Difunctions and Block-Ordered Relations

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Abstract

Seventy years ago, in a series of publications, Jacques Riguet introduced the notions of a "relation difonctionelle", the "différence" of a relation and "relations de Ferrers". He also presented a number of properties of these notions, including an "analogie frappante" between "relations de Ferrers" and the "différence" of a relation. Riguet's definitions, particularly of the central concept of the "différence" of a relation, use formulae involving nested complements. Riguet's proofs make extensive use of natural language (French) making them difficult to understand. The primary purpose of this paper is to bring Riguet's work up to date using modern calculational methods. Other goals are to document and extend Riguet's work as fully as possible, and to correct extant errors in the literature.

We call a "relation difonctionelle" a "difunctional relation", the "différence" of a relation we call the "diagonal" of a relation and a "relation de Ferrers" we call a "staircase relation" — a special case of a "block-ordered relation". We avoid as much as possible the use of nested complements by exploiting the left and right factor operators (aka division or residual operators) on relations.

We present complete, calculational proofs of two fundamental properties of difunctions: a relation is difunctional if and only if it can be represented by a pair of functional relations and that a relation is difunctional if and only if it is the union of a set of completely disjoint rectangles. The diagonal of a relation (Riguet's "différence") is a difunction that plays a very significant rôle in the study of block-ordered relations; accordingly, we study its properties in depth. For completeness, we also present a second method for constructing a difunction from an arbitrary relation: Riguet's "fermeture difonctionelle".

Riguet used an informal, mental picture of a staircase-like structure to introduce "relations de Ferrers" in the case of finite relations. Riguet also stated a necessary and sufficient condition for a "relation de Ferrers" to be the union of a totally ordered class of rectangles, where the ordering has a property that we call "polar". By omitting the totality requirement, we abstract the more general notion of a block-ordered relation. We explore conditions under which a given relation has a non-redundant, polar covering and when it is block-ordered. In doing so, we formulate and prove a

theorem establishing an equivalence between the property of a relation being block-ordered and properties of the diagonal of a relation. Our theorem generalises Riguet's "analogie frappante".

Finally, we consider the special case of staircase relations. We consider different definitions that formalise Riguet's mental picture. Contrary to claims made in the published literature, we show that the definitions are not equivalent in general.

Contents

List of Figures

1 Introduction

The interface between requirements and specifications poses a major challenge for practising programmers because it is intrinsically a social process that is largely unsupported by mathematical method: requirements are informal and customer-led whereas specifications are formal (even if, as is often the case, the "specification" is the actual implementation of the requirements). There is no mathematically verifiable "correctness" relation between requirements and specifications.

The challenge of assuring the customer that their requirements have indeed been met can be overcome in different ways. We would argue that one of the most important ways is by deriving —by mathematical calculation— properties of the specification which are then interpreted in a way that can be understood by the customer. This process is vital to the integrity of the science of computing.

Seventy years ago, in a series of publications [Rig48, Rig50, Rig51], Jacques Riguet introduced the notions of a "relation difonctionelle", the "différence" of a relation and "relations de Ferrers". In the case of finite relations, he provided an informal mental picture of a "relation de Ferrers" in the form of a staircase-like structure. But his formal definition of a "relation de Ferrers" bears little or no resemblance to the mental picture and it is difficult to see how the formal corresponds to the informal. The name "relation de Ferrers" also gives little clue as to the practical relevance of the notion. Riguet's definitions, particularly of the "différence" of a relation, use (in our view) over-complicated and outdated formulae involving nested complements that are better formulated using the factor operators (aka division or residual operators). Riguet also relies heavily on natural language (French) justifications of important properties as well as asserting several properties without proof. More recent publications, some of which do not cite Riguet but which copy his definitions, introduce errors by failing to recognise the restrictions that Riguet made clear in his account of the properties of the notions.

The writing of this paper initially began as an exercise in applying modern calculational reasoning to bring Riguet's work up to date and more accessible to a wider audience. In view of the extant errors in relatively recent publications and to try to avoid introducing yet more errors, we decided to include full details of all proofs. In the process, we decided that some changes in terminology were desirable: for reasons that we explain later, we call the "différence" of a relation the "diagonal" of the relation and we call "relations de Ferrers" staircase relations. We also realised that certain generalisations of Riguet's work were desirable, the primary one being from "staircase" relations to "block-ordered relations": the property of being a "staircase" relation demands a certain total ordering on "blocks" ("rectangles totalement ordonnées par inclusion" [Rig51]), being "block-ordered" does not require the ordering to be total. In summary, the goals of this paper are as follows:

1. To demonstrate the efficacy of modern calculational reasoning in developing a theory of block-ordered relations.

2. To document as fully as possible the precise relation between difunctions and block-ordered relations (Riguet's "analogie frappante").

3. To set the record straight with respect to the origin of the concepts and theorems relating difunctions to block-ordered relations.

4. To correct extant errors in the literature.

1.1 Mental Pictures

Partly as a consequence of our decision to include all proofs, this document has become quite long and it is inappropriate to introduce all parts in one go. In order to set the scene, this section gives a very informal account of the principle notions introduced. In doing so, we use notation that will be introduced in later sections. Readers unfamiliar with the notation are invited to read the section nevertheless,postponing full understanding until later.

For many, it is useful to have a "mental picture" of formal mathematical statements. Fig. 1 is such a mental picture of what we shall call a "staircase relation". (Riguet [Rig51] presents a similar picture of a "relation de Ferrers".) The shaded area depicts a binary relation on sets A and B, the vertical axis depicting the set A, the horizontal axis depicting the set B, and the shaded area depicting the set of pairs (a,b) for which the relation holds. Informally a staircase relation is any relation that can be depicted in such a way.

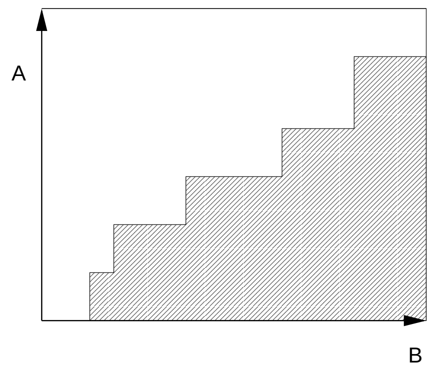


Figure 1: Mental Picture of a Staircase Relation

One of the problems we address in this paper is how to formulate the notion of a "staircase" relation in a way that is both amenable to mathematical calculation and captures the very informal definition just given. In the process of so doing, it is necessary to resolve ambiguities and/or misconceptions that inevitably arise from informal definitions.

Fig. 2 is a "mental picture" of a difunctional relation of type $A{\sim}B$. Informally, a difunctional relation is a (heterogeneous) relation that is the union of a collection of "completely disjoint rectangles[1]". The relation shown in fig. 2 is what we call the "diagonal" of the staircase relation shown in fig. 1.

