Interpolation for a sequent calculus of generalized quantifiers

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Abstract. Van Lambalgen (1991) proposed a translation from a language containing a generalized quantifier Q into a first-order language enriched with a family of predicates R_i, for every arity i (or an infinitary predicate R) which takes $Qx\phi(x, y_1, \ldots, y_n)$ to $\forall x(R(x, y_1, \ldots, y_n) \rightarrow \phi(x, y_1, \ldots, y_n))$ ($y_1, \ldots, y_n$ are precisely the free variables of $Qx\phi$). The logic of Q (without ordinary quantifiers) corresponds therefore to the fragment of first-order logic which contains only specially restricted quantification. We prove that this logic (and the corresponding fragment) has the interpolation property.

1 Introduction

It is rather difficult to develop proof systems for generalized quantifiers. Van Lambalgen (1991) proposed a natural deduction system for several generalized quantifiers, such as 'for many', 'for almost all', 'for uncountably many'. As a technical device in adequacy proofs he used a satisfiability-preserving translation from the language containing a generalized quantifier Q into a first-order language enriched with a family of predicates R_i, for every arity i (or an infinitary predicate R) which takes $Qx\phi(x, y_1, \ldots, y_n)$ to $\forall x(R(x, y_1, \ldots, y_n) \rightarrow \phi(x, y_1, \ldots, y_n))$, where $y_1, \ldots, y_n$ are precisely the free variables of $Qx\phi$. R stands for dependence relation: intuitively, a quantifier ranges over objects which are dependent on the parameters of the formula ϕ. This can be used to design a proof system with elegant left- and right- introduction rules for the quantifier, with side conditions involving the dependence relation.

In (van Benthem and Alechina 1993) an alternative 'dependence semantics' for generalized quantifiers based on the intuition above was introduced, and parallels with modal logic investigated. To stress this analogy with modal logic, we use $\Diamond_x$ and $\Box_x$ to denote generalized quantifiers.

In this paper we study one particular property of the logic of $\Box_x$ and $\Diamond_x$ and of the corresponding fragment of first order logic, namely the interpolation property. We show that if $\varphi \rightarrow \psi$ is a valid implication, then there is a formula χ which contains only the predicates which occur in both φ and ψ (and probably a fixed tautology $\top$) such that $\varphi \rightarrow \chi$ and $\chi \rightarrow psi$ are valid. Since the logic of $\Box_x$ and $\Diamond_x$ can be translated into first order logic, a first order interpolant for any two formulas in our logic exists, but it does not have to belong to the same logic (to the same fragment of first order logic). So it is not immediately evident that the interpolation theorem should hold. In fact there is indeed a slight difference between the formulation of the theorem in our case and for the first order logic which will be mentioned later.

Interpolation is a natural property of a consequence relation and it also might be useful for various technical purposes. This paper does not contain any applications of the interpolation theorem to proof search methods in the logic; our main purpose was to illustrate how the interpolation property survives in a somewhat non-standard (with respect to quantifier rules) inference system.

2 Language and Models

The language $\mathcal{L}(\Diamond)$ with a generalized quantifier $\Diamond$ is the ordinary language of first-order predicate logic without ordinary $\forall$ and $\exists$, equality and functional symbols but with an existential generalized quantifier $\Diamond$. The notion of a w.f.f. is defined standartly; if φ is a w.f.f., then so is $\Diamond_x\varphi$. A universal dual of $\Diamond$ is defined as usual: $\Box_x\varphi =_{df} \neg\Diamond_x\neg\varphi$.

We will call the models which we use to interpret these languages, *dependence models*, to distinguish them from the standard models for generalized quantifiers.

Definition 1. A *dependence model* is a structure of the form $M = \langle D, R, V\rangle$ where D is a domain, V a valuation (a function assigning n-ary predicate symbols subsets of D^n), and R is a relation between elements and finite sets of elements of D, called the dependence relation. We shall write $R(a, b_1, \ldots, b_n)$ or $R(a, \bar{b})$ for $R(a, \{b_1, \ldots, b_n\})$, tacitly assuming that the second argument of R is invariant under permutations and repetitions.

