Quantification over interdependent variables

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1 Introduction

In standard first order logic, individual variables take values in domains of points which do not have any structure except the one expressible in the language. Each variable can take any value; the assignments are completely independent. We are going to give up this assumption and look what happens to quantifiers in the new setting.

In many-sorted logic the assumption *is* already weakened: each sort has its own domain, therefore it is no longer true that any variable can take any value. In other words, while in the standard case $\exists x\varphi$ is true in a model M under an assignment s iff there is an element d in the domain of M such that $M, s^x_d \models \varphi$, for many sorted logic we should add that d should be in the domain associated with the sort of x. Within each sort the situation is as before. Moreover, it is easy to translate many-sorted logic back into ordinary predicate logic extended with unary predicates for each sort. There is almost no difference between the two, except for the fact that the translation mentioned above leads to increase in complexity.

A more radical variation would be the following (cf. (Németi 1992)): assume that not all assignments to individual variables are allowed. This has a clear intuitive motivation which can be illustrated by the following example from programming languages (cf. (Wirth 1976)). Consider a variable *date* which is an array of variables *day*, *month*, *year*. The values of *date* are triples of numbers, where the first number is the value of *day*, the second number is the value of *month*, and the third number is the value of *year*. The variable *day* may take values in the set $1, \ldots, 31$; the values of *month* come from $1, \ldots, 12$ and year can be any whole number. Each of these three variables can be updated (or, in our setting, quantified over) separately. It is clear however that if the value of *month* is 2, then the value of *day* cannot be 30; depending on the year, *day* may or may not take the value 29. Observe that this is not reducible to the case of many-sorted logic. Some assignments to *day, month* and *year* are not allowed while they respect the domains of the variables. In the truth definition for $\exists$ one might require that s^x_d is an allowed assignment. Here the *names* of variables become important; there can be a witness for $\exists x P(x)$ but no witness for $\exists y P(y)$ in a model.

Again this can be translated into first order logic by adding a special predicate for 'allowed assignment', in the example above, a ternary predicate *Date* would do. But in this case, the difference in complexity is even more striking. It was proved by Németi that the logic corresponding to the set of models equipped with arbitrarily restricted sets of assignments is decidable!

Yet another variation on the standard way of looking at quantifiers and variables is evaluating quantifiers 'locally' (cf. (Blackburn and Seligman 1995)). Consider the truth definition of a modal operator:

$$M \models \Diamond\varphi[w] \Leftrightarrow \exists v(wRv \wedge M, v \models \varphi)$$

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A modal operator can be seen as a quantifier binding a variable; the name of this variable is not important, but the range of a bound variable (the set of values it may take) depends on the point where the formula is evaluated.

In (van Lambalgen 1991) some generalized quantifiers were given an analogous interpretation, as quantifiers over objects dependent on the parameters of the formula: $Q^d x\varphi(x, y_1, \ldots, y_n)$ is interpreted as $\exists x(R(x, y_1, \ldots, y_n) \wedge \varphi(x, y_1, \ldots, y_n))$; the meaning of a quantifier depends on an assignment to the free variables and is therefore local.

In this paper we try to develop a uniform approach to the phenomena mentioned above.

2 Basic logic

2.1 Language and models

Here we investigate in general the idea that the range of a variable can be restricted by the values of some other 'relevant' variables. We introduce a logic corresponding to the following truth definition: $\exists x\varphi(x, y_1, \ldots, y_n)$ is true in a model M under an assignment of values d_i to y_i, if there is an object d, such that $\varphi(d, d_1, \ldots, d_n)$ holds in M and d is a possible value for x given the assignment of values to y_i. For the time being we assume that only the free variables of the formula $\exists x\varphi$ are relevant for determining the range of x.

Consider a language $\mathcal{L}(\Diamond)$, which contains countably many individual variables $v_0, \ldots, v_n, \ldots$, predicate symbols, $\neg$, $\wedge$, $=$ and a quantifier $\Diamond$. A well formed formula of $\mathcal{L}(\Diamond)$ is defined as follows:

1. if $x_1, \ldots, x_n$ are individual variables and P an n-place predicate symbol, then $P(x_1, \ldots, x_n)$ is a w.f.f.; if x, y are individual variables, then $x = y$ is a w.f.f.;
2. if φ_1 and φ_2 are w.f.f.'s, then $\neg\varphi_1$ and $\varphi_1 \wedge \varphi_2$ are w.f.f.'s;
3. if φ is a w.f.f. and x an individual variable, then $\Diamond_x\varphi$ is a w.f.f.;
4. nothing else is a w.f.f.

We define $\Box_x\varphi$ to denote $\neg\Diamond_x\neg\varphi$; $\vee, \rightarrow, \equiv$ are defined as usual.

Definition 1 A *structured dependence model* is a structure of the form $M = \langle D, R, V\rangle$ where D is a domain, V an interpretation function (a function assigning n-ary predicate symbols subsets of D^n), and R is a relation between a pair $\langle$*variable*, *object*$\rangle$ and a finite set of such pairs. Intuitively, R tells which value a variable can take given the values which other variables have taken. We could have demanded from the start that if $\langle x, d\rangle R\{\langle y_i, d_i\rangle : 1 \leq i \leq n\}$, then $x, y_1, \ldots, y_n$ are all different variables; cf. Lemma 6.

The relation $M, s \models \varphi$ ('φ is *true* in M under assignment s') is defined as follows:

- $M, s \models P_i^n(x_{j1} \ldots x_{jn}) \Leftrightarrow \langle s(x_{j1}) \ldots s(x_{jn})\rangle \in V(P_i^n)$;
- $M, s \models x = y \Leftrightarrow s(x) = s(y)$;
- $M, s \models \neg\varphi \Leftrightarrow M, s \not\models \varphi$;
- $M, s \models \varphi \wedge \psi \Leftrightarrow M, s \models \varphi$ and $M, s \models \psi$;
- $M, s \models \Diamond_x\psi(x, y_1, \ldots, y_n) \Leftrightarrow$ there exists $s' =_x s$ such that
$$\langle x, s'(x)\rangle\ R\ \{\langle y_1, s'(y_1)\rangle, \ldots, \langle y_n, s'(y_n)\rangle\}$$
and $M, s' \models \psi(x, \bar{y})$, where $\bar{y}$ are precisely the free variables of $\Diamond_x\psi$. (This will be denoted by $FV(\Diamond_x\psi) = \bar{y}$.)

We say that $M \models \varphi$ iff $M, s \models \varphi$ for all variable assignments s. $\square$

If Γ and Δ are sets of formulas, we say that $\Gamma \models \Delta$ if for every model M and assignment s, if $M, s \models \Gamma$, then $M, s \models \Delta$. The logic corresponding to this notion of consequence is called L_{str}.

Observe that the truth definition for van Lambalgen's generalized quantifiers can be obtained from the truth definition above by erasing the first element in each pair (or dismissing some information about the variables):

$$s'(x)\ R\ \{s'(y_1), \ldots, s'(y_n)\}$$

instead of

$$\langle x, s'(x)\rangle\ R\ \{\langle y_1, s'(y_1)\rangle, \ldots, \langle y_n, s'(y_n)\rangle\}$$

We will see later that this move strengthens the logic by making the axiom $\diamond_x\varphi \rightarrow \diamond_y\varphi[x/y]$ valid (given that y is free for x in φ).

Another special case of structured dependence models are *partial assignments models* defined in section 3.

To finish this section, we prove that structured dependence models share a property which is taken for granted in classical first order logic but, as we shall see later, does not hold for all modal quantifiers.

Lemma 2 *(Locality) For every structured dependence model M, pair of assignments s and z and formula φ, if $s\lceil FV(\varphi) = z\lceil FV(\varphi)$, then*

$$M, s \models \varphi \Leftrightarrow M, z \models \varphi.$$

Proof. Completely analogous to the proof for the classical first order logic, the proof goes by induction on the complexity of φ. For atomic formulas the proposition obviously holds, and the steps for $\neg$ and $\wedge$ are also trivial.

