The Definability of the Qualitative Independence of Events in terms of Extended Indicator Functions

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Footnote

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Abstract

The main purpose of this paper is to define the relation of probabilistic independence of events in terms of a qualitative ordering of extended indicator functions of events.

1 INTRODUCTION

An ordering $\succeq$ over events is often interpreted as a qualitative probabilistic ordering, that is, the intuitive sense of $A \succeq B$ is $P(A) \geq P(B)$. There exists an extended literature on examining the conditions that guarantee the existence of such a probabilistic representation for some set of ordered events (for a brief survey see, for example, Suppes and Zanotti, 1976). In this paper, we consider the qualitative relation of probabilistic independence $\perp$; the desired interpretation of $\perp$ is

$$A \perp B \text{ iff } P(A \cap B) = P(A)\, P(B) \tag{1}$$

The independence relation $\perp$ was studied in the literature since at least Domotor (1969). Domotor gives a definition of $\perp$ in terms of a qualitative ordering on the Cartesian product of elements from $\mathcal{F}$: $A \perp B$ iff $A \cap B \times \Omega \sim A \times B$ for all $A, B \in \mathcal{F}$, where $\sim$ is an equivalence relation induced by $\succeq$ and $A \times B \succeq C \times D$ iff $P(A)P(B) \geq P(C)P(D)$. He proved from this definition some basic properties of $\perp$. In Krantz, Luce, Suppes, and Tversky (1971) a four-place relation also was used to define independence. Another approach was taken by Fine (1973) who proposed a set of axioms for $\succeq$ and $\perp$, which are not sufficient to prove (1).

In this paper the relation of probabilistic independence of events is defined in terms of the qualitative ordering of extended indicator functions of events.

2 UNDEFINABILITY OF $\perp$ IN TERMS OF QUALITATIVE ORDERING OVER EVENTS

Some natural elementary axioms on independence were given by Fine (1973):

I1. $A \perp B \Rightarrow B \perp A$;

I2. $A \perp \Omega$;

I3. $A \perp B \Rightarrow A \perp \neg B$;

I4. $B \cap C = \emptyset, A \perp B, A \perp C \Rightarrow A \perp (B \cup C)$;

I5. $A \perp B, C \perp D, B \succeq D \Rightarrow (A \succeq C \Rightarrow A \cap B \succeq C \cap D)$,

$(A \succ C, B \succ \emptyset \Rightarrow A \cap B \succ C \cap D)$.

He proved that these conditions guarantee the existence of a probability assignment P, agreeing with $\succeq$, and of an operation G such that

$A \perp B$ iff $P(A \cap B) = G(P(A), P(B))$.

In order to make G associative, Fine suggests the following axiom:

I6. $A \perp B \cap C,\ B \perp C,\ A' \cap B' \perp C',\ A' \perp B',\ A \approx A',\ B \approx B',\ C \approx C' \Rightarrow$

$\Rightarrow A \cap B \cap C \approx A' \cap B' \cap C'$.

Another property, not derivable from *I1 - I6* , is

I7. $A \perp B, B \perp C, A \perp B \cap C \Rightarrow C \perp A \cap B$.

It is still an open question whether $\perp$ (interpreted as $A \perp B$ iff $P(A \cap B) = P(A) \cdot P(B)$) could be given a finite axiomatization. In Fine's system, the independence of events does not follow from $P(A \cap B) =$ $= G(P(A),\ P(B))$ (intuitively, from the product factorization of probability). Fine's axioms *I1 - I7* do not imply the distributive property of G over addition, i.e., do not provide a numerical representation in which G stands for multiplication.

However, it is easy to show that the independence relation is undefinable in terms of events and a qualitative ordering over them, a result that is part of the folklore of the subject, but, not, as far as we know, written down anywhere.

THEOREM 1. *Let* $(\Omega, \succeq, \perp)$ *be such that*

1. Ω *is a finite nonempty set;*

2. $\succeq$ *is a relation on the subsets of* Ω *satisfying Scott's conditions:*

   S1. $A \succeq B$ *or* $B \succeq A$*;*

   S2. $A \succeq \emptyset$*;*

   S3. $\Omega \succ \emptyset$*;*

S4. for all subsets $A_0, \ldots, A_n$, $B_0, \ldots, B_n$ of Ω, if $A_j \succeq B_j$ for $0 \le j < n$, and for all ω in Ω

$$A_0^i(\omega) + \cdots + A_n^i(\omega) = B_0^i(\omega) + \cdots + B_n^i(\omega),$$

then $A_n \preceq B_n$, where A^i is the indicator function of event A.