The relation $M, s \models \varphi$ ('φ is *true* in M under assignment s') is defined standartly, except for

- $M, s \models \Diamond_x\psi(x, y_1, \ldots, y_n) \Leftrightarrow$ there exists $s' =_x s$ such that $R(s'(x), s'(y_1), \ldots, s'(y_n))$ and $M, s' \models \psi(x, \bar{y})$, where $\bar{y}$ are all (and just the) free variables of $\Diamond_x\psi$.
  It is easy to see that
- $M, s \models \Box_x\psi(x, \bar{y}) \Leftrightarrow$ if for all $s' =_x s$: $R(s'(x), s'(\bar{y})) \Rightarrow M, s' \models \psi(x, \bar{y})$.

We say that $M \models \varphi$ iff $M, s \models \varphi$ for all variable assignments s. □

We will call *the minimal logic of dependence models* (or just the minimal logic) the set of all valid formulas in the language of $\mathcal{L}(\Diamond)$.

Observe that the minimal logic corresponds to a fragment of first order logic enriched with a family of predicates R^i, for every arity i, under the following standard translation ST:

- $ST(P_i^n(x_1 \ldots x_n)) := P_i^n(x_1 \ldots x_n)$;
- ST commutes with classical connectives;
- $ST(\Diamond_x\varphi(x, y_1, \ldots, y_n)) := \exists x(R^{n+1}(x, y_1, \ldots, y_n) \wedge ST(\varphi(x, y_1, \ldots, y_n)))$.

Claim 1 *If φ is a formula of $\mathcal{L}(\Diamond)$, then*

$$M, s \models \varphi \;\Leftrightarrow\; M', s \models ST(\varphi),$$

for the classical model $M' = (D, V')$, where V' extends V to interpret the predicate R as R_M, and R_M satisfies, for every permutation τ,

$$R_M(a, \bar{b}) \Leftrightarrow R(a, \tau(\bar{b}))$$

$$R_M(a, c\bar{b}) \Leftrightarrow R(a, cc\bar{b}))$$

Proof. Note that our semantics can be reformulated so that R is a relation between an element and a sequence, not an element and a set, satisfying the two conditions above. □

3 Sequent Calculus

To define a sequent calculus, we introduce a new kind of variables: indexed variables (cf. van Lambalgen 1991). Thus we avoid explicitely mentioning R in the derivation. Intuitively, $x_{\bar{y}}$, where $\bar{y}$ is a finite set of variables, ranges over the objects dependent on the objects assigned to $\bar{y}$. We use them as a technical device in the proofs.

Definition 2. Let Var be the set of ordinary variables. Define a new set of variables $IVar$ by recursion, as follows:

$IVar_0 = Var$,
$IVar_{n+1} = IVar_n \cup \{x_{\bar{z}} : x \in Var,\ \bar{z} \subseteq IVar_n \text{ is a finite set}\}$
$IVar = \bigcup_n IVar_n$.

We shall refer to variables in $IVar \backslash Var$ as indexed variables. Elements of $IVar_n$ are called indexed variables of depth n. □

Definition 3. The sequent calculus for the minimal logic without equality and without ordinary quantifiers is is a formal system including *sequent axioms* of the form $\varphi \Longrightarrow \varphi$, *structural rules*:

$$\textit{Permutation}\quad \frac{\Gamma_1,\ \varphi_1,\ \varphi_2,\ \Gamma_2 \Longrightarrow \Delta}{\Gamma_1,\ \varphi_2,\ \varphi_1,\ \Gamma_2 \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta_1,\ \varphi_1,\ \varphi_2,\ \Delta_2}{\Gamma \Longrightarrow \Delta_1,\ \varphi_2,\ \varphi_1,\ \Delta_2}$$

$$\textit{Weakening}\quad \frac{\Gamma \Longrightarrow \Delta}{\varphi,\ \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta,\ \varphi}$$

$$\textit{Contraction}\quad \frac{\varphi,\ \varphi,\ \Gamma \Longrightarrow \Delta}{\varphi,\ \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta,\ \varphi,\ \varphi}{\Gamma \Longrightarrow \Delta,\ \varphi}$$

and *left- and right-introduction rules* for connectives:

$$\frac{\Gamma,\ \varphi \Longrightarrow \Delta}{\Gamma \Longrightarrow \neg\varphi,\ \Delta}\,\neg_r \qquad \frac{\Gamma \Longrightarrow \varphi, \Delta}{\Gamma,\ \neg\varphi \Longrightarrow \Delta}\,\neg_l$$

$$\frac{\Gamma,\ \varphi, \psi \Longrightarrow \Delta}{\Gamma,\ \varphi \wedge \psi \Longrightarrow \Delta}\ \neg_l \quad \frac{\Gamma \Longrightarrow \varphi, \Delta \quad \Gamma \Longrightarrow \psi,\ \Delta}{\Gamma \Longrightarrow \varphi \wedge \psi,\ \Delta}\ \neg_r,$$

dually for $\vee$;

$$\frac{\Gamma, \varphi(u_{\bar{y}}, \bar{y}) \Longrightarrow \Delta}{\Gamma, \Box_x \varphi(x, \bar{y}) \Longrightarrow \Delta}\ \Box_l \qquad \frac{\Gamma \Longrightarrow \varphi(u_{\bar{y}}, \bar{y}), \Delta}{\Gamma \Longrightarrow \Box_x \varphi(x, \bar{y}), \Delta}\ \Box_r$$

