

Logic Puzzles

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Outline

Logic puzzles have been developed to test students' skill in logical reasoning.

Two classes discussed here:

- The Island of Knights and Knaves
- Portia's Casket

Exploitation of the associativity of equivalence simplifies the problems considerably.

The Island of Knights and Knaves

The island has two types of inhabitants, “knights” who always tell the truth, and “knaves” who always lie.

Suppose A is the proposition “person A is a knight” and suppose A makes a statement S . Then A is true is the same as S is true. That is,

$$A \equiv S .$$

Examples

If A says “I am a knight” then what we can infer from the statement is $A \equiv A$. But since this is always **true** we get no information from the statement.

Similarly, it cannot be that a native says “I am a knave” because we would then conclude $A \equiv \neg A$ which is always **false**.

If A says “I am the same type as B ”. we infer $A \equiv (A \equiv B)$ which simplifies to B .

Examples (Continued)

It is rumoured that there is gold buried on the island. You ask one of the natives, A, whether there is gold on the island. He makes the following response: “There is gold on this island equivalent to I am a knight.” The problem is

- (a) Can it be determined whether A is a knight or a knave?
- (b) Can it be determined whether there is gold on the island?

Solution

Let G denote the proposition “There is gold on the island”. A’s statement is $A \equiv G$. So what we are given is:

$$A \equiv A \equiv G .$$

This simplifies to G . So we deduce that there is gold on the island but it is not possible to tell whether A is a knight or a knave.

Formulating Questions

If native A is asked a yes/no question Q then the response to the question is

$$A \equiv Q .$$

That is, the response will be yes if A is a knight and the answer is really yes, or A is a knave and the answer is really no. Otherwise the response will be no.

For example, asked the question “are you a knight” all natives will answer yes, as $A \equiv A$.

Asked the question “is B a knight” A will respond yes if they are both the same type, otherwise no. That is, A 's response is yes or no depending on the truth or falsity of $A \equiv B$.

Example

A tourist comes to a fork in the road, where one branch leads to a restaurant and one doesn't. A native of the island is standing at the fork. Formulate a single yes/no question that the tourist can ask such that the answer will be yes if the left fork leads to the restaurant, and otherwise the answer will be no.

Solution

Let Q be the question.

Let A be “the native is a knight”. Let L be the proposition “the left fork leads to the restaurant”.

We require that L equivaless the response to the question is yes. But the response to the question Q is yes equivaless $Q \equiv A$.

So we require that

$$L \equiv (Q \equiv A) .$$

Equivalently,

$$Q \equiv (L \equiv A) .$$

The question is thus: Is it the case that the statement that the left fork leads to the restaurant is equivalent to your being a knight?

Exercise 1 Suppose you come across two of the inhabitants. You ask both of them whether the other one is a knight. Will you get the same answer in each case?

□

Exercise 2 There are three natives A, B and C. Suppose A says “B and C are the same type.” What can be inferred about the number of knights?

□

Exercise 3 Suppose C says “A and B are as like as two peas in a pod”. What question should you pose to A to determine whether or not C is telling the truth?

□

Exercise 4 What single question allows you to determine whether A is a knight? Justify your question using the construction given above.

□

Exercise

The following is a list of statements that might be made by person A on the island of knights and knaves. In each case the problem is to determine what can be deduced about A and B.

- (a) If I am a knight then B is a knight.
- (b) If I am a knave then B is a knight.
- (c) If I am a knight then B is a knave.
- (d) If I am a knave then B is a knave.
- (e) If B is a knight then I am a knight.
- (f) If B is a knave then I am a knight.
- (g) If B is a knight then I am a knave.
- (h) If B is a knave then I am a knave.

Sample Solutions

(a) We are required to simplify $A \equiv A \Rightarrow B$.

$$\begin{aligned}
 & A \equiv A \Rightarrow B \\
 = & \quad \{ \text{definition of "only if"} \} \\
 & A \equiv A \equiv A \wedge B \\
 = & \quad \{ \text{reflexivity of equivalence} \} \\
 & A \wedge B .
 \end{aligned}$$

So both A and B are knights.

(e) We are required to simplify $A \equiv B \Rightarrow A$.

$$\begin{aligned}
 & A \equiv B \Rightarrow A \\
 = & \quad \{ \text{definition of "only if"} \} \\
 & A \equiv A \equiv A \vee B \\
 = & \quad \{ \text{reflexivity of equivalence} \} \\
 & A \vee B .
 \end{aligned}$$

So at least one of A and B is a knight but which it is is not clear.