Let $\varphi = \diamond_x\psi(x, \bar{y})$, and the proposition hold for $\psi(x, \bar{y})$. Assume that $M, s \models \diamond_x\psi(x, \bar{y})$. Then there is an assignment $s' =_x s$ such that the pair $\langle x, s'(x)\rangle$ is in the relation R to the set of pairs $\{\langle y_i, s'(y_i)\rangle : y_i \in FV(\diamond_x\psi)\}$ and $M, s' \models \psi$. Consider an assignment $z' =_x z$ with $z'(x) = s'(x)$. Obviously, for every variable u in $FV(\psi)$, $z'(u) = s'(u)$; therefore by the inductive hypothesis $M, z' \models \psi$. Also, since $\langle x, s'(x)\rangle = \langle x, z'(x)\rangle$ and $\langle y_i, s'(y_i)\rangle = \langle y_i, z'(y_i)\rangle$ for every $y_i \in FV(\diamond_x\psi)$, $\langle x, z'(x)\rangle$ is in the relation R to $\{\langle y_i, z'(y_i)\rangle : y_i \in FV(\diamond_x\psi)\}$, which gives $M, z \models \diamond_x\psi(x, \bar{y})$. $\square$

2.2 Hilbert-style axiomatization

To give a Hilbert-style axiomatization for the logic of structured dependence models, we need to extend the language by countably many individual constant symbols. The idea is to repeat a standard Henkin argument; however, this extension is less trivial than in case of the first-order logic. The difficulty comes from the fact that the variables have much more 'individuality' than in first order logic. As a result, the constants should also acquire some.

We add to $\mathcal{L}(\diamond)$ a set of individual constants d^i_j, countably many for each i. We call individual variables and constants *terms*. Intuitively, d^i_j 'behaves approximately as the variable v_i'. The precise meaning of this will become clear below. Formally, we introduce a function $\eta : TERMS \longrightarrow VARIABLES$, such that $\eta(v_i) = v_i$ and $\eta(d^i_j) = v_i$. We call a term t a *n-term*, if $\eta(t) = v_n$.

A well formed formula is defined as usual. A formula without free variables is called a *sentence*.

The interpretation of constants is the same as in first order logic. If $M = \langle D, R, V\rangle$, and d^i_j is a constant, then $V(d^i_j) \in D$.

The meaning of a term t in a model M given an assignment s, $[t]_{M,s}$ is as expected: if $t = v_i$, $[t]_{M,s} = s(v_i)$, and if $t = d^i_j$, $[t]_{M,s} = V(d^i_j)$. The following clauses of the truth definition are extended to the case of formulas containing constants:

- $M, s \models P^n_i(t_1 \dots t_n) \Leftrightarrow \langle [t_1]_{M,s} \dots [t_n]_{M,s}\rangle \in V(P^n_i)$;
- $M, s \models \Diamond_x \psi(x, t_1, \dots, t_n) \Leftrightarrow$ there exists $s' =_x s$ such that
$$\langle x, s'(x)\rangle \; R \; \{\langle \eta(t_1), [t_1]_{M,s'}\rangle, \dots, \langle \eta(t_n), [t_n]_{M,s'}\rangle\}$$
and $M, s' \models \psi(x, \bar{t})$.

It is in the last clause that the idea that constants are instances of a particular variable becomes clear. For example,

$$M, s \models \Diamond_{v_3} P(d^1_1, v_3) \Leftrightarrow$$
$$\Leftrightarrow \exists d \in D(\langle v_3, d\rangle R\{\langle v_1, V(d^1_1)\rangle\} \wedge M, s^3_d \models P(d^1_1, v_3))$$

Note that $\Diamond_{v_3} P(d^4_1, v_3)$ can be false in the same model, even if $V(d^1_1) = V(d^4_1)$. Analogously, $\Diamond_{v_3} P(v_1, v_3)$ can be true and $\Diamond_{v_3} P(v_4, v_3)$ false even if $v_1 = v_4$. This corresponds to the failure of the following axiom (even in the language without constants):
$$t_1 = t_2 \wedge \varphi(t_1) \rightarrow \varphi(t_2),$$
given that t_2 is free for t_1 is φ. Instead, two weaker versions of this axiom are valid: the version where φ is an atomic formula and the version where t_1 and t_2 are terms of the same type.

Definition 3 *L_{str} is the following axiom system:*

A0 *propositional logic;*

A1 $\Box_x(\varphi \rightarrow \psi) \rightarrow (\Box_x\varphi \rightarrow \Box_x\psi)$*, given that* $\Box_x\varphi$ *and* $\Box_x\psi$ *contain the same parameters (free variables and constants);*

A2 $\varphi \rightarrow \Box_x\varphi$*, given that x is not free in φ;*

=1 $t = t$*;*

=2 $t_1 = t_2 \rightarrow t_2 = t_1$*;*

=3 $t_1 = t_2 \wedge t_2 = t_3 \rightarrow t_1 = t_3$*;*

=4 $t_1 = t_2 \wedge \varphi(t_1) \rightarrow \varphi(t_2)$, *given that t_2 is free for t_1 in $\varphi(t_1)$ and either* (=4a) *φ is atomic, or* (=4b) $\eta(t_1) = \eta(t_2)$*;*

R1 $\varphi,\ \varphi \rightarrow \psi/\psi$*;*

R2 $\varphi(t^i)/\Box_{v_i}\varphi[t^i/v_i]$*, where t^i is an i-term, i.e. v_i or d^i_j.*

The definition of a derivation in L_{str} is standard. We will write $\Gamma \vdash_{L_{str}} \varphi$ for 'φ is derivable from Γ in L_{str}'. In the sequel, the subscript is omitted when it is clear which system weare talking about.

In L_{str} the deduction theorem holds with restrictions (analogous to the restrictions in classical first order logic; cf. for example (Mendelson 1979)).

The following formulas are derivable in L_{str}:

A1b $\Box_x(\varphi \to \psi) \to (\varphi \to \Box_x\psi)$, given that $\Box_x\varphi$ and $\Box_x\psi$ contain the same parameters and x is not free in φ. This formula follows immediately from A1 and A2.

T $\Box_x(\Box_x\varphi \to \varphi)$.

Observe that $\Box_x\varphi \to \varphi$ is not valid in L_{str} since it is possible that φ is false on an assignment s, but true on every other assignment which can be obtained from s by changing the value of x to some d, so that $\langle x, d\rangle$ is in the relation R to the assignment of values to the rest of free variables of φ; this just means that $\langle x, s(x)\rangle$ does not satisfy this condition. Prefixing $\Box_x\varphi \to \varphi$ by $\Box_x$ means that we switch to the assignments s with $\langle x, s(x)\rangle$ accessible from the free variables of $\Box_x\varphi \to \varphi$; on such assignments, if $\Box_x\varphi$ is valid, so is φ.
Proof.

1. $\varphi(x) \to (\Box_x\varphi(x) \to \varphi(x))$ — A0
2. $\Box_x(\varphi(x) \to (\Box_x\varphi(x) \to \varphi(x)))$ — 1, R2
3. $\Box_x\varphi(x) \to \Box_x(\Box_x\varphi(x) \to \varphi(x)))$ — 2, A1, R1
4. $\neg\Box_x\varphi(x) \to (\Box_x\varphi \to \varphi(x))$ — A0
5. $\Box_x(\neg\Box_x\varphi(x) \to (\Box_x\varphi \to \varphi(x)))$ — 4, R2
6. $\neg\Box_x\varphi(x) \to \Box_x(\Box_x\varphi(x) \to \varphi(x))$ — 5, A1b, R1
7. $\Box_x(\Box_x\varphi(x) \to \varphi(x))$ — 3, 6, A0, R1 $\Box$

Observe that if ψ_1 and ψ_2 have the same free variables (with the possible exception of x), then from $\psi_1 \to (\psi_2 \to (\psi_1 \wedge \psi_2))$ by two applications of A1 we can derive $\Box_x\psi_1 \wedge \Box_x\psi_2 \to \Box_x(\psi_1 \wedge \psi_2)$. This implies that if all formulas χ_i have the same free variables, then by T the following formula (used in the proof below) is derivable:

$$\Box_x \bigwedge_i (\Box_x\chi_i \to \chi_i).$$

Now we prove completeness of L_{str} for the original language, not containing constants. We prove that for every set of formulas Γ and formula φ, if Γ and φ do not contain constants,

$$\Gamma \vdash \varphi \Leftrightarrow \Gamma \models \varphi.$$

Theorem 4 *L_{str} is sound and complete with respect to the class of structured dependence models: $\Gamma \vdash \varphi \Leftrightarrow \Gamma \models \varphi$.*

Proof. The proof of soundness is easy. As usual, we show that axioms are valid and inference rules preserve validity.