(in $\Box_r$ $u_{\bar{y}}$ does not occur in Γ, Δ, also not in the indices; in $\Box_l$ $u_{\bar{y}}$ *does* occur free in Γ or Δ);

$$\frac{\Gamma, \varphi(u_{\bar{y}}, \bar{y}) \Longrightarrow \Delta}{\Gamma, \Diamond_x \varphi(x, \bar{y}) \Longrightarrow \Delta}\ \Diamond_l \qquad \frac{\Gamma \Longrightarrow \varphi(u_{\bar{y}}, \bar{y}), \Delta}{\Gamma \Longrightarrow \Diamond_x \varphi(x, \bar{y}), \Delta}\ \Diamond_r$$

(in $\Diamond_l$ $u_{\bar{y}}$ does not occur in Γ, Δ, also not in the indices; in $\Diamond_r$ $u_{\bar{y}}$ occurs free in Γ or Δ). $\Box$

Definition 4. A *derivation* of a sequent $\Gamma \Longrightarrow \Delta$ (which does not contain indexed variables) in the sequent calculus for the minimal logic without equality and without ordinary quantifiers is a tree where each branch ends with an axiom, successor nodes are obtained from the predecessors by one of the rules, and the root of the tree is $\Gamma \Longrightarrow \Delta$. $\Box$

The introduction rules for quantifiers contain a hidden substitution rule:

$$\frac{\Gamma, \varphi(u_{\bar{y}}, \bar{y}) \Longrightarrow \Delta}{\Gamma, \varphi(x_{\bar{y}}, \bar{y}) \Longrightarrow \Delta}\ SUB_{av}\ l \qquad \frac{\Gamma \Longrightarrow \varphi(u_{\bar{y}}, \bar{y}), \Delta}{\Gamma \Longrightarrow \varphi(x_{\bar{y}}, \bar{y}), \Delta}\ SUB_{av}\ r$$

both with the restriction that $u_{\bar{y}}$ does not occur in Γ, Δ (also not in the indices).

Given this rule, the introduction rules for quantifiers can be written as

$$\frac{\Gamma, \varphi(x_{\bar{y}}, \bar{y}) \Longrightarrow \Delta}{\Gamma, \Box_x \varphi(x, \bar{y}) \Longrightarrow \Delta}\ \Box_l \qquad \frac{\Gamma \Longrightarrow \varphi(x_{\bar{y}}, \bar{y}), \Delta}{\Gamma \Longrightarrow \Box_x \varphi(x, \bar{y}), \Delta}\ \Box_r$$

with the same restrictions on $x_{\bar{y}}$ as above; analogously for $\Diamond$.

Note that if we do not impose the restriction that $\Box_l$ can be applied to $\varphi(x_{\bar{y}}, \bar{y})$ only if $x_{\bar{y}}$ occurs in some other formula, then it means that we assume the set of objects dependent on $\bar{y}$, for every $\bar{y}$, to be non-empty.

In (Alechina 1995a) we give a tableau calculus with formulas signed by T and F (cf. Smullyan 1968) which is complete for the minimal logic and for which holds

Theorem 5. *Every closed tableau for $F\varphi$ corresponds to a derivation of $\Longrightarrow \varphi$ in the sequent calculus.*

The proof can be found in (Alechina 1995b).

There are two important consequences:

Consequence 1 *(Completeness) The sequent calculus is complete.*

Since the tableau calculus mentioned above does not have the CUT rule, we have

Consequence 2 *(Cut elimination) The Cut rule is eliminable in the sequent calculus for the minimal logic.*

In (Alechina 1995a) we also prove that the minimal logic is decidable and has the finite model property. The proof is based on Lemma 8 below. All proofs in this paper except for Cut elimination are completely self contained and do not make any use of the tableau calculus mentioned above. We use a sequent calculus here since it is more suitable for proving the theorem.