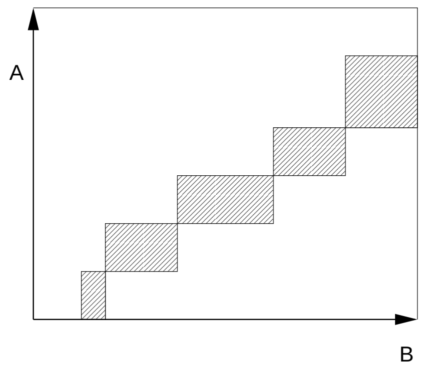


Figure 2: Mental Picture of a Difunctional Relation

The mental picture of a difunctional relation suggests a second property that appears to be folklore: each point a in the left domain and each point b in the right domain of a difunctional relation defines a rectangle whereby related pairs define the same rectangle. In this way, a difunctional relation is characterised by a pair of functional relations.

As mentioned earlier, Riguet [Rig51] uses the name "différence" for what we call the "diagonal". Fig. 3 explains in picture-form the reasoning behind the naming as well as how our formulation differs from Riguet's.

The four parts of fig. 3 depict in turn

(a) a relation R (coloured green),

(b) the factor $R^{\cup}\backslash R^{\cup}/R^{\cup}$ (in red, where $R^{\cup}$ denotes the converse of R),

(c) the diagonal of R (in blue — more precisely, the relation $R\cap R^{\cup}\backslash R^{\cup}/R^{\cup}$),

(d) the relation $R\circ\neg R^{\cup}\circ R$.

Informally, the diagonal of R (shown in fig. 3(c)) is that part of the relation R (shown in fig. 3(a)) that is common to the factor $R^{\cup}\backslash R^{\cup}/R^{\cup}$ (shown in fig. 3(b)).

Riguet formulated the diagonal as the "différence" between R and the relation $R\circ\neg R^{\cup}\circ R$, i.e. as $R\cap\neg(R\circ\neg R^{\cup}\circ R)$. (Note the nested complements, denoted by the

[1]See definition 80 for a formal definition of "completely disjoint rectangles".

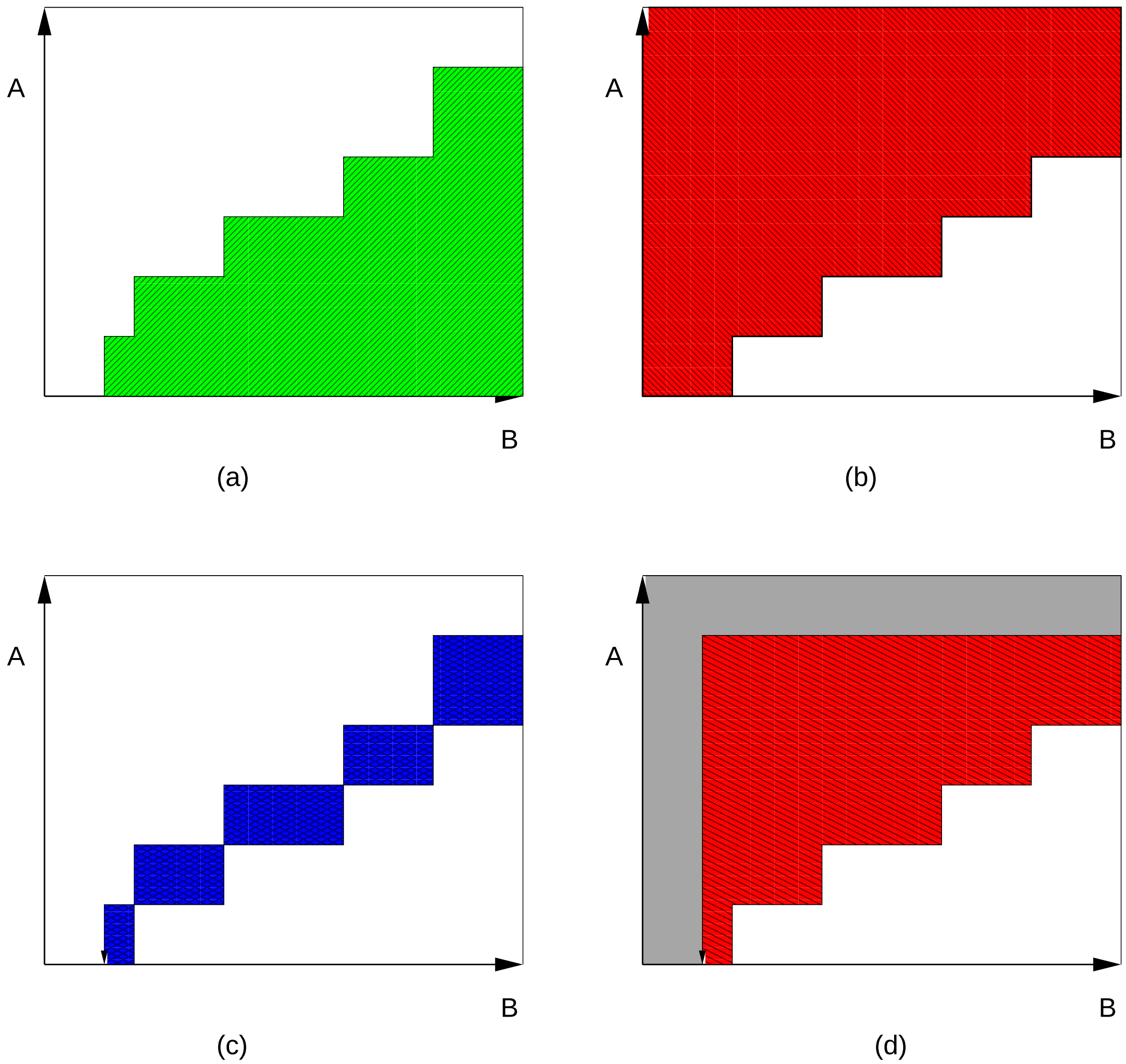


Figure 3: Riguet's "Différence"

symbol "$\neg$".) Fig. 3(d) shows the relation $R \circ \neg R^{\cup} \circ R$. It has two parts: the parts *not* coloured red, i.e. the shaded part and the white part. The part coloured red is the "useful" part $R{<} \circ R^{\cup} \backslash R^{\cup} / R^{\cup} \circ R{>}$ of the relation depicted in (b). Here $R{<}$ and $R{>}$ denote the left and right "domains" of R (not to be confused with the target and source of R). The shaded part of fig. 3(d) depicts the relation $R{\bullet}{<} \circ \top\!\!\top \cup \top\!\!\top \circ R{>}{\bullet}$: the set of pairs (a, b) such that either a is not related by R to any element of B or no element of A is related by R to b. Riguet's "différence" is the difference between the green part of fig. 3(a) and the non-red part of fig. 3(d).

Hopefully, by way of these informal pictures, we can now give an overview of the remainder of the paper.

