{e} \not\models \bigwedge\{B_2(\varphi) : \varphi \in X\}$. Therefore there are φ_1 and φ_2 in X such that $M, \bar{e} \not\models B_1(\varphi_1)$ and $M, \bar{e} \not\models B_2(\varphi_2)$. Since B_1 is also positive and therefore monotone, [6] $M, \bar{e} \not\models B_1(\varphi_1 \wedge \varphi_2)$ and $M, \bar{e} \not\models B_2(\varphi_1 \wedge \varphi_2)$, thus $M, \bar{e} \not\models B_1(\varphi_1 \wedge \varphi_2) \vee B_2(\varphi_1 \wedge \varphi_2)$. Note that X is closed under $\wedge$, therefore $\varphi_1 \wedge \varphi_2 \in X$. But this means that $M, \bar{e} \not\models \bigwedge\{B_1(\varphi) \vee B_2(\varphi) : \varphi \in X\}$.

- Let $B = \forall x B_1$. $\bigwedge\{\forall x B_1(\varphi) : \varphi \in X\} = \forall x \bigwedge\{B_1(\varphi) : \varphi \in X\}$ (because $\forall$ distributes over $\bigwedge$), and by the inductive hypothesis this is equivalent to $\forall x B_1(\bigwedge\{\varphi : \varphi \in X\})$.

- Let $B = \Box_x B_1$. This case is analogous, but since $\Box$ distributes only over conjunctions of formulas with the same free variables, it is important that all formulas in X (and therefore in $\{B_1(\varphi) : \varphi \in X\}$) have the same free variables.

- Let $B = \Diamond_x B_1$. This is the only non-trivial part, and here we need the fact that the model is ω-saturated. First of all, it would be convenient if the set $Y = \{B_1(\varphi) : \varphi \in X\}$ had the following property: for every n, $M \models \psi_n \rightarrow \psi_1 \wedge \ldots \wedge \psi_{n-1}$, $\psi_i \in Y$. This is not so in general, but we can consider instead of Y the set $Y' = \{B_1(\varphi_1), B_1(\varphi_1 \wedge \varphi_2), B_1(\varphi_1 \wedge \varphi_2 \wedge \varphi_3), \ldots : \varphi_i \in X\}$. For every formula in Y, there is a formula in Y' which implies it (due to the monotonicity of B_1). Therefore $\bigwedge Y' \rightarrow \bigwedge Y$. On the other hand, $Y' \subseteq Y$ (because X is closed under $\wedge$). Therefore $\bigwedge Y \rightarrow \bigwedge Y'$. Analogously $\bigwedge\{\Diamond_x \psi : \psi \in Y\} = \bigwedge\{\Diamond_x \psi : \psi \in Y'\}$. So, it is suffices to prove that

  $$\bigwedge\{\Diamond_x \psi : \psi \in Y'\} = \Diamond_x \bigwedge\{\psi : \psi \in Y'\}$$

  Since $\bigwedge\{\psi : \psi \in Y'\}$ implies ψ for any $\psi \in Y'$, $\Diamond_x \bigwedge\{\psi : \psi \in Y'\}$ implies $\Diamond_x \psi$ and therefore $\bigwedge\{\Diamond_x \psi : \psi \in Y'\}$. This proves the implication right to left.

  For the other direction, assume that the model satisfies $\Diamond_x \psi$ for every $\psi \in Y'$. Due to the definition of Y' and monotonicity of B_1, this means that for every n the model

[6] Monotonicity in $\mathcal{L}_{\forall\Box}$ is restricted to the formulas with the same free variables, but this is always the case in this proof.

satisfies $\Diamond_x(\psi_1 \wedge \ldots \wedge \psi_n)$, i.e. that for every n there is an element x satisfying $R(x, \bar{e})$ and $\psi_1(x, \bar{e}), \ldots, \psi_n(x, \bar{e})$. Since the model is ω-saturated, there is an element which satisfies the whole set: $R(u, \bar{e}) \wedge \bigwedge\{\psi(u, \bar{e}) : \psi \in Y'\}$. Then $\Diamond_x \bigwedge\{\psi : \psi \in Y'\}$ is true.

- Let $B = \exists x B_1$: the proof is analogous to the previous case. $\Box$

4 Closure Lemma

Let A be a conjunction as in the condition of the Closure Lemma. We also assume that all quantifiers are non-vacuous and that every quantifier prefix contains at least one $\Box$-quantifier (cf. the end of section 2).

Let A' be the subformula of A which contains all and only occurrences of the predicate symbol P. We shall use both the $\mathcal{L}_{\forall\Box}$-form of A', namely $\bigwedge_i \bar{Q}_i \bar{x} P(\bar{x}, \bar{y})$, and its standard translation $\bigwedge_i \forall \bar{x}(\mathcal{R}_i \rightarrow P(\bar{x}, \bar{y}))$, where i runs over the occurrences of P. In the sequel we call the $\mathcal{R}_i$ *R-conditions*. The standard translation of A' is thus equivalent to $\forall \bar{x}(\bigvee_i \mathcal{R}_i \rightarrow P(\bar{x}, \bar{y}))$. $P(\bar{x}, \bar{y})$ has a good minimal substitution in A if

$$\bigvee_i \mathcal{R}_i = \bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigvee_i \mathcal{R}_i \rightarrow \varphi(\bar{x}, \bar{y}))\}.$$

Note that good minimal substitutions are the same expressions which were used in the proof of Theorem 2 as minimal substitutions for occurrences of predicate symbols in the scope of $\Box$- and $\forall$-quantifiers.

Example 2 The R-condition corresponding to $\Box_x \Box_y P(x, y)$ is $R(x) \wedge R(y, x)$.