Before proving the interpolation theorem, we need several technical definitions and facts.

The definition of a *main formula* of the application of a rule is standard; for example, $\varphi \wedge \psi$ is the main formula of the application of the rule $\wedge_r$ as formulated in definition 3, and φ and ψ are called main formulas in the premises.

We use a standard notion of a *successor of a formula in the derivation*: if φ is main formula in an application of the rule, it is a successor of the main formulas in the premises, and the successor relation is transitive. Note that each formula in a derivation can have many successors on a given branch, but only one at every node.

Definition 6. A formula χ is in *normal form* if in χ every subformula immediately in the scope of a quantifier contains the quantified variable of this quantifier free. □

For example, $\Diamond_y \Diamond_x (P(x,y) \wedge \Diamond_z S(x,z))$ is in normal form, and $\Diamond_x (P(x) \wedge \Diamond_z S(z))$ is not (since $\Diamond_z S(z)$ is immediately in the scope of $\Diamond_x$ and does not have x free).

If all formulas in the derived sequent are in normal form and indexed variables do not occur only in the root sequent, some useful properties hold for any formula which occurs in the derivation.

Claim 2 *Every formula in the derivation is in normal form.*

Proof. Obvious. □

Claim 3 *For any two indexed variables x and y in one formula, either x occurs in the index of y (we write it $x < y$), or y in the index of x; in other words, the set of indexed variables of one formula is linearly ordered.*

Proof. Let $\varphi(x,y)$ occur on some branch of the derivation tree and x does not occur in the index of y, and vice versa. By assumption, the sequent at the root does not contain indexed variables free. Therefore below on the branch quantifier rules must be used to replace x and y by ordinary bound variables. Assume that

x is the first variable to be replaced. Then the premise of the rule, which is a successor of $\varphi(x, y)$, $\varphi'(x, y)$, contains y free. But y is not in the index of x, therefore no quantifier rule is applicable to $\varphi'(x, y)$. Similarly, y cannot be the first variable to be replaced. This implies that x and y are free in the derived sequent, contrary to the assumption. $\square$

Claim 4 *In every formula there is an indexed variable in whose index all other indexed variables of the formula occur. It will be called the top indexed variable of the formula.*

Proof. Immediately from the previous claim. $\square$

Claim 5 *The top indexed variable of φ contains in its index all variables which are used to instantiate the quantifiers in whose scope φ occurs in the derived sequent (i.e. quantifiers which are introduced on the successors of φ in the derivation).*

Proof. Let x be the top indexed variable of φ. Let φ' be the first successor of φ which is a main premise of a quantifier rule. Since no quantifier rules were applied yet, x is free in φ'. Suppose x is not the variable to be replaced by this rule. Then there is some other variable z in φ' which is to be replaced, and $x < z$. Since x is the top variable of φ, z is not free in φ. Let $Qz_1\varphi'$ ($Q \in \{\Box, \Diamond\}$) be the result of application of the rule. Observe that φ is immediately in the scope of Qz_1 (since no quantifiers were introduced on the successors of φ before), and φ does not contain z free. This means that a derivation contains a formula not in normal form: a contradiction. Therefore, x is the variable to be replaced. Since x is the variable used to introduce a quantifier, it should contain in its index all the free variables of φ', including the top indexed variable x' of $Qz_1\varphi'$. The same argument gives that x' is the next variable to be replaced, and it contains in its index the next top indexed variable, etc. $\square$

Claim 6 *Given an arbitrary formula φ in the derivation, one can determine from the top indexed variable of φ how many quantifiers can be introduced on the successors of φ in the derivation.*

Proof. It is easy to check that this number is equal to the depth of the top indexed variable plus 1. $\square$

Claim 7 *Given an arbitrary formula φ in the derivation, one can determine from the top indexed variable of φ what are the free variables of the successor formulas of φ which are main premises of a quantifier introduction rule.*

Proof. In the index of the top variable, the indexed variables are also linearly ordered. The index of the top variable of level i are the free variables of the formula which served as a main premise of the ith quantifier introduction rule. $\square$

From the claims, we have

Lemma 7. *If two formulas φ_1 and φ_2 have the same top indexed variable, then in the derived sequent they are in the scope of the same number of quantifiers $Q_1x_1, \ldots, Q_nx_n$ and $O_1x_1, \ldots, O_nx_n$ (where $Q_i, O_i \in \{\Diamond, \Box\}$), and the biggest subformula in the scope of Q_i has the same free variables as the biggest subformula in the scope of O_i.*

This lemma will be used in the proof of the interpolation theorem below.