It is obvious that A1 is valid. Observe that A2 is valid due to the fact that structured dependence models satisfy locality. We have already given an example showing that unrestricted =4 is not valid. Now we prove that =4a and =4b are valid. Let φ be atomic. Observe that $M, s \models t_1 = t_2 \Leftrightarrow [t_1]_{M,s} = [t_2]_{M,s}$. $M, s \models P(\ldots t_1 \ldots)$ iff $\langle \ldots [t_1]_{M,s} \ldots\rangle \in V(P)$ iff $\langle \ldots [t_2]_{M,s} \ldots\rangle \in V(P) \Leftrightarrow M, s \models P(\ldots t_2 \ldots)$. This shows that =4a is valid. To prove that =4b is valid, assume that $\eta(t_1) = \eta(t_2)$, and reason by induction on complexity of φ. We have already proved the atomic case. The cases for $\neg$ and $\wedge$ are trivial. Finally,

$$M, s \models \Diamond_x\varphi(\ldots t_1 \ldots) \Leftrightarrow \exists d(\langle x, d\rangle\ R\ \{\ldots \langle \eta(t_1), [t_1]_{M,s}\rangle \ldots\} \wedge M, s^x_d \models \varphi);$$

since t_2 is free for t_1 and $\eta(t_1) = \eta(t_2)$, this is equivalent to

$$\exists d(\langle x, d\rangle\ R\ \{\ldots\langle\eta(t_2), [t_2]_{M,s}\rangle\ldots\} \wedge M, s^x_d \models \varphi(t_1/t_2)) \Leftrightarrow M, s \models \Diamond_x\varphi(\ldots t_2\ldots).$$

To show that R2 preserves validity, assume that $\Box_{v_i}\varphi(v_i)$ is not valid, that is, there exist a model M and an assignment s such that $M, s \not\models \Box_{v_i}\varphi(v_i)$. Then there is a sequence s', which differs from s at most in i, such that $M, s' \not\models \varphi$. If t^i is v_i, we are done: $\varphi(v_i)$ is not valid. If t^i is some d^i, interpret d^i as $s(v_i)$ (observe that d^i is not free in $\varphi(v_i)$!). An argument analogous to the proof of validity of $= 4b$ above shows that $\neg\varphi(d^i)$ is satisfiable, thus $\varphi(d^i)$ is not valid.

For completeness, it is enough to show that every set Δ not containing constants and consistent with L_{str} has a model.

Assume that we have such a set Δ. We are going to construct a canonical model for Δ. The construction is analogous to the proof of completeness theorem for the minimal logic of dependence models by van Benthem given in (van Benthem and Alechina 1993).

We extend the language by adding countably many new constants

$$u^0_0, u^0_1, u^0_2, \ldots, u^0_n, \ldots$$
$$\ldots$$
$$u^m_0, u^m_1, u^m_2, \ldots, u^m_n, \ldots$$
$$\ldots$$

As usual, the domain of the model for Δ will consist of the equivalence classes of constants. Replace each free occurrence of each variable v_i in Δ by u^i_0. The resulting set of sentences is called Δ'. If we construct a model for Δ', then this model and the assignment $s_0 : v_i \mapsto [u^i_0]$ will satisfy Δ.

Make a list of sentences in the new language. Let us call this list Φ.

Let $\Sigma_0 = \Delta$, and $\Sigma_n = \Sigma_{n-1}$ if the nth formula in Φ is not consistent with Σ_{n-1}. If φ_n is consistent with Σ_{n-1}, we have two possibilities. If φ_n is not of the form $\Diamond_{v_i}\psi$, then $\Sigma_n = \Sigma_{n-1} \cup \{\varphi_n\}$.

If φ_n is of the form $\Diamond_{v_i}\psi$, we add a witness for $\Diamond_{v_i}$ as follows. We take the first new constant u^i_j which does not occur in Σ_{n-1} and add to Σ_{n-1} $\psi[v_i/u^i_j]$ and the set $\{\Box_{v_i}\chi \to \chi(v_i/u^i_j) : \Box_{v_i}\chi$ contains precisely the same constants as $\Diamond_{v_i}\psi\}$.

Let $\Sigma = \bigcup_n \Sigma_n$. Assume that Σ is inconsistent. Then a contradiction is derivable at some finite step n. Let Σ_{n-1} be consistent, and assume that Σ_n is not. Then for some $\Diamond_{v_i}\psi$,

$$\Sigma_{n-1} \cup \{\bigwedge_\chi(\Box_{v_i}\chi \to \chi(u^i_j)) \wedge \Diamond_{v_i}\psi \wedge \psi(u^i_j)\}$$

is inconsistent. Again, a contradiction is derivable from some finite conjunction of formulas in Σ_{n-1} and a finite conjunction of $\Box_{v_i}\chi_k \to \chi_k$ for $1 \leq k \leq m$.

By R2 we obtain

$$\Sigma_{n-1} \vdash \Box_{v_i} \bigwedge_{k\leq m} (\Box_{v_i}\chi_k \to \chi_k(v_i)) \to \Box_{v_i}(\Diamond_{v_i}\psi(v_i) \to \neg\psi(v_i))$$

Now we apply A1 (all χ_k have the same parameters) and T to derive

$$\Box_{v_i} \bigwedge_{k\leq m} (\Box_{v_i}\chi_k \to \chi_k).$$

Together with the statement above, this gives

$$\Sigma_{n-1} \vdash \Box_{v_i}(\Diamond_{v_i}\psi(v_i) \to \neg\psi(v_i)),$$

which by A1b implies

$$\Sigma_{n-1} \vdash \Diamond_{v_i}\psi(v_i) \to \Box_{v_i}\neg\psi(v_i),$$

that is,

$$\Sigma_{n-1} \vdash \Diamond_{v_i}\psi(v_i) \to \neg\Diamond_{v_i}\psi(v_i),$$

i.e. contrary to our assumption Σ_{n-1} is inconsistent (note that Σ_{n-1} does not contain u^i_j free, therefore the deduction theorem is applicable).

Now we have established that Σ is a maximal consistent set, i.e. it is consistent, it contains, for every formula $\varphi \in \Phi$, either φ or $\neg\varphi$, it is closed deductively, and it contains $\varphi_1 \wedge \varphi_2$ iff it contains both φ_1 and φ_2.

We define a model M_Σ for Δ using Σ as follows. Let the domain be the set of equivalence classes of constants, i.e. elements of the form $[u^i_j] = \{u : u = u^i_j \in \Sigma\}$.

For the interpretation function, we put

$$\langle [d_1], \ldots, [d_n] \rangle \in V(P) \text{ iff } P(d_1, \ldots, d_n) \in \Sigma.$$

This is well defined by the axiom =4a. The interpretation of constants is quite natural: $V(u) = [u]$.

The definition of R is slightly less straightforward:

$$\langle v_{i_1}, [d_1] \rangle R \{ \langle v_{i_2}, [d_2] \rangle, \ldots, \langle v_{i_n}, [d_n] \rangle \}$$

holds in M_Σ iff for some $e_1, \ldots, e_n$, such that $e_j \in [d_j]$ and$\eta(e_j) = i_j$ and for all formulas χ, such that $\Box_{v_{i_1}}\chi$ contains precisely the constants $e_2, \ldots, e_n$,

$$\forall\chi(\Box_{v_{i_1}}\chi(v_{i_1}, e_2, \ldots, e_n) \to \chi(e_1, e_2, \ldots, e_n) \in \Sigma).$$

Observe that this is equivalent to

$$\forall\chi(\chi(e_1, e_2, \ldots, e_n) \to \Diamond_{v_{i_1}}\chi(v_{i_1}, e_2, \ldots, e_n) \in \Sigma)$$

Also, observe that by =4b, if

$$\forall\chi(\Box_{v_{i_1}}\chi(v_{i_1}, e_2, \ldots, e_n) \to \chi(e_1, e_2, \ldots, e_n) \in \Sigma)$$

holds for *some* $e_1, \ldots, e_n$ with $e_j \in [d_j]$ and $\eta(e_j) = i_j$, then it holds for *all* such constants.

Lemma 5 *(Truth Lemma) For every sentence φ, such that φ contains not more than oneterm of each type in the scope of each quantifier, $M_\Sigma \models \varphi \Leftrightarrow \varphi \in \Sigma$.*

Proof. The proof of the lemma goes by induction. The inductive hypothesis is somewhat stronger than the statement of the lemma; it says

$$\varphi(d_1, \ldots, d_n) \in \Sigma \Leftrightarrow M_\Sigma, s \models \varphi(v_{i_1}, \ldots, v_{i_m}, d_{m+1}, \ldots, d_n),$$

$0 \leq m \leq n$, where $s(v_{i_j}) = [d_j]$ and $\eta(d_j) = i_j$ for $j \leq m$, for every formula φ which satisfies the condition of the lemma.

Without this condition, the assignment s above is not well defined. Namely, if φ contains two different constants d and e of type i, which are not in the same equivalence class, s would have to assign v_i two different objects $[d]$ and $[e]$.

The clause for atomic formulas follows from the definition of V_Σ. The clauses for negation and conjunction are trivial. Assume that φ is of the form $\Diamond_{v_{i_1}}\psi(v_{i_1}, d_2, \ldots, d_n)$ with $\eta(d_j) = i_j$.