1.2 Overview

To begin, we present the axiomatic basis for our formal reasoning in section 2. The basis for the axiom system originated in the work of De Morgan, Pierce, Schröder, Tarksi and, no doubt, many others. This section is an abbreviated version of the presentation in [BDGvdW19] to which the reader is referred for full details (including proofs of the stated theorems).

Section 3 goes into more detail on basic elements of relation algebra. At this point, we adhere to our maxim of providing proofs of all stated properties. Whilst the topics in this section —in particular factors (section 3.2), the domain operators (definition 42) and "provisional orderings" (definition 74)— all play a significant rôle later, we recommend that the reader skim the section briefly in the first instance, returning to it later as and when necessary. (The notion of a "provisional ordering" is new but the motivation for its introduction only becomes apparent later.)

Section 4 is the beginning of topics specific to block-orderings. "Blocks" or "rectangles" are particular sorts of relations that are pictured as rectangles. As pictured in fig. 2, a difunctional relation can be characterised as a collection of "completely disjoint rectangles". Section 4.1 presents a number of elementary properties of squares and rectangles whilst section 4.2 introduces some important definitions and properties: the notion of an "indexed set" of rectangles (definition 85), the notion of "completely disjoint rectangles" (definition 86) and the characterisation of an indexed set of completely disjoint rectangles by a pair of functional relations (theorem 100).

In section 5 we formulate properties of partial equivalence relations that will be familiar to most readers. The main topic is a theorem characterising a partial equivalence relation as a collection of disjoint squares. In more familiar terminology, a partial equivalence relation partitions its domain into disjoint equivalence classes. Note that we focus on *partial* equivalence relations (of which equivalence relations form a special case). In general, we are obliged to reason about the left and right domains of relations, particularly when reasoning about the diagonal of a relation (definition 151) — a topic that is central to this investigation. Recall our discussion of the shaded area of fig. 3(d).

We formulate several proofs of the per characterisation theorem, theorem 105, in section 5. Later we do the same for the characterisation of difunctional relations, theorem 132, one of the proofs being based on theorem 105. We do so in order to evaluate different calculational methods. In this case, contrary to the view we ourselves have propagated, the calculations exploiting points and the saturation axiom are preferable to the point-free calculations. Our formalism allows us to mitigate the negative aspects of pointwise reasoning so that points appear in formulae only where this is desirable. This is discussed further in section 10.

The main results of this investigation are presented in section 6 on difunctional relations, section 7 on the "diagonal" of a relation and sections 8 and 9 on block-ordered and staircase relations, respectively.

Section 6 is about the basis for the name "difunction": a difunctional relation is characterised by a pair of functional relations (theorem 132); moreover, such a characterisation is unique (theorem 136). This is a well-known generalisation of the properties of partial equivalence relations and, as mentioned above, is included in order to evaluate different calculational methods.

For completeness, section 6.4 documents the properties of the "difunctional closure" of a relation: the "fermeture difonctionelle" introduced by Riguet [Rig50].

Section 7 is a detailed examination of the properties of the diagonal of a relation. Riguet's account of "relations de Ferrers" includes a theorem characterising such relations as the "réunion" of "rectangles" that have a very special property. Referring to fig. 1, each individual "tread" of a staircase relation defines a unique rectangle (exact details of which are given later) and the relation is the "réunion" of them all. With this as motivation, we abstract the notion of a "polar covering" and we prove a theorem that *every* relation has a polar covering. See definition 155 and theorem 156 in section 7.2. As a step towards Riguet's characterisation of "relations de Ferrers", we define the notion of a "non-redundant" polar covering. For finite relations, it is straightforward to show that a non-redundant polar covering can always be constructed from a given polar covering of the relation. The algorithm may, however, not be practical; moreover, there are infinite relations that do not have a non-redundant polar covering. (The less-than relation on real numbers is an example.) A focus of section 7 is to investigate when the diagonal of a relation defines a non-redundant polar covering of the relation. The main result in this section is thus theorem 167 (which we believe to be original to this paper).

Block-ordered relations are defined in section 8. Although we don't discuss it in any detail, the practical application of block-ordering a relation is efficient storage and recovery of information. Dividing the left and right domains of a relation into "blocks" is an obvious first step. We take the opportunity in section 8.1 to point out the pioneering contribution to information science made by Hartmanis and Stearns [HS64, HS66] in their study of so-called "pair algebras". The relevance to block-ordered relations is that so-called "perfect" Galois connections provide a rich source of examples. Section 8.2 relates block-orderings to diagonals. The section is entitled "analogie frappante" because the concluding theorem of the section (theorem 196) is a necessary and sufficient condition for a relation to be block-ordered expressed as a property of the diagonal of a relation, thus generalising Riguet's "analogie frappante" between the properties of a "relation de Ferrers" and difunctional relations. Theorem 184 proves that every block-ordered relation has a non-redundant polar covering, the non-redundancy of which is witnessed by the relation's diagonal. Section 8.3 introduces a less-restrictive notion of "imperfect" block-orderings.

Section 9 was the starting point of this investigation: principally, how should the informal mental picture of a "staircase" relation be made precise and what then are its properties? Unsurprisingly (at least to us) it turns out that pictures can be deceiving. We have been able to verify that all the claims made by Riguet are valid and much of the section is devoted to that task; in particular, theorem 234 establishes the (unproven) theorem in [Rig51] that every staircase relation has a linear polar covering. On the other hand, we provide examples showing that other claims in the extant literature are not valid. In particular, theorem 220 proves, by way of concrete examples, that not every staircase relation is block-ordered. It is the case, however, as correctly stated by Riguet [Rig51], that every finite staircase relation is block-ordered. Theorem 236 is a slight generalisation.

Section 10 concludes the paper with a brief summary and discussion of publications in the last thirty years. (We have been unable to fill the forty-year gap —in respect of non-finite relations— from 1950 to 1990 and would welcome receiving information about relevant publications in that period.)

Section 11 is a last-minute addition. In it I suggest how a substantial rewrite of this report could be undertaken.

2 The Axiom System

We assume familiarity with a number of basic concepts of relation algebra: composition, converse, left and right domains, and left and right factors (aka "residuals"). Our presentation is based on the system of axioms formulated by Voermans [Voe99]; full details can be found in [BDGvdW19]. In addition to the axioms we give a pointwise *interpretation* of each of the operators. That is, we say, for each operator that we introduce, how the operator defines a set of pairs. In giving the interpretation we use the notation $[\![E]\!]$ to mean "the interpretation of E". Thus we write $x[\![R]\!]y$ instead of xRy; this enhances readability and also emphasises the difference between the objects of an abstract relation algebra and the interpretation of such objects as binary relations.

2.1 Point-Free Relation Algebra

We begin with a point-free axiomatisation of homogeneous relations. Later we extend the axiomatisation to heterogeneous relations (section 2.4) and to points (section 2.5).