Example 3 Let $A' = \forall x \Box_y P(x, y) \wedge \Box_x \forall y P(x, y)$, then

$$A'^* = \forall x \forall y(R(y, x) \rightarrow P(x, y)) \wedge \forall x \forall y(R(x) \rightarrow P(x, y))$$

which is equivalent to $\forall x \forall y(R(y, x) \vee R(x) \rightarrow P(x, y))$. The good minimal substitution for P in A must be therefore $R(y, x) \vee R(x)$.

We are going to prove the existence of good minimal substitutions for all non-vacuous quantifier prefixes containing at least one $\Box$. From this and from the considerations at the end of the section 2 the Closure Lemma will follow. But first we need several propositions.

Proposition 2 *Let $\mathcal{R}$ be an R-condition, such that*

$$\mathcal{R}(\bar{x}, \bar{y}) \equiv \bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\mathcal{R}(\bar{x}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\};$$

then

$$\mathcal{R}(\bar{x}, \bar{y}) \equiv \bigwedge\{\psi(x, \bar{y}\bar{z}) : \forall \bar{x} \forall \bar{z}(\mathcal{R}(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y}\bar{z}))\},$$

and vice versa: if

$$\mathcal{R}(\bar{x}, \bar{y}) \equiv \bigwedge\{\psi(x, \bar{y}\bar{z}) : \forall \bar{x} \forall \bar{z}(\mathcal{R}(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y}\bar{z}))\},$$

then

$$\mathcal{R}(\bar{x}, \bar{y}) \equiv \bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\mathcal{R}(\bar{x}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\}.$$

Proof. Assume $\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathcal{R}(\bar{x},\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$. For every $\varphi(\bar{x},\bar{y})$ holds: $\varphi(\bar{x},\bar{y}) \equiv \forall\bar{z}(\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z}))$. Therefore

$$\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\forall\bar{z}(\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z})) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to (\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z}))\}.$$

Since for every φ, $\forall\bar{z}(\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z})) \equiv \varphi(\bar{x},\bar{y}) \wedge \top(\bar{z})$,

$$\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to (\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z}))\}.$$

Now we prove that $\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\psi(\bar{x},\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\}$. Trivially,

$$\bigwedge\{\psi(\bar{x},\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z})\} \to \bigwedge\{\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to (\varphi(\bar{x},\bar{y}) \wedge \top(\bar{z}))\}$$

and this implies that $\bigwedge\{\psi(\bar{x},\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\} \to \mathcal{R}(\bar{x},\bar{y})$.

Since $\mathcal{R}(\bar{x},\bar{y}) \to \bigwedge\{\psi(\bar{x},\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\}$, we have $\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\psi(\bar{x},\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\}$.

For the other direction of the proposition, let

$$\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\psi(\bar{x},\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathcal{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\}.$$

It is easy to check that $\mathcal{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\forall\bar{z}\psi(\bar{x},\bar{y},\bar{z}) : \forall\bar{x}(\mathcal{R}(\bar{x},\bar{y}) \to \forall\bar{z}\psi(\bar{x},\bar{y},\bar{z}))\}$, and then the reasoning goes as above: the set of φ's with free variables $\bar{x},\bar{y}$ satisfying the same condition is larger than the set of $\forall\bar{z}\psi(\bar{x},\bar{y},\bar{z})$, therefore its conjunction implies the given one; on the other hand, the set of φ's satisfying the condition is implied by $\mathcal{R}$, therefore the two sets are equivalent. $\Box$

To prove the next proposition, we shall use the following tautology of the minimal logic: $L_{min} \vdash \Box_x(\Box_x\theta \to \theta)$ (the proof is easy).

Proposition 3 $R(z,\bar{x}\bar{y}) \equiv \bigwedge\{\varphi(z,\bar{x},\bar{y}) : \forall z\forall\bar{x}(R(z,\bar{x}\bar{y}) \to \varphi(z,\bar{x},\bar{y}))\}$.

Proof. The direction from left to right is trivial. For the converse direction, we have to prove $\bigwedge\{\varphi(z,\bar{x},\bar{y}) : \forall z\forall\bar{x}(R(z,\bar{x}\bar{y}) \to \varphi(z,\bar{x},\bar{y}))\} \to R(z,\bar{x}\bar{y})$, in other words,

$$\bigwedge_{\varphi}(\forall\bar{x}\Box_z\varphi(z,\bar{x},\bar{y}) \to \varphi(z,\bar{x},\bar{y})) \to \bigwedge_{\psi}(\Box_z\psi(z,\bar{x},\bar{y}) \to \psi(z,\bar{x},\bar{y})).$$

It suffices to derive $\bigwedge_{\psi}\Box_z\psi(z,\bar{e},\bar{y}) \to \psi(d,\bar{e},\bar{y})$ from $\bigwedge_{\varphi}\forall\bar{x}\Box_z\varphi(z,\bar{x},\bar{y}) \to \varphi(d,\bar{e},\bar{y})$. Take an arbitrary $\psi(z,\bar{x},\bar{y})$. We substitute this formula for θ in the tautology above:

$$\forall\bar{x}\Box_z(\Box_z\psi(z,\bar{x},\bar{y}) \to \psi(z,\bar{x},\bar{y})).$$

We assume that the conjunction $\bigwedge_{\varphi}\forall\bar{x}\Box_z\varphi(z,\bar{x},\bar{y}) \to \varphi(d,\bar{e},\bar{y})$ holds. As a special case we obtain

$$\forall\bar{x}\Box_z(\Box_z\psi(z,\bar{x},\bar{y}) \to \psi(z,\bar{x},\bar{y})) \to (\Box_z\psi(z,\bar{e},\bar{y}) \to \psi(d,\bar{e},\bar{y})).$$