The following lemma constitutes the main part of the decidability proof from (Alechina 1995a).

Lemma 8. *Let Γ, Δ be sets of formulas in normal form. If $\Gamma \Longrightarrow \Delta$ is derivable, then it has a derivation where all quantifier rules involving a variable of the form $x_{\bar{y}}$ - for a fixed $\bar{y}$ - at every branch are applied simultaneously, i.e. are not separated by any other rules.*

Proof. Consider a branch in a derivation where a formula φ_1 has $x_{\bar{y}}$ free, at step m some quantifier rule is applied to φ_1, producing a formula $Qx\varphi_1(x/x_{\bar{y}})$, and at step $m+n$ some quantifier rule is applied to a formula φ_2 which has $u_{\bar{y}}$ free, producing $Qu\varphi_2(u/u_{\bar{y}})$. Let us call these two particular applications of rules $q1$ and $q2$, respectively. If $m = 1$ or all rules applied on steps $n+1, \ldots, n+m-1$ involved a variable of the form $z_{\bar{y}}$, we are done. Suppose that some other rules were applied. We prove by induction that they can be permuted either with $q1$ or with $q2$, decreasing the distance between $q1$ and $q2$ by one.

The proof is trivial for the cases when the rules in between do not involve both a successor of $Qx\varphi_1$ and a predecessor of $\varphi_2(u_{\bar{y}})$ as main premises. We are going to show that only such rules were applied. For assume otherwise. Then $Qx\varphi_1$ is a subformula of $Qu\varphi_2$. If $Qx\varphi_1$ is immediately in the scope of Qu, $Qu\varphi_2$ is not in normal form, since u is not free in $Qx\varphi_1$: a contradiction. Assume that there quantifiers $Qz_1, \ldots, Qz_k$ between Qu and Qx. In the derivation some indexed variables $z'_1, \ldots, z'_k$ are used to instantiate $Qz_1, \ldots, Qz_k$. Since all formulas in the derivation are in normal form, u is free in the scope of Qz_1. Therefore $u_{\bar{y}}$ is in the index of z'_1. Analogously z'_i is in the index of z'_{i+1}, $1 \leq i < k$, and z'_k is in the index of $x_{\bar{y}}$. Since the relation 'to be in the index of' is transitive, $u_{\bar{y}}$ is in the index of $x_{\bar{y}}$, which is impossible. $\Box$

4 Interpolation

The interpolation theorem is proved here for the logic of $\Box$ and $\Diamond$. However, it is also tells something about the corresponding fragment of first-order logic. Since it is a fragment of first order logic, every derivable sequent has an interpolant in the language containing R and the predicates occurring both in the consequent and antecedent; however, this interpolant does not have to be in the same fragment, i.e. does not have to be a translation of a quantifier formula. We prove that there is always an interpolant in the same fragment.

The proof is more complicated than the proof for the first order logic, although it follows the same strategy. For every rule we assume that the premises have interpolants, and construct an interpolant for the conclusion. For example, consider the following derivation in first order logic:

$$\dfrac{\dfrac{\varphi \Longrightarrow \psi \quad \varphi' \Longrightarrow \psi}{\varphi \vee \varphi' \Longrightarrow \psi}}{\varphi \vee \varphi' \Longrightarrow \exists x \psi}$$

Assume that the interpolants for $\varphi \Longrightarrow \psi$ and $\varphi' \Longrightarrow \psi$ are χ and χ', respectively. Then the interpolant for $\varphi \vee \varphi' \Longrightarrow \psi$ is $\chi \vee \chi'$ and the interpolant for $\varphi \vee \varphi' \Longrightarrow \exists x \psi$ is $\exists x(\chi \vee \chi')$. Since for the quantifier rules in our calculus some additional side conditions hold, the same straightforward strategy does not work. For example, if the free variables of ψ are x_y, y, the free variables of χ - x_y, y and the free variables of χ' - z, then the introduction rule for $\Diamond_x$ is applicable for ψ but not for $\chi \vee \chi'$:

$$\dfrac{\dfrac{\chi(x_y, y) \vee \chi'(z) \Longrightarrow \psi(x_y, y)}{\chi(x_y, y) \vee \chi'(z) \Longrightarrow \Diamond_x \psi(x, y)}}{\ldots}$$

Therefore some more work is needed to construct and interpolant.