Let $\Diamond_{v_{i_1}}\psi(v_{i_1}, d_2, \ldots, d_n) \in \Sigma$. By the construction of Σ, there is a witness for $\Diamond_{v_{i_1}}$, namely a constant d_1 with $\eta(d_1) = i_1$, such that $\psi(d_1, \ldots, d_n) \in \Sigma$ and

$$\forall\chi(\Box_{v_{i_1}}\chi(v_{i_1}, d_2, \ldots, d_n) \to \chi(d_1, d_2, \ldots, d_n) \in \Sigma).$$

By the definition of R, the latter means

$$\langle v_{i_1}, [d_1]\rangle R\{\langle v_{i_2}, [d_2]\rangle, \ldots, \langle v_{i_n}, [d_n]\rangle\}.$$

By the inductive hypothesis, $M_\Sigma, s \models \psi(v_{i_1}, \ldots, v_{i_m}, d_{m+1}, \ldots, d_n)$, where $s(v_{i_j}) = [d_j]$. Thus $M_\Sigma, s' \models \Diamond_{v_{i_1}}\psi(v_{i_1}, \ldots, v_{i_m}, d_{m+1}, \ldots, d_n)$, where s' is any assignment which differs from s in the i_1st coordinate.

Let $M_\Sigma, s \models \Diamond_{v_{i_1}}\psi(v_{i_1}, \ldots, v_{i_m}, d_{m+1}, \ldots, d_n)$, where $s(v_{i_j}) = [d_j]$ for $1 < j \leq m$, and $\eta(d_j) = i_j$ for $m < j \leq n$. Then for some $[d_1]$

$$\langle v_{i_1}, [d_1]\rangle R\{\langle v_{i_2}, [d_2]\rangle, \ldots, \langle v_{i_n}, [d_n]\rangle\}$$

and $M_\Sigma, s^{i_1}_{[d_1]} \models \psi(v_{i_1}, \ldots, v_{i_m}, d_{m+1}, \ldots, d_n)$. By the definition of R, there are $e_1, \ldots, e_n$, with $e_j \in [d_j]$ and $\eta(e_j) = i_j$, $1 \leq j \leq n$, such that

$$\forall\chi(\chi(e_1, e_2, \ldots, e_n) \to \Diamond_{v_{i_1}}\chi(v_{i_1}, e_2, \ldots, e_n) \in \Sigma).$$

By the inductive hypothesis, $\psi(e_1, \ldots, e_n) \in \Sigma$. Therefore $\Diamond_{v_{i_1}}\psi(v_{i_1}, e_2, \ldots, e_n) \in \Sigma$. By =4b, the same holds for any $e'_2, \ldots, e'_n$ in the eqivalence classes $[d_2], \ldots, [d_n]$ if $\eta(e_j) = i_j$.

This ends the proof of the truth lemma. □

Observe that the set Δ' consists of sentences satisfying the condition of the lemma; so, the lemma gives $M_\Sigma \models \Delta'$.

Coming back to the restriction on the set of formulas for which the truth lemma holds, note that since the extended language contains formulas of the form $\varphi(u^1_0, u^1_1)$, it may happen that the following set of pairs is in R:

$$\langle v_1, u^1_2\rangle R\{\langle v_1, u^1_0\rangle, \langle v_1, u^1_1\rangle\}.$$

This looks quite counterintuitive given that in accordance with the intended meaning of R this means: 'if the variable v_1 takes the value u^1_0 *and* the value u^1_1, u^1_2 is a possible value for v_1'. Later, when we interpret

$$\langle x_1, d_1\rangle R\{\langle x_2, d_2\rangle, \ldots, \langle x_n, d_n\rangle\}$$

as '$\{\langle x_i, d_i\rangle : 1 \leq i \leq n\}$ is a good *assignment*', i.e. a function, this becomes completely unacceptable. So the last step in the construction of the canonical model is to show that we may leave only such decent, or functional sets of pairs in the relation R and throw away all non-functional sets of pairs in R, where some variable occurs with different objects.

Lemma 6 *Let φ be a formula which contains at most one term of each type in the scope of each quantifier. Then for every structured dependence model M and assignment s, $M, s \models \varphi \Leftrightarrow M', s \models \varphi$, where M' is obtained from M by leaving only functional sets of pairs in the relation R.*

Proof. The proof goes by induction on the complexity of φ. The only non-trivial step is the quantifier clause. Let φ be $\Diamond_{v_i}\psi(v_i, t_1, \dots, t_n)$. By assumption, $\eta(t_j) \neq v_i$ for $1 \leq j \leq n$, and $\eta(t_j) \neq \eta(t_k)$ for $1 \leq j \neq k \leq n$. But then there is an element d satisfying

$$\langle v_i, d\rangle \ R \ \{\langle \eta(t_1), [t_1]_{M,s}\rangle, \dots \langle \eta(t_n), [t_n]_{M,s}\rangle\}$$

if and only if there is a corresponding element in M', since the set of pairs above is functional. Therefore

$$M, s \models \Diamond_{v_i}\psi(v_i, t_1, \dots, t_n) \Leftrightarrow M', s \models \Diamond_{v_i}\psi(v_i, t_1, \dots, t_n).$$

This finishes the proof of the lemma and of the Theorem 4. □

3 Obtaining stronger logics

3.1 In the direction of generalized quantifiers

Every generalized quantifier satisfies the axiom of alphabetic variants: $\Box_x\varphi \rightarrow \Box_y\varphi(x/y)$, with y free for x in φ. It turns out that this axiom is the only one which we need to add to L_{str} to obtain the minimal logic of *dependence models* defined in (van Benthem and Alechina 1996):

Definition 7 A *dependence model* is a structure of the form $M = \langle D, R, V\rangle$ where (D, V) is as before, and R is a relation between elements and finite sets of elements of D, called the dependence relation.

The relation $M, s \models \varphi$ is defined as usual, except for $M, s \models \Diamond_x\psi(x, y_1, \dots, y_n) \Leftrightarrow$ there exists $s' =_x s$ such that $R(s'(x), \{s'(y_1), \dots, s'(y_n)\})$ and $M, s' \models \psi(x, \bar{y})$, where $\bar{y}$ are precisely the free variables of $\exists x\psi$. □

Proposition 8 *$L_{str} + \Box_x\varphi \rightarrow \Box_y\varphi(x/y)$ is complete with respect to the class of dependence models.*

Proof. Follows from the completeness theorem for the minimal logic of dependence models, given in (van Benthem and Alechina 1996), (Alechina 1995). □

Observe that the *names* of the variables do not matter in dependence models, where R is a relation between an object and a finite set of objects. Not surpisingly, the axiom of renaming bound variables corresponds to a sort of permutation equivalence for variables:

Consequence 9 *$L_{str} + \Box_x\varphi \rightarrow \Box_y\varphi(x/y)$ is complete with respect to the class of structured dependence models satisfying the following property: for any permutation of variables in $\mathcal{L}$ τ,*

$$\langle v_{i_1}, d_{i_1}\rangle R\{\langle v_{i_2}, d_{i_2}\rangle, \dots, \langle v_{i_n}, d_{i_n}\rangle\} \equiv$$

$$\equiv \langle \tau(v_{i_1}), d_{i_1}\rangle R\{\langle \tau(v_{i_2}), d_{i_2}\rangle, \dots, \langle \tau(v_{i_n}), d_{i_n}\rangle\} \equiv R(d_{i_1}, \{d_{i_2}, \dots, d_{i_n}\}).$$

Other examples of strengthening the logic to obtain 'real' generalized quantifiers are given in (Alechina and van Lambalgen 1995a).

The dependence relation in dependence models has a sort of built-in antisymmetry; in a derivation, relations between parameters can be represented as a directed acyclic graph (cf. (Alechina and van Lambalgen 1995b)). The situation is very different for assignments models.

3.2 Circular dependencies, or assignments

In this section we take a somewhat different approach to modal quantifiers, based on the intuition that the truth of a quantified formula depends on availability of a certain assignment (rather than a certain object). First, we consider logics for partial assignments models.

Definition 10 A *partial assignment* is a finite set of pairs $\langle x, d\rangle$, where x is a variable and d an element of the domain. $\square$

There are two possibilities for defining partial assignments models. Andréka and Németi proposed (for assignments models) to distinguish models where only part of assignments are present (Crs) and models where all assignments are present, but only part of them is available for the quantifier (Crs^+). The first kind of models satisfies $\varphi \to \Diamond_x \varphi$, and the second kind does not. The latter +-variant is closer to the intuitions underlying the logic of structured dependence models; we shall see that one additional condition on R yields the logic corresponding to the +-variant of partial assignments models. To obtain the variant without +, we'll have to change the truth definition instead of just adding one more condition on R. Most of the proofs below go by way of first proving a certain property for the + variant and then extending the result to the proper (partial) assignments models. This is the reason why we consider both usual and + variants.