Since this holds for every ψ, we can derive $\bigwedge_{\psi}\Box_z\psi(z,\bar{e},\bar{y}) \to \psi(d,\bar{e},\bar{y})$. $\Box$

Proposition 4 *If $Qx_1,\ldots,Qx_n$ contains at least one $\Box$-quantifier, and all quantifiers are non-vacuous, and the only occurrence of $P(\bar{x},\bar{y})$ in A is of the form $Qx_1\ldots Qx_nP(\bar{x},\bar{y})$, then P has a good minimal substitution in A.*

Proof. The general form of the prefix described in the condition of this proposition, is

$$\forall(\bar{u})_1\Box_{z_1}\forall(\bar{u})_2\Box_{z_2}\ldots\forall(\bar{u})_k\Box_{z_k}\forall(\bar{u})_{k+1}P(\bar{x},\bar{y}),$$

where $k > 0$ (that is, there is at least one $\Box$ in the prefix), and $\bar{u}\bar{z} = \bar{x}$. The standard translation of this formula is

$$\forall\bar{x}(R(z_1,(\bar{u})_1,\bar{y})\wedge\ldots\wedge R(z_k,(\bar{u})_k,z_{k-1},(\bar{u})_{k-1},\ldots,z_1,(\bar{u})_1,\bar{y})\rightarrow P(\bar{x},\bar{y}))$$

(since there is at least one $\Box$-quantifier in the prefix).

We have to show that

$$\mathcal{R}\equiv\bigwedge_{i=1}^{i=k}R(z_i,(\vec{uz})_{\leq i},\bar{y}),$$

where $(\vec{uz})_{\leq i}$ are the variables bound by the quantifiers preceding $\Box_{z_i}$, is a good minimal substitution for P.

By propositions 2 and 3 (observe that $z_i(\vec{uz})_{\leq i}\subseteq\bar{x}$),

$$R(z_i,(\vec{uz})_{\leq i},\bar{y})\equiv\bigwedge\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(R(z_i,(\vec{uz})_{\leq i},\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\}$$

Note that here we essentially use the fact that the z_i occur in $P(\bar{x},\bar{y})$, that is, that $\Box$-quantifiers are non-vacuous.

It is easy to see that $\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y})\rightarrow\bigwedge\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\}$.

To prove the other direction, namely

$$\bigwedge\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\}\rightarrow\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y}),$$

we argue as before:

$$\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(R(z_i,(\vec{uz})_{\leq i},\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\}\subseteq\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\}$$

therefore

$$\bigwedge\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\}\rightarrow\bigwedge\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(R(z_i,(\vec{uz})_{\leq i},\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\}$$

and this means that $\bigwedge\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\}\rightarrow R(z_i,(\vec{uz})_{\leq i},\bar{y})$. Since this holds for every i,

$$\bigwedge\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\}\rightarrow\bigwedge_i R(z_i,(\vec{uz})_{\leq i},\bar{y}),$$

that is, $\mathcal{R}$ is a good minimal substitution. $\Box$

Proposition 5 *A disjunction of good minimal substitutions is a good minimal substitution, i.e. if for every i, $1\leq i\leq n$, $\mathcal{R}_i(\bar{z}_i,\bar{y})=\bigwedge\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(\mathcal{R}_i(\bar{z}_i,\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\}$, $\bar{z}_i\subseteq\bar{x}$, then $\bigvee_i\mathcal{R}_i(\bar{z}_i,\bar{y})=\bigwedge\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(\bigvee_i\mathcal{R}_i(\bar{z}_i,\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\}$.*

Proof. Since for every $\mathcal{R}_i$

$$\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(\bigvee_i\mathcal{R}_i(\bar{z}_i,\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\}\subseteq\{\varphi(\bar{x},\bar{y}):\forall\bar{x}(\mathcal{R}_i(\bar{z}_i,\bar{y})\rightarrow\varphi(\bar{x},\bar{y}))\},$$

$$\bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\} \to \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\},$$

that is, for every $\mathcal{R}_i$, $\mathcal{R}_i \to \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$, and this implies $\bigvee_i \mathcal{R}_i \to \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$.

Now we prove that the implication holds also in the other direction. From Proposition 2 follows that if $\mathcal{R}_i(\bar{z}_i,\bar{y}) = \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$, $\bar{z}_i \subseteq \bar{x}$, then $\mathcal{R}_i(\bar{z}_i,\bar{y}) = \bigwedge\{\psi(\bar{z}_i,\bar{y}) : \forall\bar{z}_i(\mathcal{R}_i(\bar{z}_i,\bar{y}) \to \psi(\bar{z}_i,\bar{y}))\}$.

Now, assume that $\bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$ holds and none of the $\mathcal{R}_i(\bar{z}_i,\bar{y})$ holds. Then as we have just seen, there are formulas $\psi_1,\ldots,\psi_n$, such that for every i, $\forall\bar{z}_i(\mathcal{R}_i(\bar{z}_i,\bar{y}) \to \psi_i(\bar{z}_i,\bar{y}))$ and $\neg\psi_i(\bar{z}_i,\bar{y})$. Take the disjunction of these formulas, $\bigvee_i \psi_i(\bar{z}_i,\bar{y})$. The resulting formula is also false. An equivalent formula with free variables $\bar{x},\bar{y}$, namely, $\bigvee_i \psi_i(\bar{z}_i,\bar{y}) \vee \bot(\bar{x})$, belongs to the set $\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i,\bar{y}) \to \varphi(\bar{x},\bar{y}))\}$; but this formula is false, a contradiction. $\Box$

Lemma 3 *(Closure Lemma.) Let A be a conjunction of atomic formulas prefixed by $\forall$ and $\Box$-quantifiers, so that all occurrences of a predicate symbol in A have the same free variables. Then every atomic formula in A has a good minimal substitution, and this minimal substitution is the same as the one used to obtain a frame correspondent.*

Proof. The lemma follows from the four propositions proved above. $\Box$

5 Syntactic transformations

We now describe the syntactic transformations which reduce the task of finding a correspondent for an axiom χ to simple applications of the Intersection and Closure Lemmas, thus finishing the proof of the Theorem 3.