3. $\perp$ is a relation satisfying Fine's Axioms I1-I7 given above.

Then the independence relation $\perp$ is not definable in terms of the structure $(\Omega, \succeq)$.

Proof. Consider the following structure: $\Omega = \{a, b, c, d\}$, $\succeq$ is as follows:

$0 \prec a \prec b \prec c \prec a + b \prec a + c \prec d \prec b + c \prec a + d \prec a + b + c \prec b + d \prec$
$\prec c + d \prec a + b + d \prec a + c + d \prec b + c + d \prec a + b + c + d.$

Let event $A = \{a, b\}, B = \{b, d\}$. We shall show that this structure can be given two different numerical interpretations, both preserving the ordering, but in one of them A and B are independent:

$$P(A \cap B) = P(A)P(B),$$

and in the second they are not; the notion of independence agrees with Fine's axioms.

P_1: $a = 10/80,\ b = 15/80,\ c = 22/80,\ d = 33/80;\ a + b + c + d = 1.$

P_2: $a = 10/80,\ b = 16/80,\ c = 21/80,\ d = 33/80.$

In the first case $15/80 = 25/80 \cdot 48/80$, in the second $16/80 \neq 26/80 \cdot 49/80$. Whence by Padoa's principle $\perp$ is not definable in terms of $(\Omega, \succeq)$. Q.E.D.

3 DEFINITION OF INDEPENDENCE IN TERMS OF EXTENDED INDICATOR FUNCTIONS

The notion of extended indicator function was used in Suppes and Zanotti (1976) to give a simple set of conditions for the existence of a probability measure agreeing with the qualitative ordering.

Let Ω be the set of possible outcomes, $\mathcal{F}$ an algebra of events on Ω. Let A^i be the indicator function (or characteristic function) of event A. The algebra

$\mathcal{F}^*$ of extended indicator functions relative to $\mathcal{F}$ is the smallest semigroup (under functional addition) containing the indicator functions of all events in $\mathcal{F}$.

It is easy to see that extended indicator functions are random variables of a restricted sort (taking nonnegative integer values). The intuitive sense of qualitative ordering is that $A^* \succeq B^*(A^*, B^* \in \mathcal{F}^*)$ iff the expected value of A^* is greater or equal to the expected value of B^*. Certainly, for $A, B \in \mathcal{F}$, $A^i \succeq B^i$ iff $A \succeq B$.

DEFINITION 1.(Suppes & Zanotti, 1976). *Let Ω be a nonempty set, let $\mathcal{F}$ be an algebra of sets on Ω, and let $\succeq$ be a binary relation on $\mathcal{F}^*$, the algebra of extended indicator functions relative to $\mathcal{F}$. Then the qualitative algebra $(\Omega,\ \mathcal{F}^*,\ \succeq)$ is qualitatively satisfactory if and only if the following axioms are satisfied for every A^*, B^*, and C^* in $\mathcal{F}^*$:*

Axiom 1. The relation $\succeq$ is a weak ordering of $\mathcal{F}^*$;

Axiom 2. $\Omega^i \succ \emptyset^i$;

Axiom 3. $A^* \succeq \emptyset^i$;

Axiom 4. $A^* \succeq B^*$ iff $A^* + C^* \succeq B^* + C^*$;

Axiom 5. If $A^* \succ B^*$ then for every C^* and D^* in $\mathcal{F}^*$ there is a positive integer n such that $nA^* + C^* \succeq nB^* + D^*$.

(The strict ordering $\succ$ is defined in the usual way: $A^* \succ B^*$ iff $A^* \succeq B^*$ and not $B^* \succeq A^*$; nA^* stands for $A^* + \cdots + A^*$ n times).