The first unit is a lattice structure. Specifically, let $(\mathcal{A}, \subseteq)$ be a partially-ordered set. We postulate that $\mathcal{A}$ forms a complete, universally distributive lattice. The infimum and supremum operators will be denoted by $\cap$ and $\cup$, respectively. The top and bottom elements of the lattice will be denoted by $\top\!\!\top$ and $\bot\!\!\bot$, respectively. We call elements of $\mathcal{A}$ *relations* and denote them by variables R, S and T. The interpretation of $\mathcal{A}$ is the set of relations of some fixed type. The interpretation of a relation is a set; so $\mathcal{A}$ is a powerset.

As suggested by the choice of notation, the interpretation of $\subseteq$ is the subset ordering, the interpretation of $\cap$ is set intersection, and the interpretation of $\cup$ is set union. Formally,

$$[\![R \subseteq S]\!] \;\equiv\; \langle \forall x,y : x[\![R]\!]y : x[\![S]\!]y \rangle \quad ,$$

$$x\,[\![R \cap S]\!]\,y \;\equiv\; x[\![R]\!]y \wedge x[\![S]\!]y \quad \text{, and}$$

$$x\,[\![R \cup S]\!]\,y \;\equiv\; x[\![R]\!]y \vee x[\![S]\!]y \quad .$$

The interpretation of $\top\!\!\top$ is the universal relation and the interpretation of $\bot\!\!\bot$ is the empty relation. That is,

$$\langle \forall x,y :: x[\![\top\!\!\top]\!]y \equiv \mathsf{true} \rangle \;\wedge\; \langle \forall x,y :: x[\![\bot\!\!\bot]\!]y \equiv \mathsf{false} \rangle \quad ,$$

This is the most complicated unit in the framework but one which should be familiar to the reader.

Every binary relation has a converse; the converse operator, denoted by a postfix "$^{\cup}$" symbol (pronounced "wok"), is interpreted by

$$x\,[\![R^{\cup}]\!]\,y \;\equiv\; y[\![R]\!]x$$

for all x and y. Axiomatically, we postulate the existence of a (total) unary function from relations to relations such that, for all relations R and S

(1) $\qquad R^{\cup} \subseteq S \;\equiv\; R \subseteq S^{\cup} \quad .$

The Galois connection (1) is all that is necessary to define the converse operator and its interface with the lattice structure. Its being a Galois connection makes it so attractive.

The set of homogeneous binary relations over some universe includes the identity relation, I, with the interpretation

$$x[\![I]\!]y \;\equiv\; x=y$$

for all x and y. Relations may also be composed via the binary composition operator, $\circ$, defined at the point level by

$$x\,[\![R\circ S]\!]\,z \;\equiv\; \langle\exists y :: x[\![R]\!]y \wedge y[\![S]\!]z\rangle \quad .$$

We capture these two notions axiomatically by demanding the existence of a relation I and a binary operator, $\circ$, mapping a pair of relations to a relation, such that $(\mathcal{A},\circ,I)$ is a monoid.

There are two interfaces to be specified. The interface with the converse operator is soon dealt with. Bearing in mind the intended relational interpretations of converse and composition we postulate

(2) $\qquad (R\circ S)^{\cup} \;=\; S^{\cup}\circ R^{\cup} \quad ,$

for all relations R and S. For the interface with the lattice structure we postulate that a relation algebra is a regular algebra. In particular, we postulate that for all relations R the functions $(R\circ)$ and $(\circ R)$ are universally distributive. This is equivalent to postulating the existence of two factor operators; these are discussed in detail in section 3.2.

In the theory developed in this paper, the converse operator plays a very significant rôle. Because converse has such strong distributivity properties, it is frequently possible to "dualise" a property by simply applying the converse operator to obtain a property that is the mirror image of the original. (See, for example, (3) and (4).) Also, operators we define frequently have left and right variants with mirror properties. (See, for example, the domain operators introduced in definition 42.)

2.2 Operator Precedence

We have now introduced quite a large number of operators. In order to reduce the number of parentheses in formulae we should agree on a precedence between the different operators.

A general rule we use throughout is that all prefix and postfix operators as well as subscripting and superscripting take precedence over infix operators and infix operators in turn take precedence over multifix operators. When both prefix and postfix operators are applied to an expression, we use parentheses to clarify the order of evaluation. An exception is when a prefix and postfix operator obey an "associative" law, in which case we omit the parentheses. For example —as observed by De Morgan— complement and converse "associate". So we can safely write $\neg R^{\cup}$, parsing it as $\neg(R^{\cup})$ or as $(\neg R)^{\cup}$

depending on the calculational needs. Thus it remains to discuss the relative precedence of the infix operators.

For infix operators, the general rule is that metaoperators (operators like $\equiv$ and $\wedge$) have the lowest precedence. Next come relations like $\leq$ and $\subseteq$. The operators of relation algebra have the next highest precedence, and function application (which we denote by an infix dot) has the highest precedence of all. Among the infix operators of relation algebra the precedence is: intersection and union have the same, lowest precedence, and the highest precedence is given to composition.

2.3 Modularity Rule and Cone Rule

Although composition distributes through suprema, it does *not* distribute through infima. This creates difficulties in calculations that combine infima with composition. The rule we now introduce to overcome this difficulty acts as an interface between all three units of the framework. Riguet [Rig48] named the rule after the famous mathematician J.W.R. Dedekind (he called it "la relation de Dedekind") because of its resemblance to the modular identity, a property of normal subgroups attributed to Dedekind. Schmidt and Ströhlein [SS93] have adopted Riguet's terminology (they refer to "the Dedekind formula") whereas Freyd and Ščedrov [Fv90] call it the *law of modularity* (possibly for the same reason as Riguet). We call it the *modularity rule*.

The modularity rule is that, for all relations R, S and T,

(3) $\quad R\circ S\cap T \;\subseteq\; R\circ(S \,\cap\, R^{\cup}\circ T) \quad .$

The dual property, obtained by exploiting properties of the converse operator, is, for all relations R, S and T,

(4) $\quad S\circ R\cap T \;\subseteq\; (S \,\cap\, T\circ R^{\cup})\circ R \quad .$

(This the first of many examples of mirror-image duality that we forewarned of in section 2.1.)

An additional rule, sometimes called "Tarski's rule", is called the *cone rule* below: for all relations R,

(5) $\quad \langle\forall R \;::\; \top\!\!\top\circ R\circ\top\!\!\top = \top\!\!\top \;\;\equiv\;\; R\neq\perp\!\!\perp\rangle \quad .$

Axiom systems for relation algebra often include a complementation (negation) operator and, instead of the modularity rule, the so-called Schröder rule is postulated. Our formulation of Schröder's rule is slightly different from standard accounts, as we now explain.