Step 1. Assume that a formula χ is of the form $\bigwedge_i \chi_i$, where each χ_i is of the form $Qz_1\ldots Qz_m(A \to B)$, A and B as in the Theorem 3. Then if every conjunct in χ corresponds to a first-order condition on R in the canonical model, say $\chi_i^\dagger$, χ itself corresponds to a first-order condition on R (namely, a conjunction of $\chi_i^\dagger$). So, it suffices to prove that every χ_i corresponds to a first-order condition on R.

Step 2. Writing down the validity conditions of χ_i, we obtain

$$\forall\varphi_1\ldots\forall\varphi_n Qz_1\ldots Qz_m(A \to B),$$

where $\forall\varphi_j$ are the quantifiers over definable relations corresponding to the predicate symbols in χ_i. If P_j is an n-place predicate symbol, then $\forall\varphi_j$ can be instantiated on any formula with precisely n variable places.

Assume that $m > 0$ (if not, we can move to the next step). Then translating the truth conditions for the quantifiers in first-order logic, we obtain

$$\forall\varphi_1\ldots\forall\varphi_n\forall z_1\ldots\forall z_m(\Gamma \to (A^* \to B^*)),$$

where Γ is a conjunction of R-conditions corresponding to the $\Box$-quantifiers in the prefix (if there are such quantifiers). This is equivalent to

$$\forall z_1\ldots\forall z_m(\Gamma \to \forall\varphi_1\ldots\forall\varphi_n(A^* \to B^*)).$$

Step 3. A may contain disjunctions; we use the fact that $\forall\varphi_1 \ldots \forall\varphi_n(\bigvee_i A_i^* \to B^*)$ is equivalent to $\forall\varphi_1 \ldots \forall\varphi_n \bigwedge_i(A_i^* \to B^*)$, and since $\forall$ distributes over $\wedge$, to $\bigwedge_i \forall\varphi_1 \ldots \forall\varphi_n(A_i^* \to B^*)$. Substituting this in the formula obtained on the previous step and applying the same reasoning (now with conjunction in the consequent), we obtain

$$\bigwedge_i \forall z_1 \ldots \forall z_n(\Gamma \to \forall\varphi_1 \ldots \forall\varphi_n(A_i^* \to B^*)).$$

It now suffices to prove that each conjunct corresponds to a first-order condition on R in an ω-saturated canonical model.

Step 4. A_i may contain constant formulas (without predicate symbols other than $=$, $\top$ and $\bot$). We move those to Γ. Let $A^* = A' \wedge \alpha$, where α are constant formulas:

$$(\Gamma \to (\alpha \wedge A' \to B^*)) \equiv (\Gamma \wedge \alpha \to (A' \to B^*))$$

Let us denote $\Gamma \wedge \alpha$ as Γ'. Note that Γ' is still first-order.

Step 5. A' may contain negative formulas: those we move to the consequent using

$$(\eta \wedge \nu \to B^*) \equiv (\nu \to \neg\eta \vee B^*)$$

Note that the consequent is still positive.

Step 6. Finally, we have a formula

$$(*) \quad \forall z_1 \ldots \forall z_m(\Gamma' \to \forall\varphi_1 \ldots \forall\varphi_n(A' \to B')),$$

where Γ' is first-order, B' is a positive first-order formula, and A' is a conjunction of formulas $(Qx_1 \ldots Qx_k\varphi_i)^*$, where all quantifiers are non-vacuous.

Assume that there is only one predicate letter P in χ. Then the reasoning goes as follows: P occurs in A' in subformulas of the form $(Qx_1 \ldots Qx_kP(\bar{x}, \bar{y}))^*$, where the $\bar{x}$ are bound and the $\bar{y}$ free (rename the bound variables if necessary). The condition (*) can be rewritten as $\forall z_1 \ldots \forall z_m(\Gamma' \to \bigwedge\{B'(\varphi(\bar{x}, \bar{y})) : \bigwedge_i(\bar{Q}_i\bar{x}\varphi(\bar{x}, \bar{y}))^*\})$.

Applying the Intersection Lemma, $\forall z_1 \ldots \forall z_m(\Gamma' \to B'(\bigwedge\{\varphi(\bar{x}, \bar{y}) : \bigwedge_i(\bar{Q}_i\bar{x}\varphi(\bar{x}, \bar{y}))^*\}))$, and by the Closure Lemma (P has a good minimal substitution in A', say p), this is equivalent to $\forall z_1 \ldots \forall z_m(\Gamma' \to B'(p))$, which is a first-order statement in R.