Definition 11 A *partial assignments model* is a structure of the form $M = \langle D, W, V\rangle$, where $\langle D, V\rangle$ is a first order model and W is an arbitrary non-empty set of partial assignments. The relation $M, s \models \varphi$, where $s : FV(\varphi) \longrightarrow D$ is a partial assignment, is defined as follows:

- The clause for atoms and negation is the same as for structured dependence models (with assignments restricted to the free variables of a formula);
- $M, s\lceil FV(\psi_1 \wedge \psi_2) \models \psi_1 \wedge \psi_2 \Leftrightarrow M, s\lceil FV(\psi_1) \models \psi_1 \ \wedge \ M, s\lceil FV(\psi_2) \models \psi_2$;
- $M, s\lceil(y_1, \dots, y_n) \models \Diamond_x \psi(x, y_1, \dots, y_n) \Leftrightarrow$
  $\Leftrightarrow \ \exists d \in D(s' = \{\langle x, d\rangle, \langle y_1, s(y_1)\rangle, \dots, \langle y_n, s(y_n)\rangle\} \in W \ \wedge \ M, s' \models \psi(x, \bar{y}))$

We say that a formula φ is true in a partial assignments model M ($M \models \varphi$) if for all assignments s of the right length *in* W $M, s \models \varphi$. The set of formulas valid in this class of models is called $L_{p.a.}$. For a more familiar notion of validity, namely $M \models \varphi$ if *all* assignments of the right length satisfy φ, we use the notation $L^+_{p.a.}$, following the notation proposed by Andréka and Németi for Crs^+_n. $\square$

This truth definition demands something stronger than that the pair $\langle x, d\rangle$ is accessible from the rest. The fact that

$$\{\langle x, d\rangle, \langle y_1, s(y_1)\rangle, \dots, \langle y_n, s(y_n)\rangle\}$$

is an admissible partial assignment implies that also $\langle y_1, s(y_1)\rangle$ is accessible from $\{\langle x, d\rangle, \langle y_2, s(y_2)\rangle, \dots, \langle y_n, s(y_n)\rangle\}$, and so on. The relation '$\langle x, d\rangle$ and the set of pairs D' form a good assignment' is not directed; in a sense, every pair in $\{\langle x, d\rangle\} \cup D'$ depends on the rest.

Lemma 12 *The class of structured dependence models satisfying the* circularity property*: if*

$$\langle x_{i_1}, d_{i_1}\rangle R \{\langle x_{i_2}, d_{i_2}\rangle, \dots, \langle x_{i_n}, d_{i_n}\rangle\}$$

holds, then for every pair $\langle x_{i_j}, d_{i_j}\rangle$,

$$\langle x_{i_j}, d_{i_j}\rangle R \{\langle x_{i_1}, d_{i_1}\rangle, \dots, \langle x_{i_n}, d_{i_n}\rangle\} \backslash \{\langle x_{i_j}, d_{i_j}\rangle\},$$

defines the same logic as $L^+_{p.a.}$.

Proof. Let φ be any formula, and assume there are a structured dependence model M and an ordinary assignment s, such that $M, s \models \varphi$. Define $M_{p.a.}$ to have the same domain and interpretation function as M and put

$$\{\langle x_1, d_1\rangle, \ldots, \langle x_n, d_n\rangle\} \in W \;\Leftrightarrow\; \langle x_1, d_1\rangle \; R \; \{\langle x_2, d_2\rangle, \ldots, \langle x_n, d_n\rangle\}$$

This is well defined given the circularity property. An easy induction shows that

$$M, s \models \varphi \Leftrightarrow M_{p.a.}, s\lceil FV(\varphi) \models \varphi.$$

Conversely, let M be a partial assignments model, s a partial assignment and $M, s \models \varphi$. We define M_{str} to have the same domain and interpretation, as M, and

$$\langle x, d\rangle \; R \; z \Leftrightarrow \{\langle x, d\rangle\} \cup z \in W,$$

where z is a partial assignment. Let s' agree with s on the free variables of φ, and assign some fixed element of D to the rest of the variables. Again, it is easy to show by induction that $M_{str}, s' \models \varphi$. □

Proposition 13 *L_{str} + $\Box_{v_i}(\varphi \to \Diamond_{v_j}\varphi)$, where $v_i, v_j \in FV(\varphi)$ (A3) is complete with respect to the class of structured dependence models satisfying circularity:*

$$\langle v_i, d_i\rangle R\{\ldots \langle v_j, d_j\rangle \ldots\} \Longrightarrow \langle v_j, d_j\rangle R\{\ldots \langle v_i, d_i\rangle \ldots\}$$

Proof. To show that the axiom is valid in the class of models satisfying circularity, consider an arbitrary model M from this class, an assignment s and a formula φ with $FV(\varphi) = v_i, v_j, \bar{v}$. Note that if $v_i = v_j$, the axiom is valid already in L_{str} (it is equivalent to T). Consider a sequence $s' =_i s$ with $\langle v_i, s'(v_i)\rangle \; R \; s\lceil\bar{v} \cup \{\langle v_j, s(v_j)\rangle\}$, such that $M, s' \models \varphi$. Then by circularity $\langle v_j, s(v_j)\rangle \; R \; s\lceil\bar{v} \cup \{\langle v_i, s'(v_i)\rangle\}$, that is, $M, s' \models \Diamond_{v_j}\varphi$. But then $M, s \models \Box_{v_i}(\varphi \to \Diamond_{v_j}\varphi)$.

To prove completeness, we use the fact that any set of sentences consistent with $L_{str} + A3$ has a canonical model M_Σ as constructed above. Assume that in M_Σ

$$\langle v_i, d_i\rangle R\{\langle v_1, d_1\rangle, \ldots, \langle v_n, d_n\rangle\}\backslash\{\langle v_i, d_i\rangle\}$$

holds, that is, for every formula χ with constants $d_1, \ldots, d_n$, $\chi \to \Diamond_{v_i}\chi \in \Sigma$ (or, equivalently, $\Box_{v_i}\chi \to \chi \in \Sigma$). It suffices to show that

$$\langle v_j, d_j\rangle R\{\langle v_1, d_1\rangle, \ldots, \langle v_n, d_n\rangle\}\backslash\{\langle v_j, d_j\rangle\},$$

that is, that for every formula χ with constants $d_1, \ldots, d_n$, $\chi \to \Diamond_{v_j}\chi \in \Sigma$. Take an arbitrary formula χ with constants $d_1, \ldots, d_n$. Consider the formula $\chi \to \Diamond_{v_j}\chi$. This formula (let us call it θ) contains constants $d_1, \ldots, d_n$. By assumption, $\Box_{v_i}\theta \to \theta \in \Sigma$. But $\Box_{v_i}\theta$ is an axiom, so $\theta \in \Sigma$. □

Due to Lemma 12, this proves the completeness of L_{str} + A3 wrt $L^+_{p.a.}$.

Proposition 14 *Adding to L_{str} the axiom $\varphi \to \Diamond_{v_j}\varphi$ (A4) yields a logic complete wrt $L_{p.a.}$.*

Proof. Note that A4 implies A3, so we can use the previous proof to show that there is a partial assignments model M_Σ and an assignment s_0 satisfying a formula φ consistent with A0 – A4, R1, R2. But we do not know yet whether this assignment is in W.

If A4 is true in the canonical model M_Σ, then every assignment

$$\{\langle v_{i_1}, d_{i_1}\rangle, \ldots, \langle v_{i_n}, d_{i_n}\rangle\}$$

is a good assignment given that all d_{i_j} are of type i_j. Note that the assignment s_0 satisfying the original formula φ (which contains only the variables and no constants) is of such type, therefore s_0 is in W. This gives us $M_\Sigma, s_0 \in W \models \Delta$. □

4 Crs_n-models

Now we move to classes of models where the notion of assignment stands really central; assignments have a fixed length and do not shrink and extend depending on a formula.

To define assignments models, in particular models with assignments of length α, we need first to define formally a language with at most α variables.

Definition 15 $\mathcal{L}_\alpha(\Diamond)$ is a first order language containing α many variables v_i, $0 \leq i < \alpha$, $=$, $\neg, \wedge$ and $\Diamond$ and countably many predicate symbols. Well formed formulas (w.f.f.'s) are defined in a standard way. $\Box$

In the meta-language we will use $x_1, \ldots, x_n, \ldots$ to denote arbitrary variables of $\mathcal{L}_\alpha(\Diamond)$.