It was proved in Suppes and Zanotti (1976) that a necessary and sufficient condition for existence of a strictly agreeing probability measure on $\mathcal{F}$ is that there is an extension of $\succeq$ from $\mathcal{F}$ to $\mathcal{F}^*$ such that $(\Omega, \mathcal{F}^*, \succeq)$ is qualitatively satisfactory and this measure is unique when extended to $\mathcal{F}^*$. However, in their representation theorem and others of a like nature, the requirement that the measure P of probability norm to 1 has an arbitrary character, i.e., the form of the representation permits multiplication by an arbitrary positive constant. In the case of Suppes and Zanotti (1976), the representation theorem is formulated in terms of an expectation function E for extended indicator functions satisfying the following obvious properties for any A^* and B^* in $\mathcal{F}^*$:

E1. $E(A^*) \geq E(B^*)$ iff $A^* \succeq B^*$

E2. $E(A^* + B^*) = E(A^*) + E(B^*)$.

It is clear that if a function E defined on $\mathcal{F}^*$ satisfies *E1* and *E2*, so will αE, where α is any positive real number. On the other hand, the one 'universal' positive probability case of independence, namely, the independence of Ω with respect to Ω, can be formulated as an additional requirement on E, and this additional requirement fixes E uniquely:

E3. $E((\Omega \cap \Omega)^i) = E(\Omega^i) \cdot E(\Omega^i)$.

¿From the axioms of Definition 1, it easily follows that $E(\emptyset^i) = 0$, $E(\Omega^i) > 0$, and thus from *E3* that $E(\Omega^i) = 1$. We then also have a nonarbitrary norming of the probability measure: for A in $\mathcal{F}$

E4. $P(A) = E(A^i)$.

The preceding remarks are necessary to avoid any air of paradox in our definition of independence of events within the framework of extended indicator functions, for a satisfactory definition of independence for which one can prove

$$A \perp B \text{ iff } P(A \cap B) = P(A)\, P(B)$$

clearly does require the probability measure be normed to 1. It would be paradoxical if this could be done only after the definition of $\perp$. As the above considerations show, this is not the case.

These remarks establish the following representation theorem, which is stronger than the one given in Suppes and Zanotti (1976).

THEOREM 2. *If the structure $(\Omega, \mathcal{F}, \succeq)$ satisfies Definition 1, then there is a unique expectation function E on $\mathcal{F}^*$ and a unique probability measure P which together satisfy E1 – E4.*

Within the framework of Definition 1, we now define independence as follows:

DEFINITION 2. *$A \perp B$ iff*

a. $\forall m \forall m' \forall n \forall n' (m\Omega^i \preceq nA^i \ \&\ m'\Omega^i \preceq n'B^i \Rightarrow mm'\Omega^i \preceq nn'(A \cap B)^i)$

b. $\forall m \forall m' \forall n \forall n' (m\Omega^i \succeq nA^i \ \&\ m'\Omega^i \succeq n'B^i \Rightarrow mm'\Omega^i \succeq nn'(A \cap B)^i)$

It was proved in Suppes, Krantz, Luce, & Tversky (1989) (based on Suppes & Zanotti, 1991) (for a bit different setup) that the conditions of the left side of the definition guarantee that this property holds. We shall follow this proof here.

THEOREM 3. *Given* $(\Omega,\ \mathcal{F},\ \succeq)$ *satisfying Definition 1 with a unique expectation function* E *and measure* P *satisfying E1 – E4, then for* A, B *in* $\mathcal{F}$

$$P(A \cap B) = P(A)P(B) \text{ iff } A \perp B.$$

Proof.

To prove the theorem, we define, for every A^i in $\mathcal{F}^*$, the set S_A of numbers:

$$S_A = \{\frac{m}{n} : m\Omega^i \succeq nA^i\ \}.$$

It is easy to show that S_A is nonempty and has a greatest lower bound, which is in fact just the unique expectation function E:

$$E(A^i) = g.l.b.S_A.$$

For independent A and B we have (see Def.2a): $\frac{m}{n} \in S_A, \frac{m'}{n'} \in S_B \Rightarrow \frac{mm'}{nn'} \in S_{A\cap B}$, and thus

$$E((A \cap B)^i) \geq E(A^i)E(B^i)$$

Then we define $T_A = \{\frac{m}{n} : m\Omega^i \preceq nA^i\}$. Each set T_A is obviously nonempty and has a least upper bound. We need to show that