Suppose we consider an algebra that obeys all the axioms of relation algebra except for the modularity rule. Suppose that the algebra is complemented (i.e. every relation has a complement); we denote the complement of relation R by $\neg R$. Then the *middle-exchange rule*: for all R, S, X and Y,

$$(6) \qquad R\circ\neg X\circ S \subseteq \neg Y \;\equiv\; R^{\cup}\circ Y\circ S^{\cup} \subseteq X$$

is equivalent to the modularity rule. Occasionally, its equivalent, the *rotation* rule:

$$(7) \qquad R\circ S \subseteq \neg T^{\cup} \;\equiv\; T\circ R \subseteq \neg S^{\cup}$$

is used.

The middle-exchange rule gets its name from the fact that the middle term in a composition is exchanged with the right side of an inclusion. It has an attractive, symmetric form, making it easy to remember in spite of having four free variables. The standard rule, due to Schröder, is the conjunction of the two equivalences obtained by instantiating R and S to the identity relation. The rotation rule (so called because of the way the variables are rotated) also has an attractive form.

This concludes our discussion of the point-free algebraic framework. In a few sentences, a relation algebra is a complete, universally distributive lattice on which is defined a monoid structure and a unary converse operator. Composition on the left and on the right are both universally distributive (with the implication that they both have upper adjoints: the factor operators to be introduced in section 3.2). Converse is a lattice isomorphism that preserves the unit of composition and distributes contravariantly through composition. Finally, the lattice structure, converse and the monoid structure are all interrelated via the modularity rule.

2.4 Heterogeneous Relations

A *heterogeneous* relation R has a *type* given by two sets A and B, which we call the *target* and *source* of R. We use the notation $A{\sim}B$ to denote the type of a relation. Formally, a relation of type $A{\sim}B$ is a subset of $A{\times}B$. (Equivalently, it is a function with domain $A{\times}B$ and range Bool.) A *homogeneous* relation is a relation of type $A{\sim}A$ for some A.

The operators in the algebra of heterogeneous relations are typed. For example, the composition of two relations R and S, denoted as always by $R\circ S$, is only defined when the source of R equals the target of S. Moreover, the target of $R\circ S$ is the target of R and the source of $R\circ S$ is the source of S. That is, if R has type $A{\sim}B$ and S has type $B{\sim}C$ then $R\circ S$ has type $A{\sim}C$. We assume the reader is familiar with such rules.

The rules of the untyped calculus are applicable in the typed calculus, with some restrictions on types. Restrictions are necessary on types for, for example, the middle-exchange rule: (6).

Care must be taken with the overloading of notation. It is tempting, for example, to state the rule:

$$\top\!\!\top^{\cup} = \top\!\!\top$$

without qualification. But, if R has type $A\sim B$, its converse $R^{\cup}$ has type $B\sim A$. Thus the notation "$\top\!\!\top$" on the left side of the equation denotes the universal relation of type $A\sim B$, for some types A and B; on the other hand, the notation "$\top\!\!\top$" on the right side of the equation denotes the universal relation of type $B\sim A$. Rather than overload the notation in this way, we could decorate every occurrence of $\top\!\!\top$ with its type. For example, we could rephrase the rule as

$$({}_{A}\top\!\!\top_{B})^{\cup} \;=\; {}_{B}\top\!\!\top_{A} \quad .$$

The same applies to $\perp\!\!\perp$. We prefer not to do so because the type information is usually easy to infer. An exception is that we occasionally decorate the identity relation I with its type: I_A denotes the identity relation of type $A\sim A$.

Typed relation algebra, as briefly summarised above, extends category theory to what has been called *allegory theory*. See Freyd and Ščedrov [Fv90] for more details.

2.5 Points

The relations of a given type form a powerset. A powerset forms a complete, universally distributive, complemented lattice under the subset ordering. However, these properties do not characterise the properties of the *elements* of the sets in the powerset. For this, we need the notion of a "saturated", "atomic" lattice: elements of a set are modelled by so-called "atoms".

Let us recall the appropriate definitions , first in an arbitrary lattice and later specialising to relations.

Definition 8 (Atom and Atomicity) Consider an arbitrary poset ordered by the relation $\subseteq$ and with least element $\perp\!\!\perp$. Then the element p is an *atom* iff

$$\langle \forall q \;::\; q\subseteq p \;\equiv\; q=p \,\vee\, q=\perp\!\!\perp\rangle \quad .$$

Note that $\perp\!\!\perp$ is an atom according to this definition. If p is an atom that is different from $\perp\!\!\perp$ we say that it is a *proper* atom. A lattice is said to be *atomic* if

$$\langle \forall q \;::\; q\neq\perp\!\!\perp \;\equiv\; \langle \exists a : \mathrm{atom}.a \wedge a\neq\perp\!\!\perp : a\subseteq q\rangle\rangle \quad .$$

In words, a lattice is atomic if every proper element includes a proper atom.

□

Definition 9 (Saturated) A complete lattice is *saturated* iff

$$\langle \forall p :: p = \langle \cup a : atom.a \wedge a \subseteq p : a \rangle \rangle \ .$$

□

The following theorem is central to the use of saturated lattices as a model of powersets.

Theorem 10 Suppose $\mathcal{A}$ is a complete, universally distributive lattice. Then the following statements are equivalent.

(a) $\mathcal{A}$ is saturated,

(b) $\mathcal{A}$ is atomic and complemented,

(c) $\mathcal{A}$ is isomorphic to the powerset of its atoms.

□

Given a type A, the homogeneous relations of a given type $A \sim A$ form a powerset. A *coreflexive* relation is a relation of type $A \sim A$, for some A, that is a subset of the identity relation. (Coreflexives are also called partial identities, monotypes and tests.) To our axiom system, we add the following postulates.

1. For each type A, the poset of coreflexives is a complete, universally distributive, saturated lattice.
2. The *all-or-nothing* rule:

   $$\langle \forall a,b,R : AC.a \wedge AC.b : a \circ R \circ b = \perp\!\!\perp \vee a \circ R \circ b = a \circ \top\!\!\top \circ b \rangle$$

   where AC abbreviates "atomic and coreflexive".

The all-or-nothing rule (which we owe to Glück [Glü17]) is equivalent to the postulate that the lattice of relations of a given type is atomic and saturated. The atoms are events of the form $a \circ \top\!\!\top \circ b$ where a and b are atomic coreflexives; such an event models the pair (a, b) in conventional pointwise formulations of relation algebra.

Theorem 11 Suppose that, for all types A, the lattice of coreflexives of type $A \sim A$ is a complete, universally distributive, saturated lattice. Then, if the all-or-nothing rule is universally valid, the lattice of relations of type $A \sim B$ (for arbitrary types A and B) is also a saturated, atomic lattice; the atoms are elements of the form $a \circ \top\!\!\top \circ b$ where a and b are atoms of the lattice of coreflexives of types A and B, respectively. It follows that the lattice of relations is isomorphic to the powerset of the set of elements of the form $a \circ \top\!\!\top \circ b$ where a and b are atoms of the lattice of coreflexives.