Definition 16 A Crs_α*-model* for $\mathcal{L}_\alpha(\Diamond)$ is a structure M of the following form: $M = \langle D, W, V \rangle$, where D is a non-empty domain, $W \subseteq D^\alpha$ is non-empty, and V is an interpretation function. Since there are α many variables, an *assignment* of elements of D to the variables is any element of D^α. The relation 'a formula φ is true in M under assignment s ($M, s \models \varphi$)' is defined *only for assignments in* W. The clauses for atomic formulas, $\neg$ and $\wedge$ are as before, and

$$M, s \models \Diamond_x \psi \Leftrightarrow \exists s' =_x s(s' \in W \wedge M, s' \models \psi). \qquad \Box$$

We write $M \models \varphi$ (φ true in M) for $\forall s \in W(M, s \models \varphi)$.

Definition 17 A Crs^+_α*-model* for $\mathcal{L}_\alpha(\Diamond)$ is a structure M of the following form: $M = \langle D, W, V \rangle$, where D is a non-empty domain, $W \subseteq D^\alpha$ is non-empty, and V is an interpretation function. The relation 'a formula φ is true in M under assignment s ($M, s \models \varphi$)' is defined inductively as above.

Now $M \models \varphi$ (φ true in M) stands for $\forall s \in D^\alpha(M, s \models \varphi)$. $\Box$

A formula is $Crs_\alpha(Crs^+_\alpha)$-valid, if it is true in all models. A formula is $Crs_\alpha(Crs^+_\alpha)$-satisfiable, if its negation is not $Crs_\alpha(Crs^+_\alpha)$-valid.

It is clear that for $\alpha = \omega$ assignments models cannot be reduced to a special case of structured dependence models, since R must become an infinite relation. For $\alpha < \omega$, there is a resemblance between partial assignments models and assignments models. But there is an important property of structured dependence models which holds also for dependence models and partial assignments models (being special cases of structured dependence models), but not for assignments models, namely locality.

To get a feeling why locality fails in assignments models, consider the following example for $\mathcal{L}_2(\Diamond)$. Let M be a Crs_2 (Crs^+_2)-model with domain $D = \{0, 1\}$, $V(P) = \{\langle 1 \rangle\}$ and W consisting of the following assignments:

$$\begin{array}{rcc} & v_0 & v_1 \\ s = & 0 & 0 \\ s' = & 1 & 0 \\ s'' = & 0 & 1 \end{array}$$

Then $M, s \models \Diamond_{v_1} P(v_1)$ and $M, s' \not\models \Diamond_{v_1} P(v_1)$, although these assignments agree on the free variables of $\Diamond_{v_1} P(v_1)$.

The failure of locality is the reason why the axiom $\varphi \rightarrow \Box_x \varphi$, given that x is not free in φ, is not valid in assignments models. Namely, consider the model above; $M, s \models \Diamond_{v_1} P(v_1)$ but $M, s \not\models \Box_{v_0} \Diamond_{v_1} P(v_1)$.

Since we cannot give an axiomatization of assignments models based on L_{str}, we define an *embedding* of Crs^+_n-logic into $L^+_{p.a.}$ and of Crs_n-logic into $L_{p.a.}$. Both embeddings use the following simple idea. Consider the formulas of the n variable language $\mathcal{L}_n$. The difference in the truth conditions for these formulas in assignments models and in partial assignments models is that in the former a quantifier takes into account the assignment of values to all n variables, whether they occur free under the quantifier or not, whereas in the latter only the free variables are important. For the formulas which contain all n variables free under a quantifier, the truth conditions are the same. We take a formula φ of $\mathcal{L}_n$ and add dummy variables under every quantifier in φ. The result is called φ^n. We show that φ^n is equivalent to φ in assignments models, and that φ^n is satisfiable in assignments models if, and only if, it is satisfiable in partial assignments models. This gives the desired embedding.

Observe that formulas which have all n variables under each quantifier satisfy locality in Crs_n:

Lemma 18 *Let φ be a formula of $\mathcal{L}_n(\Diamond)$ which contains all n variables in the scope of each quantifier. Then for every Crs (Crs^+)-model M and a pair of assignments s, z which agree on the free variables of φ, $M, s \models \varphi \Leftrightarrow M, z \models \varphi$.*

Proof. The proof goes by induction on the complexity of φ; for atomic formulas the claim is obviously true, the clauses for $\neg$ and $\wedge$ use the induction hypothesis in a standard way. Assume $M, s \models \Diamond_{v_i}\psi$. Then there is an assignment $s' =_i s$ in W such that $M, s' \models \psi$. Since all variables except v_i are free in $\Diamond_{v_i}\psi$, and z agrees with s on the free variables of $\Diamond_{v_i}\psi$, $z =_i s'$, which gives $M, z \models \Diamond_{v_i}\psi$. $\Box$

This property will be also used in the proof of embedding results below.

4.0.1 Embedding of Crs_n^+ into $L_{p.a.}^+$

Define a translation function ${}^n : \mathcal{L}_n(\Diamond) \longrightarrow \mathcal{L}_n(\Diamond)$ ($\mathcal{L}_n(\Diamond)$ is the language using only the variables $v_0, \ldots, v_{n-1}$) as follows: ^{n} commutes with atomic formulas and propositional connectives, and

$$(\Diamond_{v_i}\psi)^n = \Diamond_{v_i}(\top(v_0, \ldots, v_{n-1}) \wedge \psi).$$

Proposition 19 *$Crs_n^+ \models \varphi \Leftrightarrow L_{p.a.}^+ \models \varphi^n$.*

Proof. We will prove $Crs_n^+ \not\models \varphi \Leftrightarrow L_{p.a.}^+ \not\models \varphi$. Assume that $Crs_n^+ \not\models \varphi$. Then there is a Crs_n^+ model for $\neg\varphi$. Since in Crs_n^+ $\neg\varphi$ and $(\neg\varphi)^n$ are equivalent, the same model satisfies $(\neg\varphi)^n$. To prove the proposition, it is enough to show that for every $\theta \in \mathcal{L}_n(\Diamond)$, θ^n is Crs_n^+-satisfiable iff θ^n is $L_{p.a.}^+$-satisfiable.

Assume that θ^n is Crs_n^+-satisfiable, i.e. there is a Crs_n^+-model M and an assignment (of length n) s_0, such that $M, s_0 \models \theta^n$. Define a partial assignments model $M_{p.a.+}$ as follows: $V_{p.a.+} = V$, $D_{p.a.+} = D$, and $W_{p.a.+} = W$.

We prove by induction that for every subformula ψ of θ^n

$$M, s \models \psi \Leftrightarrow M_{p.a.+}, s\lceil FV(\psi) \models \psi.$$

The clauses for atoms and $\neg$ are obvious. The clause for $\wedge$ is also easy:

$$M, s \models \psi_1 \wedge \psi_2 \Leftrightarrow M, s \models \psi_1 \;\wedge\; M, s \models \psi_2 \Leftrightarrow$$

$$\Leftrightarrow M_{p.a.+}, s\lceil FV(\psi_1) \models \psi_1 \;\wedge\; M_{p.a.+}, s\lceil FV(\psi_2) \models \psi_2 \Leftrightarrow$$
$$\Leftrightarrow M_{p.a.+}, s\lceil FV(\psi_1 \wedge \psi_2) \models \psi_1 \wedge \psi_2$$

Finally, $M, s \models \Diamond_{v_i}\psi \Leftrightarrow \exists s' =_i s(s' \in W \;\wedge\; M, s' \models \psi)$; since ψ is a subformula of θ in the scope of a quantifier, $FV(\psi) = \{v_0, \ldots, v_{n-1}\}$, and $s'\lceil FV(\psi) = s'$. Therefore the condition above is equivalent to

$$\exists d(s' = s\lceil FV(\Diamond_{v_i}\psi) \cup \{\langle v_i, d\rangle\} \in W_{p.a.+} \;\wedge M_{p.a.+}, s' \models \psi),$$

i.e. $M_{p.a.+}, s\lceil FV(\Diamond_{v_i}\psi) \models \Diamond_{v_i}\psi$.

This shows $M_{p.a.+}, s_0\lceil FV(\theta^n) \models \theta^n$, i.e. if θ^n is Crs_n^+-satisfiable, then it has a $L_{p.a.}^+$-model.

To prove the opposite direction, suppose that θ^n is satisfiable in a partial assignments model M and an assignment s_0. We construct a Crs_n^+-model M' and an assignment $a(s_0)$ with $M', a(s_0) \models \theta^n$. Let M' have the same domain and interpretation function as M, and $W' = W \cap \{s : dom(s) = \{v_0, \dots, v_{n-1}\}\}$. Of course, all assignments in M' are of length n.