Now we consider the general case, when there is more than one predicate symbol. Then we eliminate the second-order quantifiers one by one in the following way. Split A' in two parts, A_1 and A_2, so that A_2 contains all and only occurrences of P_n:

$$\forall z_1 \ldots \forall z_m(\Gamma' \to \forall\varphi_1 \ldots \forall\varphi_n(A' \to B'))$$

is equivalent to $\forall z_1 \ldots \forall z_m(\Gamma' \to \forall\varphi_1 \ldots \forall\varphi_n(A_1 \wedge A_2 \to B'))$, and this in turn to $\forall z_1 \ldots \forall z_m(\Gamma' \to \forall\varphi_1 \ldots \forall\varphi_{n-1}(A_1 \to \forall\varphi_n(A_2 \to B')))$. And we apply the Intersection Lemma and the Closure Lemma to $\forall\varphi_n(A_2 \to B')$.

$$\vdots$$

This way all second-order quantifiers which bind predicate symbols occurring both in the antecedent and in the consequent can be eliminated.

If B contains predicate symbols which are not in the antecedent, these can be replaced by a fixed contradiction having the same parameters as the original atomic formula; since B is positive, and therefore monotone, the resulting formula is equivalent to the original one. Analogously, a predicate symbol occurring only in the antecedent can be replaced by a tautology.

Assume that A contains a predicate symbol which does not have a quantifier prefix, that is, $A \to B$ can be written as $A' \wedge P(\bar{x}) \to B(P(\bar{x}))$. By assumption, the $\bar{x}$ are free in B. Since B is positive, $B(P(\bar{x}))$ can be equivalent to $B' \wedge P(\bar{x})$ or $B' \vee P(\bar{x})$.

$$A' \wedge P(\bar{x}) \to B' \vee P(\bar{x})$$

obviously corresponds to a first order condition, namely a trivial one, and

$$A' \wedge P(\bar{x}) \to B' \wedge P(\bar{x})$$

is equivalent to $A' \wedge P(\bar{x}) \to B'$, the case which we treated above.

Let $\chi^\dagger$ be the result of applying steps $1-6$ to χ. We proved that if χ is an axiom, then in a canonical ω-saturated model $\chi^\dagger$ holds. The converse holds due to the argument used in the proof of the Theorem 2: we used the same minimal substitutions as in that proof, hence a correspondent in the sense of completeness is a frame correspondent. $\square$

From the proof of the theorem follows that for weak Sahlqvist formulas the correspondents are unique and equal to the frame correspondents. For weak Sahlqvist formulas which do not have (generalized) existential quantifiers in the consequent, correspondents hold in all canonical models; the remaining weak Sahlqvist formulas have correspondents which hold in all ω-saturated canonical models.

Example 4 The characteristic axiom of the "for almost all" quantifier (the Fubini property): $\Box_x\Box_y P(x,y,\bar{z}) \to \Box_y\Box_x P(x,y,\bar{z})$, corresponds to the following condition on R: $R(y,\bar{z}) \wedge R(x,y\bar{z}) \to R(x,\bar{z}) \wedge R(y,x\bar{z})$.

Proof. Rewriting the validity conditions of the axiom gives

$$\forall\varphi(\forall x\forall y(R(x,\bar{z}) \wedge R(y,x\bar{z}) \to \varphi(x,y,\bar{z})) \to \forall y\forall x(R(y,\bar{z}) \wedge R(x,y\bar{z}) \to \varphi(x,y,\bar{z}))$$

which is equivalent to

$$\bigwedge\{\forall y\forall x(R(y,\bar{z}) \wedge R(x,y\bar{z}) \to \varphi(x,y,\bar{z})) : \forall x\forall y(R(x,\bar{z}) \wedge R(y,x\bar{z}) \to \varphi(x,y,\bar{z}))\}.$$

By the Intersection Lemma,

$$\forall y\forall x(R(y,\bar{z}) \wedge R(x,y\bar{z}) \to \bigwedge\{\varphi(x,y,\bar{z})) : \forall x\forall y(R(x,\bar{z}) \wedge R(y,x\bar{z}) \to \varphi(x,y,\bar{z}))\}),$$

while by the Closure Lemma

$$\bigwedge\{\varphi(x,y,\bar{z})) : \forall x\forall y(R(x,\bar{z}) \wedge R(y,x\bar{z}) \to \varphi(x,y,\bar{z}))\} \equiv R(x,\bar{z}) \wedge R(y,x\bar{z})$$

in every canonical model. Thus we obtain the correspondent
$\forall y\forall x(R(y,\bar{z}) \wedge R(x,y\bar{z}) \to R(x,\bar{z}) \wedge R(y,x\bar{z}))$. $\square$

Example 5 The characteristic axiom of the "co-countably many" quantifier (Keisler's axiom): $\forall x \Box_y P(x,y,\bar{z}) \wedge \Box_x \forall y P(x,y,\bar{z}) \rightarrow \Box_y \forall x P(x,y,\bar{z})$, corresponds to

$$\forall x \forall y (R(y,\bar{z}) \rightarrow R(y,x\bar{z}) \vee R(x,\bar{z})).$$

Proof. The axiom is valid iff

$$\forall \varphi(\forall x \forall y (R(y,x\bar{z}) \rightarrow \varphi(x,y,\bar{z})) \wedge \forall x \forall y (R(x,\bar{z}) \rightarrow \varphi(x,y,\bar{z})) \rightarrow \forall x \forall y (R(y,\bar{z}) \rightarrow \varphi(x,y,\bar{z})),$$

namely, $\forall \varphi(\forall x \forall y (R(y,x\bar{z}) \vee R(x,\bar{z})) \rightarrow \varphi(x,y,\bar{z})) \rightarrow \forall x \forall y (R(y,\bar{z}) \rightarrow \varphi(x,y,\bar{z}))$. This can be rewritten as

$$\bigwedge \{\forall x \forall y (R(y,\bar{z}) \rightarrow \varphi(x,y,\bar{z})) : \forall x \forall y (R(y,x\bar{z}) \vee R(x,\bar{z})) \rightarrow \varphi(x,y,\bar{z}))\},$$

by the Intersection Lemma,

$$\forall x \forall y (R(y,\bar{z}) \rightarrow \bigwedge \{\varphi(x,y,\bar{z})) : \forall x \forall y (R(y,x\bar{z}) \vee R(x,\bar{z})) \rightarrow \varphi(x,y,\bar{z}))\},$$

and by the Closure Lemma, $\forall x \forall y (R(y,\bar{z}) \rightarrow R(y,x\bar{z}) \vee R(x,\bar{z}))$. □

6 Non-existence of correspondents for completeness

6.1 Restrictions imposed in the Sahlqvist theorem

In this section we show that not all formulas which have a frame correspondent also admit a correspondent for completeness. First we prove this for the truth definition we are working with; later the result is generalized.