□

In common with all coreflexives, a point is a homogeneous relation of type $A{\sim}A$. However, in keeping with the idea that points represent elements of type A, we often abbreviate the type $A{\sim}A$ to just A.

Definition 12 (Point) A *point* is a proper, atomic, coreflexive relation.
□

For the purposes of this paper, we don't need all the details of what is meant by "atomic". If A is a type, we use a, a' etc. to denote points of type A. Similarly for points of type B. Properties we use of a point a of type A are:

(13) $a{\circ}a \;=\; a \;=\; a^{\cup}$,

(14) $\top\!\!\top{\circ}a{\circ}\top\!\!\top \;=\; \top\!\!\top$,

(15) $a{\circ}\top\!\!\top{\circ}a \;=\; a$,

(16) $\langle\forall p \;::\; p \subseteq I_A \;\equiv\; p = \langle\cup a : a \subseteq p : a\rangle\rangle$.

Also, for points a and a' of the same type,

(17) $a = a' \;\vee\; a{\circ}a' = \perp\!\!\perp$.

Property (14) is equivalent to the property that a point is non-empty ("proper"). The property is an instance of the rule we call the "cone rule" introduced earlier. In general, if a is a point of type A and b is a point of type B, the relation $a{\circ}\top\!\!\top{\circ}b$ represents the pair (a,b); given a relation R of type $A{\sim}B$ and points a and b of type A and B, respectively, the statement

$$a{\circ}\top\!\!\top{\circ}b \;\subseteq\; R$$

has the interpretation that the pair a and b are related by R. Specifically, for all relations R and points a and b of appropriate type,

(18) $(a{\circ}R{\circ}b \neq \perp\!\!\perp) \;=\; (a{\circ}\top\!\!\top{\circ}b \subseteq R) \;=\; (a{\circ}\top\!\!\top{\circ}b = a{\circ}R{\circ}b)$.

(In conformance with long-standing mathematical practice, property (18) should be read conjunctionally: that is as the equality of three terms. In this case, each term is boolean.) The *saturation* property is that

(19) $\langle\forall R \;::\; R = \langle\cup\, a,b : a{\circ}\top\!\!\top{\circ}b \subseteq R : a{\circ}\top\!\!\top{\circ}b\rangle\rangle$.

The *irreducibility* property is that, if $\mathcal{R}$ is a function with range relations of type $A{\sim}B$ and source K, then, for all points a and b of appropriate type,

(20) $a{\circ}\top\!\!\top{\circ}b \;\subseteq\; \cup\mathcal{R} \;\equiv\; \langle\exists k : k{\in}K : a{\circ}\top\!\!\top{\circ}b \subseteq \mathcal{R}.k\rangle$.

The *identity relation* I_A of type A has the property that, for all points a and a' of type A,

(21) $\quad a\circ\top\!\!\top\circ a' \subseteq I_A \;\equiv\; a = a'$.

Relations of the form $R\circ b\circ S$, where b is a point, play a central rôle in what follows. The interpretation of $R\circ b\circ S$ is a relation that holds between points a and c iff the relation R holds between a and b, and the relation S holds between b and c. This is expressed precisely by the property:

(22) $\quad a\circ\top\!\!\top\circ c \subseteq R\circ b\circ S \;\equiv\; a\circ\top\!\!\top\circ b \subseteq R \;\wedge\; b\circ\top\!\!\top\circ c \subseteq S$.

3 Basic Structures

This section contains a miscellany of topics that are referred to repeatedly in subsequent sections. We recommend that the reader scans it briefly in the first instance, postponing a more detailed reading until later.

3.1 Specifications

Sometimes we want to define functions indirectly via a property relating input and output values. The property is formalised and then it is shown that the formal specification relates each input value to exactly one output value. That is, the formal specification relates each input value to at most one and at least one output value. In order to reason within our axiom system, we then want to conclude that output values are points. See, for example, section 3.4, where we define the meaning of functionality and exhibit an expression that formulates, in very general terms, the result of applying a function to an argument.

Although the process seems to be obvious, we want to stick to our goal of validating every step within our axiom system. For this reason, we now present the technical justification. As just mentioned, we refer the reader to section 3.4 for a concrete example.

In the following lemmas, p is a coreflexive relation and dummies a and a' are points of the same type as p.

We begin with the consequence of showing that specification p has at least one solution.

Lemma 23

$$p \neq \bot\!\!\bot \;\equiv\; \langle \exists a :: a \subseteq p \rangle \;\; .$$

Proof

$p \neq \perp\!\!\perp$

$=\quad \{\quad$ cone rule: (5) $\quad\}$

$\top\!\!\top \circ p \circ \top\!\!\top = \top\!\!\top$

$=\quad \{\quad$ saturation property: (19) $\quad\}$

$\top\!\!\top \circ \langle \cup a : a \subseteq p : a \rangle \circ \top\!\!\top \;=\; \top\!\!\top$

$=\quad \{\quad$ distributivity $\quad\}$

$\langle \cup a : a \subseteq p : \top\!\!\top \circ a \circ \top\!\!\top \rangle \;=\; \top\!\!\top$

$=\quad \{\quad a$ ranges over points: (15) $\quad\}$

$\langle \cup a : a \subseteq p : \top\!\!\top \rangle \;=\; \top\!\!\top$

$\Rightarrow\quad \{\quad \langle \cup a : \mathsf{false} : \top\!\!\top \rangle = \perp\!\!\perp$ and $\perp\!\!\perp \neq \top\!\!\top \quad\}$

$\langle \exists a :: a \subseteq p \rangle$

$\Rightarrow\quad \{\quad a$ ranges over points: so $\perp\!\!\perp \neq a$

predicate calculus, (details left to the reader) $\quad\}$

$p \neq \perp\!\!\perp$.

□

Next we formulate the consequence of showing that specification p has at most one solution.

Lemma 24

$$\langle \forall a : a \subseteq p : a = p \rangle \quad \equiv \quad \langle \forall a, a' : a \subseteq p \wedge a' \subseteq p : a = a' \rangle \ .$$

Proof

$\langle \forall a : a \subseteq p : a = p \rangle$

$=\quad \{\quad$ anti-symmetry $\quad\}$

$\langle \forall a : a \subseteq p : a \supseteq p \rangle$

$=\quad \{\quad$ saturation: (16) $\quad\}$

$\langle \forall a : a \subseteq p : a \supseteq \langle \cup a' : a' \subseteq p : a' \rangle \rangle$

$=\quad \{\quad$ suprema $\quad\}$

$\langle \forall a : a \subseteq p : \langle \forall a' : a' \subseteq p : a \supseteq a' \rangle \rangle$

$\Leftarrow\quad \{\quad$ reflexivity of the subset relation $\quad\}$

$\langle \forall a : a \subseteq p : \langle \forall a' : a' \subseteq p : a = a' \rangle \rangle$

$=\quad\{\quad$ nesting of quantifications $\quad\}$

$\langle\forall\, a,a' : a\subseteq p \wedge a'\subseteq p : a=a'\rangle$

$\Leftarrow\quad\{\quad$ Leibniz and predicate calculus $\quad\}$

$\langle\forall a : a\subseteq p : a=p\rangle$.