Theorem 9. *If a sequent $\Gamma \Longrightarrow \Delta$ is provable in the minimal logic without identity and ordinary quantifiers, then there is a formula χ (called an interpolant) such that*

1. *$\Gamma \Longrightarrow \chi$ and $\chi \Longrightarrow \Delta$ are provable,*
2. *$PRED(\chi) \subseteq PRED(\Gamma) \cap PRED(\Delta)$ if $PRED(\Gamma) \cap PRED(\Delta) \neq \emptyset$, and $PRED(\chi) = \{\top\}$ otherwise;*
3. *$FV(\chi) \subseteq FV(\Gamma) \cap FV(\Delta)$.*

Proof. Without loss of generality we may assume that Γ and Δ are formulas in normal form and that negations are pushed inside to atomic formulas. Then $\Gamma \Longrightarrow \Delta$ has a cut free derivation where the only rules applied are $\Box_{l,r}$, $\Diamond_{l,r}$, $\wedge_{l,r}$ and $\vee_{l,r}$, and axioms are of the form

(a) $\Phi, (\neg)P(\bar{x}) \Longrightarrow (\neg)P(\bar{x}), \Psi$, and P occurs both in Γ and Δ,
(b) $\Phi, P(\bar{x}), \neg P(\bar{x}) \Longrightarrow \Psi$, and P occurs possibly only in one of Γ, Δ.
(c) $\Phi \Longrightarrow P(\bar{x}), \neg P(\bar{x}), \Psi$, and P occurs possibly only in one of Γ, Δ.

We prove the theorem by induction on the length of the derivation of $\Gamma \Longrightarrow \Delta$. First we prove the theorem with condition (2) weakened to $PRED(\chi) \subseteq (PRED(\Gamma) \cap PRED(\Delta)) \cup \{\top\}$.

Assume that $\Gamma \Longrightarrow \Delta$ is an axiom. Then χ equals $P(\bar{x})$ in case (a), $\neg\top(\bar{x})$ in case (b) and $\top(\bar{x})$ in case (c). We shall call the successors of $P(\bar{x})$ and $\neg P(\bar{x})$ *'successors of the axiom'*.

The inductive hypothesis says that for the $(n-1)$st step in the derivation the interpolant exists, and the following condition holds. Let φ_1 and φ_2 be successors of $P(\bar{x})$ and $\neg P(\bar{x})$, respectively. The inductive hypothesis says that every time when a quantifier introduction rule is applied to φ_1 or φ_2, $FV(\chi) \subseteq FV(\varphi_1) \cup FV(\varphi_2)$. Note that this additional condition is satisfied for the basis of induction.

Before proceeding further, we need the following

Claim 8 *Let $\Gamma \Longrightarrow \Delta$ and suppose χ is an interpolant for this sequent. Let $\Gamma' \Longrightarrow \Delta'$ be obtained from $\Gamma \Longrightarrow \Delta$ (and possibly some other sequent) by applying inference rules which do not involve the successors of any axiom as main premises. Then χ is an interpolant for $\Gamma' \Longrightarrow \Delta'$.*

Proof. Easy induction on the derivation of $\Gamma' \Longrightarrow \Delta'$. □

In the sequel, we consider only the cases when one of the main premises of the rule is a successor of an axiom.

Assume that $\Gamma \Longrightarrow \Delta$ is obtained by a rule $\wedge_l$ or $\vee_r$ from a sequent which has an interpolant. Then, if χ is an interpolant for $\Gamma', \varphi_i, \psi \Longrightarrow \Delta'$, then it is an interpolant for $\Gamma', \varphi_i \wedge \psi \Longrightarrow \Delta'$, analogously for $\vee_l$.

Consider the rules $\wedge_r$ and $\vee_l$. Let χ_1 be an interpolant for $\Gamma', \varphi_i \Longrightarrow \Delta'$ and χ_2 for $\Gamma', \psi \Longrightarrow \Delta'$. Then $\chi_1 \vee \chi_2$ is an interpolant for $\Gamma', \varphi_i \vee \psi \Longrightarrow \Delta'$. Analogously for $\wedge_r$.

Finally, we consider the quantifier rules. Since $P(\bar{x})$ and $\neg P(\bar{x})$ have the same free variables, they also have the same top indexed variable, which by Lemma 7 means that both formulas are in the scope of the same number of quantifiers in $\Gamma \Longrightarrow \Delta$ and that if φ_1 is the successor of $P(\bar{x})$ to which the ith quantifier rule is applied, and φ_2 is the successor of $\neg P(\bar{x})$ to which the ith quantifier rule is applied, then φ_1 and φ_2 have the same free variables (including the top indexed variable on which the quantifier is introduced).