Recall that the definition of Sahlqvist formulas (cf. Theorem 2) allowed existential and $\Diamond$-quantifiers in the antecedent, while for weak Sahlqvist this is not allowed. The following theorem gives the reason why.

Theorem 5 *$\Diamond_x \varphi \rightarrow \Diamond_x(\varphi \vee \psi)$ does not have a correspondent in the sense of completeness.*

Proof. Let us call $\Diamond_x \varphi \rightarrow \Diamond_x(\varphi \vee \psi)$ A. Although A does have a frame correspondent, namely $\forall x \forall \bar{y} \forall \bar{z}(R(x,\bar{y}) \rightarrow R(x,\bar{y}\bar{z}))$, we show that it does not have a correspondent for completeness. Assume that $A^\dagger$ exists. The following axioms are consistent with A: $\Diamond_x x = x$ and $\forall y \neg \Diamond_y x = y$. Hence, together with A they have a model where $A^\dagger$ holds. From Proposition 1 follows that $A^\dagger$ implies $\forall x \forall \bar{y} \forall \bar{z}(R(x,\bar{y}) \rightarrow R(x,\bar{y}\bar{z}))$. Since the correspondents of the other two axioms (being their *-translations) hold in every model, we have a model where $\forall x \forall \bar{y} \forall \bar{z}(R(x,\bar{y}) \rightarrow R(x,\bar{y}\bar{z}))$, $\exists x R(x)$ and $\forall x \neg R(x,x)$ hold simultaneously. But this is impossible. □

An immediate consequence of the theorem is that "$\Box$ over $\wedge$"-combination in the antecedent cannot be allowed, since the axiom considered above can be written equivalently as

$$\Box_x(\varphi \wedge \psi) \rightarrow \Box_x \varphi.$$

Also, we have

Corollary 1 *Extensionality, that is $\forall x(\varphi \equiv \psi) \rightarrow (\Box_x \varphi \equiv \Box_x \psi)$ does not have a correspondent in the sense of completeness.*

Proof. One can check that extensionality is consistent with $\Diamond_x x = x$ and $\neg\Diamond_y x = y$, and that extensionality implies $\Diamond_x\varphi \to \Diamond_x(\varphi \vee \psi)$. Hence if extensionality would have a correspondent for completeness, it would imply the frame correspondent of $\Diamond_x\varphi \to \Diamond_x(\varphi \vee \psi)$. The rest of the argument is the same as above. $\Box$

Corollary 2 $\Diamond_x\varphi \to \Box_x\varphi$ *does not have a correspondent in the sense of completeness.*

Proof. The proof is analogous; the consistency can be shown by constructing a model where D is infinite, $R(x, \bar{y}) \Leftrightarrow \bigwedge_i x \neq y_i$, and for each predicate P^n, $V(P^n) = D^n$. $\Box$

Corollary 3 $\exists x(x \neq y \wedge \varphi(x, y)) \to \Diamond_x(\varphi(x, y) \vee \psi)$ *does not have a correspondent in the sense of completeness.*

Proof. Analogous. $\Box$

The clause of the Theorem 3 forbidding occurrences of the same predicate letter with different free variables is also necessary. Suppose that there is a variable in the antecedent not occurring in the consequent, as in $\Box_y(\Box_x P(x, y) \to \Box_z P(x, z))$, is equivalent to $\Diamond_y(\Box_x P(x, y) \wedge \top(x)) \to \Box_z P(x, z)$, and $\forall y(\Box_x P(x, y) \to \Box_z P(x, z))$ is equivalent to $\exists y \Box_x P(x, y) \to \Box_z P(x, z)$.

This shows that the class of weak Sahlqvist formulas is strictly smaller than the class of all Sahlqvist formulas and that none of the conditions of the Theorem 3 can be dropped.

Digression. Reflection on the proofs of Theorems 2 and 3 suggests a closer look at the behaviour of singleton sets, which are used as minimal substitutions in the proof of the Theorem 2 (and in modal logic), but do not occur as good minimal substitutions in the proof of the Theorem 3. Note that singletons as minimal substitutions are used precisely in the cases ruled out in the Theorem 3. The semantical correlate of "singletons as minimal substitutions" is distinguishability. A model is called *distinguishable* if every element is uniquely determined by the set of formulas in one free variable which are true for this element. For example, canonical models for modal logic are distinguishable. But in general, our models will not be distinguishable. Suppose an ω-saturated canonical model satisfies $\Box_x x \neq y$ and extensionality for $\Box_x$, and d is the unique element satisfying $\bigwedge\{\varphi(x) : \varphi(d)\}$. We show that $\forall x(R(x) \to x \neq d)$. In a distinguishable model this would hold for every element, hence R would be empty.