□

Theorem 25 Suppose p is a coreflexive relation. Then p is a point equivales

$$\langle\exists a :: a\subseteq p\rangle \;\wedge\; \langle\forall\, a,a' : a\subseteq p \wedge a'\subseteq p : a=a'\rangle \ .$$

(As above, dummies a and a' range over points of the same type as p.)

In words, a specification p defines a point iff it has at least one solution and at most one solution.

Proof In the following dummy q ranges over coreflexives of the same type as p and a ranges over points of the same type as p.

p is atomic

$=\quad\{\quad$ definition 8 $\quad\}$

$\langle\forall q : q\subseteq p : q=p \vee q=\perp\!\!\perp\rangle$

$=\quad\{\quad$ trading $\quad\}$

$\langle\forall q : q\subseteq p\wedge q\neq\perp\!\!\perp : q=p\rangle$

$=\quad\{\quad$ lemma 23 $\quad\}$

$\langle\forall q : q\subseteq p\wedge\langle\exists a :: a\subseteq q\rangle : q=p\rangle$

$=\quad\{\quad$ distributivity (of conjunction over disjunction),
range disjunction $\quad\}$

$\langle\forall\, q,a : a\subseteq q\subseteq p : q=p\rangle$

$\Leftarrow\quad\{\quad$ anti-symmetry $\quad\}$

$\langle\forall a : a\subseteq p : a=p\rangle$

$=\quad\{\quad$ lemma 24 $\quad\}$

$\langle\forall\, a,a' : a\subseteq p \wedge a'\subseteq p : a=a'\rangle$.

Also,

$$
\begin{array}{ll}
 & p \text{ is atomic} \\
= & \{ \quad \text{definition 8} \quad \} \\
 & \langle \forall q : q \subseteq p : q = p \vee q = \perp\!\!\perp \rangle \\
\Rightarrow & \{ \quad \text{points } a \text{ and } a' \text{ are coreflexives, weakening} \quad \} \\
 & \langle \forall a,a' : a \subseteq p \wedge a' \subseteq p : (a = p \vee a = \perp\!\!\perp) \wedge (a' = p \vee a' = \perp\!\!\perp) \rangle \\
= & \{ \quad \text{points are proper (i.e. } a \neq \perp\!\!\perp \text{ and } a' \neq \perp\!\!\perp \text{)} \quad \} \\
 & \langle \forall a,a' : a \subseteq p \wedge a' \subseteq p : a = p \wedge a' = p \rangle \\
\Rightarrow & \{ \quad \text{transitivity of equality} \quad \} \\
 & \langle \forall a,a' : a \subseteq p \wedge a' \subseteq p : a = a' \rangle \quad .
\end{array}
$$

Combining the two calculations, we have established by mutual implication that

(26) $\quad p \text{ is atomic} \;\equiv\; \langle \forall a,a' : a \subseteq p \wedge a' \subseteq p : a = a' \rangle \quad .$

It follows that, for all coreflexives p,

$$
\begin{array}{ll}
 & p \text{ is a point} \\
= & \{ \quad \text{definitions 8 and 12, assumption: } p \text{ is coreflexive} \quad \} \\
 & p \neq \perp\!\!\perp \;\wedge\; p \text{ is atomic} \\
= & \{ \quad \text{lemma 23 and (26)} \quad \} \\
 & \langle \exists a :: a \subseteq p \rangle \;\wedge\; \langle \forall a,a' : a \subseteq p \wedge a' \subseteq p : a = a' \rangle \quad .
\end{array}
$$

□

3.2 Factors

If R is a relation of type $A \sim B$ and S is a relation of type $A \sim C$, the relation $R \backslash S$ of type $B \sim C$ is defined by the Galois connection, for all T (of type $B \sim C$),

$$R \backslash S \supseteq T \;\equiv\; S \supseteq R \circ T \quad .$$

Similarly, if R is a relation of type $A \sim B$ and S is a relation of type $C \sim B$, the relation R/S of type $A \sim C$ is defined by the Galois connection, for all T,

$$R/S \supseteq T \;\equiv\; R \supseteq T \circ S \quad .$$

(The existence of these operators is equivalent to the universal distributivity of composition over union.)

In relation algebra, factors are also known as “residuals”. We prefer the term “factor” because it emphasises calculational properties whereas “residual” emphasises an operational understanding (what is left after taking something away). In particular, factors have the *cancellation* properties:

$$T \circ T\backslash U \subseteq U \quad \wedge \quad R/S \circ S \subseteq R \ .$$

The factor operators (which we pronounce “under” and “over” respectively) are mutually associative. That is

(27) $\quad R\backslash(S/T) = (R\backslash S)/T \ .$

This means that it is unambiguous to write $R\backslash S/T$ — which we shall do in order to promote the associativity property by making its use invisible (in the same way that the use of the associativity of composition is made invisible).

The relations $R\backslash R$ (of type $B{\sim}B$ if R has type $A{\sim}B$) and R/R (of type $A{\sim}A$ if R has type $A{\sim}B$) play a central rôle in what follows. As is easily verified, both are *preorders*. That is, both are *transitive*:

$$R\backslash R \circ R\backslash R \subseteq R\backslash R \quad \wedge \quad R/R \circ R/R \subseteq R/R$$

and both are *reflexive*:

$$I \subseteq R\backslash R \quad \wedge \quad I \subseteq R/R \ .$$

(The notation “ I ” is overloaded in the above equation. In the left conjunct, it denotes the identity relation of type $B{\sim}B$ and, in the right conjunct, it denotes the identity relation of type $A{\sim}A$, assuming R has type $A{\sim}B$. We often overload constants in this way. Note, however, that we do not attempt to combine the two inclusions into one.) In addition, for all R,

(28) $\quad R \circ R\backslash R = R = R/R \circ R \ ,$

(29) $\quad R/(R\backslash R) = R = (R/R)\backslash R \ ,$

(30) $\quad (R\backslash R)/(R\backslash R) = R\backslash R = (R\backslash R)\backslash(R\backslash R)$ and

(31) $\quad (R/R)\backslash(R/R) = R/R = (R/R)/(R/R) \ .$

In fact, we don't use (29) directly; its relevance is as the initial step in proving the leftmost equations of (30) and (31). We choose not to exploit the associativity of the over and under operators in (30) and (31) —by writing, for example, $(R\backslash R)/(R\backslash R)$ as $R\backslash R/(R\backslash R)$— in order to emphasise their rôle as properties of the preorders $R\backslash R$ and R/R.