Let a be an arbitrary function which gives a partial assignment s from M an assignment $a(s)$ in M', so that $a(s)\lceil dom(s) = s$.

Now we prove $M, s \models \psi \Leftrightarrow M', a(s) \models \psi$. The clause for atoms and negation is trivial. For the conjunction we have:

$$M, s \models \psi_1 \wedge \psi_2 \Leftrightarrow M, s\lceil FV(\psi_1) \models \psi_1 \;\wedge\; M, s\lceil FV(\psi_2) \models \psi_2 \Leftrightarrow$$

$$\Leftrightarrow M', a(s\lceil FV(\psi_1)) \models \psi_1 \;\wedge\; M', a(s\lceil FV(\psi_2)) \models \psi_2$$

Since ψ_1 and ψ_2 are subformulas of θ^n, and $a(s)\lceil FV(\psi_i) = a(s\lceil FV(\psi_i))\lceil FV(\psi_i)$, by lemma 18 this is equivalent to

$$M', a(s) \models \psi_1 \;\wedge\; M', a(s) \models \psi_2 \Leftrightarrow M', a(s) \models \psi_1 \wedge \psi_2.$$

The clause for the quantifier is easy. Since the assignments used in the quantifier clause are of length n, we use the fact that such an assignment $s' \in W'$ iff $s' \in W$:

$$M, s \models \Diamond_{v_i}\psi \Leftrightarrow \exists d(s' = s \cup \{\langle v_i, d\rangle\} \in W \;\wedge\; M, s' \models \psi) \Leftrightarrow$$

$$\Leftrightarrow \exists s' =_i a(s) \in W'(M', s' \models \psi) \Leftrightarrow M', a(s) \models \Diamond_{v_i}\psi.$$

This gives $M', a(s_0) \models \theta^n$, that is, if θ^n has a $L_{p.a.}^+$-model, then θ^n has a Crs_n^+-model. □

4.0.2 Embedding of Crs_n into $L_{p.a.}$

Given the result above, it is easy to show that the same embedding works for Crs_n and $L_{p.a.}$.

Proposition 20 $Crs_n \models \varphi \Leftrightarrow L_{p.a.} \models \varphi$.

Proof. The beginning of the argument is the same as above; the problem is reduced to proving that θ^n is Crs_n-satisfiable iff θ^n is $L_{p.a.}$-satisfiable.

Given the definition of a partial assignments model, it is clear that $L_{p.a}$- and $L_{p.a.+}$-models are the same, and only the notions of satisfiability are different. (We cannot reformulate the definition of partial assignments models so that the satisfaction relation would be defined only for the sequences in W; cf. the clause for conjunction.) Therefore, if φ is satisfiable in an $L_{p.a.}^+$-model and the satisfying sequence is in W, then φ is $L_{p.a.}$-satisfiable.

The corresponding fact for the assignments models is also true, but needs a bit more elaborating. Let M be a Crs_n-model, and M^+ be the corresponding Crs_n^+-model (with the same D, W and V). Then for $s \in W$

$$M, s \models \varphi \Leftrightarrow M^+, s \models \varphi.$$

The proof is an obvious induction.

Consider a formula θ^n from the previous embedding proof. Without loss of generality we may assume that θ^n has all n variables free (in all logics, θ^n is satisfiable iff $\theta^n \wedge \top(v_0, \dots, v_{n-1})$ is satisfiable).

Assume that θ^n is $L_{p.a.}$ satisfiable, i.e. there is an $L_{p.a.}$ model M and an assignment (of length n) $s \in W$ such that $M, s \models \theta^n$. From the analysis above, M is an $L^+_{p.a.}$ model as well, and the embedding proof above gives a Crs^+_n model M' and assignment $a(s) = s$ which satisfy θ^n. Since s is in W and s is of length n, we have $M', s \in W' \models \theta^n$. But then there is a Crs_n-model satisfying θ^n.

Assume that there is a Crs_n-model M satisfying θ^n. Then there is a Crs^+_n-model M^+ such that $M^+, s \in W \models \theta^n$. By the embedding argument, there is an $L^+_{p.a.}$-model $M_{p.a.+}$ with $W_{p.a.+} = W$, such that $M_{p.a.+}, s\lceil FV(\theta^n) \models \theta^n$. But $s\lceil FV(\theta^n) = s$ and $s \in W$. Therefore, θ^n is $L_{p.a.}$-satisfiable. □

From the following fact (which is well-known, cf. for example (Németi 1992), but which we restate here for the sake of completeness) follows that also Crs_ω is embeddable in $L_{p.a.}$:

Proposition 21 *For every w.f.f. with at most n first variables φ, $\models_{Crs_n} \varphi$ iff $\models_{Crs_\omega} \varphi$.*

Proof. We prove that $\neg\varphi$ is Crs_n-satisfiable iff $\neg\varphi$ is Crs_ω-satisfiable.

The direction from left to right is easy: one can always transform a Crs_n model M, satisfying $\neg\varphi$, into Crs_ω model, by adding the same infinite 'tail' to all sequences in M.

For the converse, let M be a Crs_ω model, and $M, s \models \neg\varphi$. Just 'cutting the tails' of the sequences in M would not work, since it may make some sequence s' for which $s' =_x s$ does not hold in M, accessible from s in the new model. Let, for example, $n = 3$ (which means that φ contains only the variables v_0, v_1 and v_2). Assume that s' differs from s in the 0th and 5th coordinates. Then $s' =_0 s$ does not hold. But if we consider only the first 3 coordinates of s and s', then $s' =_0 s$.

To make a Crs_n model from M, which satisfies φ, we proced as follows. Let M^- be a submodel of M obtained by leaving out all sequences which differ from s in some coordinate m, $m \geq n$. An easy induction shows that

$$M, s \models \varphi \Leftrightarrow M^-, s \models \varphi.$$

(The inductive hypothesis being: for every s' which differs from s in at most n first coordinates, and every formula ψ containing at most n first variables, $M, s' \models \psi \Leftrightarrow M^-, s' \models \psi$).

M^- can be easily transformed into a Crs_n model by 'cutting the tails' of sequences in M^-, making them all of length n. □

5 The weakest logic above Crs and dependence models

So far, we have considered the following logics (with inclusions indicated by arrows, DM standing for the minimal logic of dependence models):

$$
\begin{tikzcd}
 & FOL & \\
Crs \arrow[ur] \arrow[r, squiggly] & L_{p.a.} \arrow[u] & \\
 & & DM \arrow[uul] \\
Crs^{+} \arrow[uu] \arrow[r, squiggly] & L^{+}_{p.a.} \arrow[uu] & \\
 & L_{str} \arrow[u] \arrow[uur] &
\end{tikzcd}
$$

There are lots of non-trivial logics between DM and FOL (cf. Chapter 4). But it is interesting that there is no logic which is stronger than both Crs and DM and weaker than first order logic. In this section we are going to show that $Crs+DM = FOL$. This suggests that the intuitions underlying assignments models and dependence models are in a sense orthogonal.

Although we do not give an axiomatization of Crs_n-models, it is clear that at least the following axioms are valid there:

A1' $\Box_x(\varphi \to \psi) \to (\Box_x\varphi \to \Box_x\psi)$ (unrestricted monotonicity);

A4 $\varphi \to \Diamond_x\varphi$.

The following axioms are valid in the class of dependence models:

A0 propositional logic;

A1 $\Box_x(\varphi \to \psi) \to (\Box_x\varphi \to \Box_x\psi)$, given that $\Box_x\varphi$ and $\Box_x\psi$ contain the same parameters (free variables and constants);

A2 $\varphi \to \Box_x\varphi$, given that x is not free in φ;

A5 $\Box_x\varphi \to \Box_y\varphi(x/y)$, given that y is free for x in $\varphi(x)$;

R1 $\varphi,\ \varphi \to \psi/\psi$;

R2 $\varphi(x)/\Box_x\varphi(x)$.

Putting these axioms together and replacing $\Box$ by $\forall$ and $\Diamond$ by $\exists$ we obtain the following system:

Ax0 propositional logic;

Ax1 $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$, given that x is not free in φ (from unrestricted monotonicity A1' and A2);

Ax2 $\varphi(x) \to \exists y\varphi(x/y)$, with the usual restriction (from A4 and A5);

MP $\varphi,\ \varphi \to \psi/\psi$;

Gen $\varphi(x)/\forall x\varphi(x)$ (from R1).

which axiomatizes first-order logic (cf., for example, (Mendelson 1979)). This proves the following proposition:

Proposition 22 *The weakest logic which derives all formulas valid in both dependence models and Crs_n-models, is classical first order logic.*

Both the logic of dependence models and Crs_n-models are much weaker than classical first order logic. In particular, they are decidable even in the presence of equality in the language (as we shall see below, this follows from the results of Németi (1992), Andréka and Németi (1994)).