$$l.u.b.T_A = g.l.b.S_A = E(A^i).$$

Suppose, by way of contradiction, that

$$l.u.b.T_A < g.l.b.S_A$$

(it is evident, that $l.u.b.T_A \leq g.l.b.S_A$ holds). Then there must exist integers m and n such that

$$l.u.b.T_A < \frac{m}{n} < g.l.b.S_A.$$

Thus, we have, from the left-hand inequality,

$$m\Omega^i > nA^i,$$

and from the right-hand one,

$$m\Omega^i < nA^i,$$

which together contradict the order properties of $\geq$.

Analogously with the case of S_A , the definition of T_A implies

$$E(A^i)E(B^i) \geq E((A \cap B)^i),$$

and together they give

$$E(A^i)E(B^i) = E((A \cap B)^i).$$

Q.E.D.

4 SOME PROPERTIES OF $\perp$

Now we shall show that Fine's axioms *I1 – I7* are derivable in an elementary way.

THEOREM 4. *Fine's axioms on independence I1 – I7 are derivable from Axioms 1 – 5 of Definition 1 and the definition of independence.*

Proof.

The proof of *I1 – I3* is trivial;

I4. If $B \cap C = \emptyset$, then $B^i + C^i = (B \cup C)^i$ and

$$(A \cap B)^i + (A \cap C)^i = ((A \cap B) \cup (A \cap C))^i.$$

We should prove that $\forall m \forall m^* \forall n \forall n^* (m\Omega^i \succeq nA^i \;\&\; m^*\Omega^i \succeq n^*(B \cup C)^i$ $\Rightarrow mm^*\Omega^i \succeq nn^*(A \cap (B \cup C))^i)$ (And the same for $\preceq$.)

Let $m\Omega^i \succeq nA^i$ and $m^*\Omega^i \succeq n^*(B \cup C)^i$. Since B and C are disjoint,

$$m^*\Omega^i \succeq n^*(B^i + C^i)$$

and

$$m^*\Omega^i \succeq n^*B^i + n^*C^i.$$

There exist m', m'' such that $m' + m'' = m^*$ and

$$m'\Omega \succeq n^*B^i, \; m''\Omega \succeq n^*C^i$$

(from the properties of $\succeq$ and $+$). Since $A \perp B$ and $A \perp C$,

$$mm'\Omega \succeq nn^*(A \cap B)^i$$

and

$$mm''\Omega \succeq nn^*(A\cap C)^i,$$

thus

$$m(m'+m'')\Omega \succeq nn^*((A\cap B)^i + (A\cap C)^i).$$

It gives us

$$mm^*\Omega \succeq nn^*((A\cap B)\cup(A\cap C))^i;$$

from boolean algebra we get

$$mm^*\Omega \succeq nn^*(A\cap(B\cup C))^i.$$

The proof for $\preceq$ is analogous.

Before proving I5 we need to prove the following

LEMMA. *If $A\perp B$, then:*

I5'. $A\perp B \Rightarrow \forall m\forall m'\forall n\forall n'(m\Omega^i \succeq nA^i \ \&\ m'\Omega^i \succ n'B^i \Rightarrow mm'\Omega^i \succ$
$\succ nn'A\cap B^i)$;

I5". $A\perp B \Rightarrow \forall m\forall m'\forall n\forall n'(m\Omega^i \preceq nA^i \ \&\ m'\Omega^i \prec n'B^i \Rightarrow mm'\Omega^i \prec$
$\prec nn'A\cap B^i)$.

Proof. *I5'.* Let $A\perp B$. Then

1. $\forall m\forall m'\forall n\forall n'(m\Omega^i \succeq nA^i \& m'\Omega^i \succeq n'B^i \Rightarrow mm'\Omega^i \succeq nn'A\cap B^i)$;
2. let $m\Omega^i \succeq nA^i$;
3. let $m'\Omega^i \succ n'B^i$;
4. $\exists n^*(n^*m'\Omega^i \succeq n^*n'B^i + B^i)$ from 3, Axiom 5;
5. $n^*m'\Omega^i \succeq (n^*n'+1)B^i$ from 4;
6. $mn^*m'\Omega^i \succeq n(n^*n'+1)(A\cap B)^i$ from 1, 2, 5;
7. $mn^*m'\Omega^i \succeq nn^*n'(A\cap B)^i + n(A\cap B)^i$ from 6;
8. $mn^*m'\Omega^i \succ nn^*n'(A\cap B)^i$ from 7;
9. $mm'\Omega^i \succ nn'(A\cap B)^i$ from 8.