From Lemma 8 it follows that any tableau proof can be arranged so that all decomposition rules involving a variable dependent on a given set of variables $\bar{y}$ can be applied simultaneously, not separated by other rules. Therefore, without loss of generality we may assume that the ith application of a quantifier rule to the successor formula of $P(\bar{x})$ is immediately followed by an application of a quantifier rule to the successor formula of $\neg P(\bar{x})$, or vice versa.

Let φ_1 be a successor formula of $P(\bar{x})$, φ_2 of $\neg P(\bar{x})$, χ the interpolant for

$$\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'$$

and the next rule in the derivation is the ith quantifier rule. By our assumption, φ_1 and φ_2 have the same free variables, and since $FV(\chi) \subseteq FV(\varphi_1) \cup FV(\varphi_2)$, χ has the same free variables or less. We may assume that χ has precisely the same free variables, since we can always add some tautology with the lacking variables conjunctively to χ.

The first rule must be one of the rules $\Box_l$ or $\Diamond_r$, since the variable on which the quantifier is introduced is free in another formula. Let is be $\Box_l$:

$$\frac{\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}{\Gamma', \Box_x \varphi_1(x, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}$$

The rule applied to φ_2 can be either $\Box_r$ or $\Diamond_r$.

Assume that it is $\Box_r$:

$$\dfrac{\dfrac{\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}{\Gamma', \Box_x \varphi_1(x, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}}{\Gamma', \Box_x \varphi_1(x, \bar{y}) \Longrightarrow \Box_x \varphi_2(x, \bar{y}), \Delta'}$$

The applicability of $\Box_r$ means that $x_{\bar{y}}$ is not free in Γ' and Δ'.

But then we have

$$\dfrac{\dfrac{\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \chi(x_{\bar{y}}, \bar{y})}{\Gamma', \Box_x \varphi_1(x, \bar{y}) \Longrightarrow \chi(x_{\bar{y}}, \bar{y})}}{\Gamma', \Box_x \varphi_1(x, \bar{y}) \Longrightarrow \Box_x \chi(x, \bar{y})}$$

and

$$\dfrac{\dfrac{\chi(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}{\Box_x \chi(x, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}}{\Box_x \chi(x, \bar{y}) \Longrightarrow \Box_x \varphi_2(x, \bar{y}), \Delta'}$$

that is, $\Box_x \chi$ is the interpolant for

$$\Gamma', \Box_x \varphi_1(x, \bar{y}) \Longrightarrow \Box_x \varphi_2(x, \bar{y}), \Delta'.$$

Assume that the rule applied to φ_2 is $\Diamond_r$:

$$\dfrac{\dfrac{\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}{\Gamma', \Box_x \varphi_1(x, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}}{\Gamma', \Box_x \varphi_1(x, \bar{y}) \Longrightarrow \Diamond_x \varphi_2(x, \bar{y}), \Delta'}$$

The applicability of $\Diamond_r$ means that $x_{\bar{y}}$ is free in Γ' or Δ' (or both). If $x_{\bar{y}}$ is free in Γ', but not in Δ', we can prove

$$\dfrac{\dfrac{\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \chi(x_{\bar{y}}, \bar{y})}{\Gamma', \Box_x \varphi_1(x, \bar{y}) \Longrightarrow \chi(x_{\bar{y}}, \bar{y})}}{\Gamma', \Box_x \varphi_1(x, \bar{y}) \Longrightarrow \Diamond_x \chi(x, \bar{y})}$$

(the last step is justified since $x_{\bar{y}}$ is free in Γ'), and

$$\dfrac{\dfrac{\chi(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}{\chi(x, \bar{y}) \Longrightarrow \Diamond_x \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'}}{\Diamond_x \chi(x, \bar{y}) \Longrightarrow \Diamond_x \varphi_2(x, \bar{y}), \Delta'}$$

(the last step is justified since $x_{\bar{y}}$ is not free in Δ), that is, $\Diamond_x \chi$ is the interpolant for

$$\Gamma', \Box_x \varphi_1(x, \bar{y}) \Longrightarrow \Diamond_x \varphi_2(x, \bar{y}), \Delta'.$$

Analogously, if $x_{\bar{y}}$ is free only in Δ', the interpolant is $\Box_x \chi$.