Suppose $\exists x(R(x) \wedge x = d)$, then for all φ such that $\varphi(d)$, $\exists x(R(x) \wedge \varphi(x))$. Extensionality implies for every formula ψ

$$\exists x(R(x) \wedge \psi(x)) \to \exists x(R(x, d) \wedge \psi(x, d)).$$

Hence $\exists x(R(x, d) \wedge \varphi(x))$, and by ω-saturation $\exists x(R(x, d) \wedge \bigwedge \varphi(x))$. It follows that $R(d, d)$, a contradiction. $\Box$

6.2 Other truth definitions

Let us first of all compare the results of this paper with the results previously obtained for modal logic. Recall that a modal logic L is called *first order complete* if there is a set Δ of first order sentences in the language $\{R, =\}$ such that

$$\vdash_L \varphi \text{ if and only if for every Kripke model } M\text{: if } M \models \Delta \text{ then } M \models \varphi;$$

Sahlqvist's theorem tells us that modal logics axiomatized by Sahlqvist formulas, such as K4, S4, S5, etc. are first order complete. When we transfer this concept of first order completeness to generalized quantifiers (cf. Definition 5), we see that there is an important quantifier logic axiomatized by Sahlqvist formulas which is not first order complete, namely the "almost all" quantifier axiomatized by *L1 – L4* below plus $\Diamond_x \varphi \to \Diamond_x(\varphi \vee \psi)$ (which is equivalent to *L1 – L5* given the minimal logic). We do have first order completeness for logics axiomatized by weak Sahlqvist formulas, but since extensionality is not weak Sahlqvist, this result is not very interesting taken by itself. Indeed, we can obtain even stronger negative results, as follows. The reader may have observed that for generalized quantifier logics a stronger form of first order completeness can be defined: the truth definition we considered above is universal, but nothing prevents us from considering more complex truth definitions. Might we not obtain a first order correspondent to extensionality this way? E.g. the following truth definition studied in Jervell (1975), Mijajlovic (1985) and Krynicki (1990) trivially yields extensionality:

$$\Box_x \varphi(x) \text{ if and only if } \exists y \forall x (R(x, y) \to \varphi(x)),$$

where R is a new binary predicate and y does not occur free in φ. But in this case Keisler's axiom for "co-countably many"

$$\forall x \Box_y \varphi \wedge \Box_x \forall y \varphi \to \Box_y \forall x \varphi$$

corresponds to a schema, not to a first order condition as above.

This is not accidental. For instance, for the quantifier "almost all" it can be shown that any truth definition, however complex, involving an accessibility relation R, will make at least one axiom correspond to a schema.

In this subsection, we shall consider even more general truth definitions involving a relation $R(x, Y)$, where Y is a finite *subset* of the domain. Consideration of this relation is natural, because one might argue that dependence really is a relation between objects and finite sets of objects (compair algebraic or linear dependence). Such truth definitions, where quantifiers over finite sets are allowed, will be called *weak second order*. The truth definition that we employed up till now can be expressed in the new language as follows:

$$\Box_x \varphi(x, y_1, \ldots, y_n) \;\Leftrightarrow$$

$$\Leftrightarrow\; \exists Y [\forall x (R(x, Y) \to \varphi(x, y_1, \ldots, y_n)) \wedge y_1 \in Y \wedge \ldots \wedge y_n \in Y \wedge \forall z \in Y (z = y_1 \vee \ldots \vee z = y_n)]$$

For any given weak second order definition, we may now ask whether quantifier axioms have correspondents for completeness in the language $\{R, \in, =\}$.

Theorem 6 *For any weak second order truth definition, the conjunction of the following formulas (the Friedman axioms, see Steinhorn (1985)) will not have a correspondent for completeness:*

L1 $\Diamond_x x = x$

L2 $\Box_x x \neq y$

L3 $\Box_x \varphi \wedge \Box_x \psi \to \Box_x(\varphi \wedge \psi)$

L4 $\Box_x \Box_y \varphi \to \Box_y \Box_x \varphi$

L5 $\forall x(\varphi \rightarrow \psi) \rightarrow (\Box_x \varphi \rightarrow \Box_x \psi)$

Proof. Choose a weak second order truth definition and suppose that the conjunction of the Friedman axioms has a correspondent for completeness. Then the following statement would be true:

(C) $\exists X \exists D \exists R(X = D^{<\omega} \wedge R \subseteq D \times X \wedge$ 'R satisfies the correspondent for completeness for the conjunction of the Friedman axioms')

The part in quotes is Δ_0 since all second order quantifiers range over X. Furthermore the formula $X = D^{<\omega}$ is Σ_1, hence the entire statement is Σ_1.

From the Levy - Shoenfield Absoluteness Lemma (cf. Jech (1978), p.120) it follows that (C) holds in the constructible universe **L**. We now show that this is a contradiction.

Observe first that by using our truth definition in **L**, we may add a generalized quantifier Q to the language $\{\in, =\}$ with standard interpretation; Q will satisfy the conjunction of the Friedman axioms *for this language* since R satisfies its correspondent for completeness. In **L**, define a relation S by

$$S(x, \bar{y}) =_{df} \bigwedge_{\varphi} Qx\varphi(x, \bar{y}) \rightarrow \varphi(x, \bar{y})$$

and a new quantifier $\Box$ by $\Box_x \varphi(x, \bar{y}) =_{df} \forall x(S(x, \bar{y}) \rightarrow \varphi(x, \bar{y}))$. We show that we must have $\Box_x \varphi(x, \bar{y}) \equiv Qx\varphi(x, \bar{y})$. The direction from right to left is trivial. For the converse it suffices if Q is closed under countable intersections in **L**, for in that case

a $\quad \Box_x \varphi(x, \bar{y}) \equiv \forall x((\bigwedge_{\psi} Qx\psi(x, \bar{y}) \rightarrow \psi(x, \bar{y})) \rightarrow \varphi(x, \bar{y}))$

b $\quad \bigwedge_{\psi} Qx\psi(x, \bar{y}) \rightarrow Qx\psi(x, \bar{y}) \equiv Qx(\bigwedge_{\psi} Qx\psi(x, \bar{y}) \rightarrow \psi(x, \bar{y}))$

whence (by *L5*) $Qx\varphi(x, \bar{y})$.