I5". (analogous)

Now we continue the proof of the Theorem 4.

I5. We should consider two cases:

1. $A^i \succ C^i$, $B^i \succ D^i$: it is evident, that $\exists m \exists m' \exists n \exists n' (m\Omega^i \prec nA^i,\ m\Omega^i \succeq nC^i,\ m'\Omega^i \prec n'B^i,\ m'\Omega^i \succeq n'D^i)$; thus, for some mm' and nn', $nn'(A \cap B)^i \succ mm'\Omega^i \succeq nn'(C \cap D)^i$, and it means that $(A \cap B)^i \succ (C \cap D)^i$.

2. $A^i \approx C^i$, $C^i \succ \emptyset^i$, $B^i \succ D^i$: there are m, n such that

$$nA^i \prec m\Omega^i, (n+1)A^i \succeq m\Omega^i,$$

the same should hold for C^i. Since $B^i \succ D^i$, there exist such n', m', that $n'B^i \succ m'\Omega^i \succeq (n' + n')D^i$. Now we have

$$n'(n+1)(A \cap B)^i \succ mm'\Omega^i \succ n(n' + n')(C \cap D)^i,$$

or

$$n'n(A \cap B)^i + n'(A \cap B)^i \succ n'n(C \cap D)^i + nn'(C \cap D)^i.$$

Since $n' \leq nn'$, $(A \cap B)^i \succ (C \cap D)^i$.

I6. Let $m\Omega^i \succeq nA^i$, $m'\Omega^i \succeq n'B^i$, $m''\Omega^i \succeq n''C^i$. Since $A \approx A'$, $B \approx B'$, $C \approx C'$, the same holds for A'^i, B'^i, and C'^i. Now, $m'm''\Omega^i \succeq n'n''(B \cap C)^i$ (because $B \perp C$) and $mm'm''\Omega^i \succeq nn'n''(A \cap B \cap C)^i$ (because $A \perp B \cap C$). Since $A' \perp B'$, $mm'\Omega^i \succeq nn'(A' \cap B')^i$, and, since $A' \cap B' \perp C'$,
$mm'm''\Omega^i \succeq nn'n''(A' \cap B' \cap C')^i$. We get
$\forall m \forall m' \forall m'' \forall n \forall n' \forall n'' (m\Omega^i \succeq nA^i \ \&\ m'\Omega^i \succeq n'B^i \ \&\ m''\Omega^i \succeq n''C^i \Rightarrow$
$\Rightarrow (mm'm''\Omega^i \succeq nn'n''(A \cap B \cap C)^i \ \&\ mm'm''\Omega^i \succeq nn'n''(A' \cap B' \cap C')^i))$.

It is evident that if $A \cap B \cap C \prec A' \cap B' \cap C'$, there exist m^*, n^* such that $n^*(A \cap B \cap C)^i \preceq m^*\Omega^i \prec n^*(A' \cap B' \cap C')^*$. We can always find some k such that km^* and kn^* could be represented as multiplications of m, m', m'' and n, n', n'', respectively, for which hold $m\Omega^i \succeq nA^i$, $m'\Omega^i \succeq n'B^i$, $m''\Omega^i \succeq n''C^i$. But, as we have proved, it means that $mm'm''\Omega^i \succeq nn'n''(A' \cap B' \cap C')^i$, which contradicts the assumption. The same argument works for $A \cap B \cap C \succ A' \cap B' \cap C'$. Thus, $(A \cap B \cap C)^i \approx (A' \cap B' \cap C')^i$.

I7. We need to prove that
$\forall m''\forall m^*\forall n''\forall n^*(m''\Omega^i \succeq n''C^i \;\&\; m^*\Omega^i \succeq n^*(A\cap B)^i \Rightarrow m''m^*\Omega^i \succeq$
$\succeq n''n^*(A\cap B\cap C)^i)$, and the same for $\preceq$.