6 Restricted fragments of predicate logic

A lot of insight in modal logic comes from considering it as a fragment of first order logic under the *standard translation* (cf. van Benthem (1983)). The same holds for the logics of modal quantifiers.

All logics introduced above can be embedded into fragments of first-order logic with restricted, or bounded, quantification (where every quantifier is restricted by an atomic formula).

In (Andréka, van Benthem and Németi 1995) the following fragment (called Fragment 2) is defined: a fragment of first-order language with equality where each subformula in the scope of $\exists$ is of the form $\exists\bar{x}(R(\bar{x},\bar{y}) \wedge \varphi)$, where R is any relational symbol (not necessarily fixed) and the free variables of φ are among $\bar{x},\bar{y}$. This fragment is shown to be decidable in (Andréka and Németi 1994). In this section, we show that the logics of partial assignments models and the logics of assignments models are embeddable into Fragment 2.

For Crs_n and Crs_n^+ this is a well-known fact (cf., for example, (Németi 1992)). The following translation tr from $\mathcal{L}_n(\Diamond)$ into the language of first-order logic enriched with a predicate R is used to define the embedding: tr commutes with atomic formulas and propositional symbols, and

$$tr(\Diamond_{v_i}\varphi) = \exists v_i(R(v_0,\ldots,v_{n-1}) \wedge tr(\varphi)).$$

Put for Crs_n^+ $\varphi^+ = tr(\varphi)$ and for Crs_n $\varphi^* = R(v_0,\ldots,v_{n-1}) \wedge tr(\varphi)$. Intuitively, $R(d_0,\ldots,d_{n-1})$ means that $\{\langle v_0,d_0\rangle,\ldots,\langle v_{n-1},d_{n-1}\rangle\}$ is an assignment in W. One can easily check that φ is Crs_n (Crs_n^+)-satisfiable iff φ^* (φ^+) is satisfiable in first-order logic.

For the partial assignments models, the idea is completely the same, but we have to introduce countably many relational symbols. The reason is that we must be able to distinguish assignments of the same length but to different variables (coordinates). So we introduce symbols like $R_{i_1,\ldots,i_n}$, $i_1 \leq \ldots \leq i_n$, with $R_{i_1,\ldots,i_n}(d_1,\ldots,d_n)$ meaning that d_j is assigned to the variable v_{i_j}. The translation is, as one would expect, the same as tr except for the quantifiers:

$$tr'(\Diamond_{v_{i_j}}\varphi(v_{i_1},\ldots,v_{i_n})) = \exists v_{i_j}(R_{i_1,\ldots,i_n}(v_{i_1},\ldots,v_{i_n}) \wedge tr'(\varphi))$$

For example, $\Diamond_{v_0}\Diamond_{v_2}P(v_0,v_2)$ is translated a

$$\exists v_0(R_0(v_0) \wedge \exists v_2(R_{0,2}(v_0,v_2) \wedge P(v_0,v_2)))$$

which is equivalent in first order logic to

$$\exists x\exists y(R_0(x) \wedge R_{0,2}(x,y) \wedge P(x,y));$$

$\Diamond_{v_3}\Diamond_{v_1}P(v_3,v_1)$ is translated (up to renaming of the variables) to

$$\exists x\exists y(R_3(x) \wedge R_{1,3}(y,x) \wedge P(x,y)),$$

i.e. to a different formula.

As before, we define $\varphi^+ = tr'(\varphi)$ and $\varphi^* = R_{i_1,\ldots,i_n}(v_{i_1},\ldots,v_{i_n}) \wedge tr'(\varphi))$, where $v_{i_1},\ldots,v_{i_n}$ are the free variables of φ. The claim is that φ is $L_{p.a.}(L_{p.a.^+})$-satisfiable iff $\varphi^*(\varphi^+)$ is first-order satisfiable. From this follows

Proposition 23 *Both logics of partial assignments models are decidable.*

Proof. Follows from the embedding above and the result of Andréka and Németi (1994). □

The same idea applies to L_{str}. Since the difference between L_{str} and $L^+_{p.a.}$ is in the circularity property, we must show which variable was quantified. We add countably many predicates $R^{i_j}_{i_1,\ldots,i_n}$, where the superscript stands for the quantified variable. The translation becomes

$$tr''(\Diamond_{v_{i_j}} \varphi(v_{i_1},\ldots,v_{i_n})) = \exists v_{i_j}(R^{i_j}_{i_1,\ldots,i_n}(v_{i_1},\ldots,v_{i_n}) \wedge tr''(\varphi))$$

Again, $\varphi^+ = tr''(\varphi)$ and φ is satisfiable in the structured dependence models iff φ^+ is first order satisfiable. We prove that the translation is satisfiability-preserving only for this case; the proofs for partial assignments models are analogous.

Claim 24 *For any formula φ of $\mathcal{L}(\Diamond)$, φ is satisfiable in a structured dependence model iff $tr''(\varphi)$ is first-order satisfiable.*

Proof. Assume that there is a structured dependence model $M = \langle D, R, V\rangle$ and assignment s which satisfy φ. Let $M' = \langle D, V'\rangle$, where V' agrees with V on all predicates from $\mathcal{L}(\Diamond)$, and

$$\langle d_1,\ldots,d_n\rangle \in V'(R^{i_j}_{i_1,\ldots,i_n}) \Leftrightarrow \langle v_{i_j}, d_j\rangle \ R\ \{\langle v_{i_1}, d_1\rangle,\ldots,\langle v_{i_n}, d_n\rangle\} \setminus \{\langle v_{i_j}, d_j\rangle\}.$$

For example,

$$R^4_{2,4,5}(d_1,d_2,d_3) \Leftrightarrow \langle v_4, d_2\rangle \ R\ \{\langle v_2, d_1\rangle, \langle v_5, d_3\rangle\}.$$

It is easy to prove that for every formula ψ and assignment z,

$$M, z \models \psi \leftrightarrow M', z \models tr''(\psi).$$

The proof goes by induction on the complexity of ψ. The only non-trivial case is of ψ being of the form $\Diamond_{v_{i_j}}\theta(v_{i_1},\ldots,v_{i_n})$. $M, z \models \Diamond_{v_{i_j}}\theta(v_{i_1},\ldots,v_{i_n})$ means that

$$\exists d(\langle v_{i_j}, d\rangle \ R\ \{\langle v_{i_1}, z(v_{i_1})\rangle,\ldots,\langle v_{i_n}, z(v_{i_n})\rangle\} \setminus \{\langle v_{i_j}, z(v_{i_j}\rangle\} \wedge M, z^{i_j}_d \models \theta) \Leftrightarrow$$

$$\Leftrightarrow \exists d(R^{i_j}_{i_1,\ldots,i_n}(z(v_{i_1}),\ldots,d,\ldots,z(v_{i_n})) \wedge M', z^{i_j}_d \models tr''(\theta)) \Leftrightarrow$$

$$\Leftrightarrow M', z \models \exists v_{i_j}(R^{i_j}_{i_1,\ldots,i_n}(v_{i_1},\ldots,v_{i_n}) \wedge tr''(\theta)).$$

This proves the direction from left to right.

To prove the opposite direction, assume that $tr''(\varphi)$ is first-order satisfiable. We construct a model for φ by keeping the domain and interpretation function the same as in the model for $tr''(\varphi)$ and defining R as follows:

$$\langle v_{i_j}, d_j\rangle \ R\ \{\langle v_{i_1}, d_1\rangle \ldots \langle v_{i_n}, d_n\rangle\} \setminus \{\langle v_{i_j}, d_j\rangle\} \Leftrightarrow \langle d_1,\ldots,d_j,\ldots,d_n\rangle \in V(R^{i_j}_{i_1,\ldots,i_n}).$$

The rest of the argument is the same as above. □

Proposition 25 *The logic of structured dependence models is decidable.*

Proof. Follows from the embedding above and the result of Andréka and Németi (1994). □

The standard translation for dependence models is given in (van Lambalgen 1991) and (van Benthem and Alechina 1996). It also falls within Fragment 2 and is decidable.

7 Conclusion

The logics considered above can be seen as special cases of the same principal intuition: that the range of a variable bound by a quantifier can be dependent on the values of other variables. Different logics emerge when different conditions on the dependence relation are imposed. All such logics can be translated into first order logic. But it definitely makes sense to study them on their own or as *special fragments* of first order logic, because they may have properties which full first order logic does not posess. We have seen this on the example of decidability.

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