Exercise 5 You encounter two inhabitants. Pose a question that will determine whether or not both are knights. Pose a question that will determine whether or not at least one of them is a knight.

□

Exercise 6 Inhabitant A says “either I am a knave or B is a knight”. What can you deduce about A and B?

□

Exercise 7 According to this problem, three of the inhabitants —A, B and C— were standing together in the garden. A stranger passed by and asked A, “Are you a knight or a knave?”. A answered, but rather indistinctly, so the stranger could not make out what he said. The stranger then asked B, “What did A say?”. B replied, “A said that he is a knave”. At this point the third man, C, said “Don’t believe B; he’s lying!”. The question is, what are B and C?

□

Portia's Casket

In Shakespeare's *Merchant of Venice*, Portia had three caskets: gold, silver and lead.

Inside one of these caskets Portia had put her portrait and on each was an inscription.

Portia explained to her suitor that each inscription could be either true or false but on the basis of the inscriptions he was to choose the casket containing the portrait.

If he succeeded he could marry her.

The first exercise below is a simpler version. We demonstrate how to solve the problem and then leave the remaining exercises to you.

Example

Suppose there are two caskets, gold and silver, into one of which Portia placed her portrait. The inscriptions are:

Gold: The portrait is not in here.

Silver: Exactly one of these inscriptions is true.

Which casket contains the portrait?

Solution

Let G stand for “the portrait is in the gold casket”, let S stand for “the portrait is in the silver casket”, g stand for “the inscription on the gold casket is true” and s for “the inscription on the silver casket is true”. Then we are given

$$(g \equiv \neg G) \wedge (s \equiv g \equiv \neg s) \wedge (G \neq S) .$$

The middle term simplifies to $\neg g$ and so we conclude

$$(g \equiv \neg G) \wedge \neg g \wedge (G \neq S) .$$

That is, $G \wedge \neg g \wedge \neg S$.

Solution (Continued)

Formally, the calculation is as follows:

$$\begin{aligned}
 & \text{true} \\
 = & \quad \{ \text{problem statement} \} \\
 & (g \equiv \neg G) \wedge (s \equiv g \equiv \neg s) \wedge (G \not\equiv S) \\
 = & \quad \{ \text{contrapositive applied to 1st conjunct,} \\
 & \quad \text{simplification of continued equivalences} \\
 & \quad \text{applied to 2nd conjunct,} \\
 & \quad \text{definition of inequivalence applied to 3rd conjunct} \} \\
 & (\neg g \equiv G) \wedge \neg g \wedge (\neg G \equiv S) \\
 = & \quad \{ \text{substitution of equals for equals} \} \\
 & G \wedge \neg g \wedge \neg S .
 \end{aligned}$$

This exercise will help you solve the one that follows.

Exercise 8 Prove that the following are all equivalent.

(a) $(p \vee q) \wedge (q \vee r) \wedge (r \vee p)$

(b) $p \vee q \equiv q \vee r \equiv r \vee p$

(c) $p \wedge q \equiv q \wedge r \equiv r \wedge p$

(d) $(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)$

Hint: Work towards the middle. That is, starting from (a) derive (b), and starting from (d) derive (c). (These two calculations are completely dual.) Then show that (b) and (c) are equivalent.

(The statement

$$(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)$$

expresses the fact that at least two of p , q or r is true. One of the other three equivalent expressions may be preferred in some calculations.)

□

Exercise 9 Suppose that Portia put the following inscription on the three caskets:

Gold: The portrait is in here.

Silver: The portrait is in here.

Lead: At least two of these caskets bears a false inscription.

Which casket should the suitor choose? Use the abbreviations **L** for “the portrait is in the lead casket” and **l** for “the inscription on the lead casket is true” in formulating your answer. Hint: use the previous exercise when formulating the inscription on the lead casket.

□

Exercise 10 In this version of the problem, Portia puts a dagger into one of the caskets. The suitor must choose a casket that does *not* contain the dagger. The inscriptions on the casket are as follows.

Gold: The dagger is in this casket.

Silver: The dagger is not in this casket.

Lead: At most one of these caskets bears a true inscription..

Which casket should the suitor choose?