To show that Q is closed under countable intersections in **L**, argue as follows: let Q be defined on D (i.e. $Q \subseteq \mathcal{P}(D)$ and $D \in Q$), and let $(A_n)_{n \in \omega}$ be in **L**, such that $A_n \subseteq D$ and for all n, $Qx(x \in A_n)$. We want to show $Qx(x \in \bigcap_n A_n)$.

Define a relation $T(x, y)$ by

$$T(x, y) =_{df} \forall n(x \in A_n^c \rightarrow \forall m(y \in A_m) \vee \exists m > n(y \in A_m^c)) \vee \forall n(x \in A_n \wedge y \in D)$$

Then $\forall x QyT(x, y)$: for if $x \in \bigcap_n A_n$, then $y \in D$, and if for some n, $x \notin A_n$, then $y \in \bigcap_{m \leq n} A_m$. Hence $QxQyT(x, y)$ and by *L4* $QyQxT(x, y)$.

If $y \in A_m^c$, then $\{x : T(x, y)\} = \bigcap_n A_n \cup \bigcup_{n \leq m} A_n^c$; and if $y \in \bigcap_m A_m$, then $\{x : T(x, y)\} = D$. It follows that $\bigcap_m A_m \subseteq \{y : QxT(x, y)\}$. If we have equality, also $Qy(y \in \bigcap_m A_m)$. Otherwise, let m be such that $y \in A_m^c$ and $QxT(x, y)$. It follows that $Qx(x \in \bigcap_n A_n \cup \bigcup_{n \leq m} A_n^c)$, whence $Qx(x \in \bigcap_n A_n)$.

Hence $\Box_x \varphi(x, \bar{y}) \equiv Qx\varphi(x, \bar{y})$, and it follows that $\Box$ also satisfies the Friedman axioms. Applying the Sahlqvist algorithm to the Friedman axioms (note that we now have obtained a canonical relational model, and that the Friedman axioms do not have existential quantifiers in the antecedent, hence their correspondents hold in every canonical model) shows that S satisfies

a $\exists x S(x)$

b $\neg S(x, x)$

c $S(x, \bar{y}\bar{z}) \to S(x, \bar{y})$

d $S(x, y\bar{z}) \wedge S(y, \bar{z}) \to S(y, x\bar{z})$

Furthermore, rewriting extensionality in terms of S shows that S satisfies, for every formula ψ^* which is a standard translation of a $\mathcal{L}_{\forall\Box}$ formula ψ

$$\exists x(S(x, \bar{y}) \wedge \neg\psi^*(x, \bar{y})) \to \exists x(S(x, \bar{y}\bar{z}) \wedge \neg\psi^*(x, \bar{y}))$$

that is

$$(\sharp) \qquad \forall x(S(x, \bar{y}\bar{z}) \to \psi^*(x, \bar{y})) \to \forall x(S(x, \bar{y}) \to \psi^*(x, \bar{y}))$$

We show that **L** does not have a definable well-ordering, which is a contradiction.

Define $P(x, \bar{z})$ as $S(x, \bar{z}) \wedge \forall y(S(y, x\bar{z}) \to x \leq y)$. Since $\exists \bar{z} \forall x \neg P(x, \bar{z})$ implies (with the axiom of Dependent Choice, which holds in **L**) the existence of an infinite descending chain, we must have $\forall \bar{z} \exists x P(x, \bar{z})$. Then we can choose x_0 with $S(x_0)$ and $\forall y(S(y, x_0) \to x_0 \leq y)$ and x_1 with $S(x_1, x_0)$ and $\forall y(S(y, x_0 x_1) \to x_1 \leq y)$. Due to the property ($\sharp$) the following holds:

$$\forall y(S(y, x_1) \to x_1 \leq y).$$

$S(x_0) \wedge S(x_1, x_0)$ implies together with **d** $S(x_0, x_1)$, therefore $x_1 \leq x_0$. This yields $x_0 = x_1$, a contradiction with $S(x_1, x_0)$ by **b** (cf. Theorem 1.5.3 in van Lambalgen (1994)). $\Box$

7 Conclusion

As we write in the introduction, the aim of this paper is to study how implicit dependencies between variables, imposed by generalized quantifier axioms, can be made explicit.

For this purpose, we define relational models for generalized quantifiers, where quantifiers are interpreted using a dependency relation between variables, and show that some axioms determine first order conditions on this relation. The situation is analogous to the one in modal logic, where some modal axioms determine first order properties of the accessibility relation (and some do not). We prove a Sahlqvist theorem for generalized quantifiers which defines a class of formulas having first order correspondents for completeness.

However, some important formulas such as extensionality are proved not to have a correspondent for completeness. The latter result is strengthened as follows. For *any* truth definition which involves a dependency relation between objects and finite sets of objects, the logic of the "almost all" quantifier is not first order complete. In this sense, not all dependencies between variables can be made explicit, i.e. expressed in the language containing only a dependency predicate. In a sequel to this paper (Alechina & van Lambalgen (1995)) we shall prove that the property of having a correspondent for completeness is undecidable.

8 References

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