If $x_{\bar{y}}$ is free in both Γ' and Δ', we temporarily leave the interpolant to be $\chi(x_{\bar{y}}, \bar{y})$. Note that $FV(\chi) \subseteq FV(\Gamma) \cap FV(\Delta)$, but $FV(\chi) \nsubseteq FV(\Box_x \varphi_1) \cup FV(\Diamond_x \varphi_2)$. To satisfy the induction hypothesis, $x_{\bar{y}}$ should be bound before the next rule is applied to the successors of $\Box_x \varphi_1$ and $\Diamond_x \varphi_2$. By Lemma 8 we may assume that $x_{\bar{y}}$ is bound before any other quantifier rule is applied to the successors of the axiom (since that rule would involve another variable). The formulas which are the main premises in the applications of quantifier rules on $x_{\bar{y}}$, are not successors of $\Box_x \varphi_1$ and $\Diamond_x \varphi_2$ (otherwise $\Box_x \varphi_1$ and $\Diamond_x \varphi_2$, which do not contain $x_{\bar{y}}$ free, would be immediately in the scope of a quantifier introduced on $x_{\bar{y}}$). We assume therefore that these rules may be performed before any new rule is applied to the successors of $\Box_x \varphi_1$ and $\Diamond_x \varphi_2$, and that by the time that $x_{\bar{y}}$ does not occur free any more in Γ' or in Δ', the interpolant has not changed. Then we can introduce a quantifier on χ as above.

If the first rule applied to

$$\Gamma', \varphi_1(x_{\bar{y}}, \bar{y}) \Longrightarrow \varphi_2(x_{\bar{y}}, \bar{y}), \Delta'$$

is $\Diamond_r$, the reasoning is completely analogous.

So far we have proved $PRED(\chi) \subseteq (PRED(\Gamma) \cap PRED(\Delta)) \cup \{\top\}$. To obtain clause (2) of theorem 9, we reason as follows. If $PRED(\Gamma) \cap PRED(\Delta) \neq \emptyset$, $\top$ is definable for every arity n. Namely, let P^m be both in Γ and Δ; then it is easy to define $\top^n$ for $n \geq m$. For $n < m$, take $\Diamond_{x_1} \ldots \Diamond_{x_k} P(x_1, \ldots, x_k, \ldots) \vee \neg \Diamond_{x_1} \ldots \Diamond_{x_k} P(x_1, \ldots, x_k, \ldots)$. $\Box$

Observe that if $\Gamma \Longrightarrow \Delta$ is provable, and $PRED(\Gamma) \cap PRED(\Delta) = \emptyset$, this does not mean that $\Gamma \Longrightarrow$ is provable or $\Longrightarrow \Delta$ is provable. Consider for example $\Box_x P(x) \wedge \Box_x \neg P(x) \Longrightarrow \Box_x (Q(x) \wedge \neg Q(x))$.

If we assume that for any indexed variable the domain is non-empty (which is equivalent to $\forall \bar{y} \exists x R(x, \bar{y})$), the restrictions on $\Box_l$ and $\Diamond_r$ should be dropped. The proof of the interpolation theorem remains the same, but in the step for quantifiers, we should consider one more case: when both rules are weak $\Box_l$ and $\Diamond_r$, and $x_{\bar{y}}$ is not free in Γ and Δ. Then the interpolant can be $\Box_x \chi$ as well as $\Diamond_x \chi$.

For the system without restrictions on $\Box_l$ and $\Diamond_r$, $\Gamma \Longrightarrow \Delta$ and $PRED(\Gamma) \cup PRED(\Delta) = \emptyset$ mean, as in classical predicate logic, that $\Gamma \Longrightarrow$ or $\Longrightarrow \Delta$. The proof of $\Gamma \Longrightarrow$ ($\Longrightarrow \Delta$) is obtained from the proof of $\Gamma \Longrightarrow \Delta$ by omitting the righthand side (lefthand side) of the derivation; now this does not influence the applicability of $\Box_l$ and $\Diamond_r$.

5 Conclusion

The main point of the paper can be summarized as follows: introducing more structure on the set of variables (or, more generally, objects used to instantiate quantifiers) allows to set up elegant and well behaved proof systems. Hopefully, the interpolation proof above can serve as an illustration of this. (Alechina and van Lambalgen 1995) contains some philosophical discussion of this approach as well as more technical details. The general idea of such proof systems can be found, for example, in (Fine 1985) (observe that instead of using indices on variables one might reformulate the quantifier rules above in terms of dependence graphs between variables).

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