Assume that for some m'', n'', m^*, n^* it does not hold. It means that
$m''\Omega^i \succeq n''C^i \;\&\; m^*\Omega^i \succeq n^*(A\cap B)^i \;\&\; m''m^*\Omega^i \prec n''n^*(A\cap B\cap C)^i$.

As in the previous proof, we state first that if $m^*\Omega^i \succeq n^*(A\cap B)^i$, then there is some k such that $km^* = mm'$, $kn^* = nn'$ and $m\Omega^i \succeq nA^i$, $m'\Omega^i \succeq n'B^i$. ¿From this we get

$$mm'm''\Omega^i \succeq nn'n''(A\cap B\cap C)^i$$

$$km^*m''\Omega^i \succeq kn^*n''(A\cap B\cap C)^i$$

$$m^*m''\Omega^i \succeq n^*n''(A\cap B\cap C)^i,$$

which contradicts our assumption.

After we repeat the same argument for $\preceq$, we get both conditions for $C\perp A\cap B$. Q.E.D.

5 FURTHER DEVELOPMENTS

We mention some open problems and further possible developments.

Finite axiomatizability of $\perp$. We have in mind, of course, finite axiomatizability in first-order logic. It seems likely the result should be negative, and even more so for axiomatizable by a universal sentence, if we want necessary and sufficient conditions for any finite Boolean algebra. However, some interesting sufficient conditions might be given for significant classes of qualitative probability relations. For example, in the special but significant finite case of all atoms being equiprobable, a non-first-order definition of independence can be easily given. Let $|A|$ be the cardinality of set A. Then

$$A\perp B \text{ iff } |\Omega|\cdot|A\cap B| = |A|\cdot|B|.$$

Positive relevance. It is obvious that defining probabilistic independence is closely connected with defining the notion of positive relevance ($P(A\cap B) >$

$P(A)P(B)$). Theorem 1 establishes that the notion of positive relevance is undefinable in terms of $(\Omega, \succeq)$, since independence itself is definable via positive relevance.

Let $\top$ stand for relevance, $\heartsuit$ for positive relevance, and $\spadesuit$ for negative relevance. Then

$$A \perp B \text{ iff } \textit{not } (A \top B)$$

$$A \top B \text{ iff } A \heartsuit B \textit{ or } A \spadesuit B$$

$$A \spadesuit B \text{ iff } A \heartsuit \neg B$$

$$A \perp B \text{ iff } \textit{not } (A \heartsuit B \textit{ or } A \heartsuit \neg B)$$

So, if independence is not definable, neither is positive relevance. The reverse does not hold: positive relevance is not definable via independence (although relevance is). We can give positive relevance the following definition in terms of extended indicator functions:

$$A \heartsuit B \text{ iff}$$

a. $\exists m \exists m' \exists n \exists n' (m\Omega^i \succeq nA^i \& m'\Omega^i \succeq n'B^i \& mm'\Omega^i \prec nn'(A \cap B)^i)$

b. $\forall m \forall m' \forall n \forall n' (m\Omega^i \preceq nA^i \& m'\Omega^i \preceq n'B^i \Rightarrow mm'\Omega^i \prec nn'(A \cap B)^i)$

Unlike independence, it requires existential quantifiers. Questions about axiomatizability of positive relevance, similar to those for independence, are apparently also as yet unanswered.

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succeq — $\succeq$

geq — $\geq$

bot — $\bot$

iff — iff

cap — $\cap$

cdot — $\cdot$

F — $\mathcal{F}$

times — $\times$

Omega — Ω

sim — $\sim$

Rightarrow — $\Rightarrow$

neg — $\neg$

emptyset — $\emptyset$

cup — $\cup$

succ — $\succ$

approx — $\approx$

ldots — $\ldots$

omega — ω

cdots — $\cdots$

preceq — $\preceq$

prec — $\prec$

alpha — α

forall — $\forall$

$<$

$>$

cup — $\cup$

exists — $\exists$

top — $\top$

heartsuit — $\heartsuit$

spadesuit — $\spadesuit$