In relation algebra (as opposed to regular algebra) it is possible to eliminate the factor operators altogether because they can be expressed in terms of complements and converse. The rule for doing so is given in lemma 32. Although the elimination of factors is highly undesirable, we are obliged to introduce complements and it is useful to exploit the lemma occasionally.

Lemma 32 For all R, S and T,

$$R\backslash S/T \;=\; \neg(R^{\cup}\circ\neg S\circ T^{\cup}) \quad .$$

Proof We have, for all X,

$$
\begin{array}{ll}
 & X \subseteq R\backslash S/T \\
= & \{ \quad \text{definition of factors} \quad \} \\
 & R\circ X\circ T \subseteq S \\
= & \{ \quad \text{middle-exchange: (6)} \quad \} \\
 & R^{\cup}\circ\neg S\circ T^{\cup} \subseteq \neg X \\
= & \{ \quad \text{complements} \quad \} \\
 & X \subseteq \neg(R^{\cup}\circ\neg S\circ T^{\cup}) \quad .
\end{array}
$$

The lemma follows by indirect equality (i.e. by instantiating X to the left and right sides of the claimed equality and then using reflexivity and anti-symmetry of the subset ordering).
□

For the purpose of providing examples, extreme cases are often the most illuminating. Instantiating lemma 32 with $R,S,T := \neg I\,,\neg I\,,I$, and $R,S,T := I\,,\neg I\,,\neg I$ (where I denotes an identity relation of some unspecified type), we get

$$(33) \qquad \neg I\backslash\neg I \;=\; I \;=\; \neg I/\neg I \quad .$$

Thus the equality relation on a type is the preorder of the form $R\backslash R$ (or R/R) obtained by the instantiation $R{:=}\neg I$.

Let $\mathbb{1}$ denote the type with exactly one element. Then the universal relation ${}_{\mathbb{1}}\top\!\!\top_{\mathbb{1}}$ equals the identity relation $I_{\mathbb{1}}$. Thus the type $\mathbb{1}$ is an example of a finite, non-empty type such that $\neg I_{\mathbb{1}}$ is the empty relation ${}_{\mathbb{1}}\bot\!\!\bot_{\mathbb{1}}$.

Property (28) exemplifies how much easier calculations with factors can be compared to calculations that combine complements with converses. The property is very easy to spot and apply. Expressed using lemma 32, it is equivalent to

$$R\circ\neg(R^{\cup}\circ\neg R) \;=\; R \;=\; \neg(\neg R\circ R^{\cup})\circ R \quad .$$

In this form, the property is difficult to spot and its correct application is difficult to check.

It is useful to record the distributivity properties of converse over the factor operators:

Lemma 34 For all R and S,

(35) $R^{\cup}\backslash S^{\cup} = (S/R)^{\cup} = \neg R/\neg S$.

Symmetrically,

(36) $R^{\cup}/S^{\cup} = (S\backslash R)^{\cup} = \neg R\backslash\neg S$.

Also,

(37) $(R\backslash S/T)^{\cup} = T^{\cup}\backslash S^{\cup}/R^{\cup}$.

Proof We prove the first equation of (35) using indirect equality. For any R, S and T, we have:

$$
\begin{array}{ll}
 & T \subseteq (S/R)^{\cup} \\
= & \{ \quad \text{converse: (1)} \quad \} \\
 & T^{\cup} \subseteq S/R \\
= & \{ \quad \text{Galois connection defining factors} \quad \} \\
 & T^{\cup}\circ R \subseteq S \\
= & \{ \quad \text{converse: (1) and (2)} \quad \} \\
 & R^{\cup}\circ T \subseteq S^{\cup} \\
= & \{ \quad \text{Galois connection defining factors} \quad \} \\
 & T \subseteq R^{\cup}\backslash S^{\cup} \ .
\end{array}
$$

The second equation of (35) is proved using the property

(38) $R\backslash S = \neg(R^{\cup}\circ\neg S) \quad\wedge\quad S/T = \neg(\neg S\circ T^{\cup})$.

We have:

$$
\begin{array}{ll}
 & \neg R/\neg S \\
= & \{ \quad \text{(38) with } S,T := \neg R,\neg S, \\
 & \quad\ \ \text{properties of negation and converse} \quad \} \\
 & \neg(R\circ\neg S^{\cup}) \\
= & \{ \quad \text{(38) with } R,S := R^{\cup},S^{\cup}
\end{array}
$$

$$
\begin{array}{ll}
 & \qquad \text{properties of negation and converse} \quad \} \\
 & R^{\cup} \backslash S^{\cup} \\
= & \quad \{ \quad \text{first equality} \quad \} \\
 & (S/R)^{\cup} \quad .
\end{array}
$$

Property (36) proved using symmetrical calculations and (37) is a combination of (35) and (36).

(Note how the associativity property $\neg(R^{\cup}) = (\neg R)^{\cup}$ is used silently in the above calculation.)

□

The following corollary is relevant to section 9 on staircase relations.

Corollary 39 For all R,

$$
R\backslash R \cup (R\backslash R)^{\cup} \;=\; (R^{\cup}/R^{\cup})^{\cup} \cup R^{\cup}/R^{\cup} \;=\; \neg R\backslash\neg R \cup (\neg R\backslash\neg R)^{\cup} \quad .
$$

Proof

$$
\begin{array}{ll}
 & R\backslash R \cup (R\backslash R)^{\cup} \\
= & \quad \{ \quad \text{converse and lemma 34} \\
 & \qquad \text{(in particular (35) with } R,S := R^{\cup}, R^{\cup}\text{)} \quad \} \\
 & (R^{\cup}/R^{\cup})^{\cup} \cup R^{\cup}/R^{\cup} \\
= & \quad \{ \quad \text{lemma 34} \\
 & \qquad \text{(in particular (36) with } R,S := \neg R, \neg R\text{)} \quad \} \\
 & (\neg R\backslash\neg R)^{\cup} \cup \neg R\backslash\neg R \quad .
\end{array}
$$

□

When considering concrete examples, it is sometimes necessary to know the pointwise definition of the factor operators. The following lemma is needed in theorem 220 where we exhibit a concrete counterexample to an error in the extant literature.

Lemma 40 For all relations R and points a and b (of appropriate type),

$$
a\circ\top\!\top\circ b \subseteq (R\backslash R/R)^{\cup} \;\equiv\; \langle\forall\, a',b' : a'\circ\top\!\top\circ b \subseteq R \wedge a\circ\top\!\top\circ b' \subseteq R : a'\circ\top\!\top\circ b' \subseteq R\rangle \quad .
$$

Proof

$$
\begin{array}{ll}
 & a\circ\top\!\top\circ b \subseteq (R\backslash R/R)^{\cup} \\
= & \quad \{ \quad \text{definition of converse and factors} \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & R{\circ}b{\circ}\top\!\top{\circ}a{\circ}R \subseteq R \\
= & \{ \quad \text{saturation property: (19)} \quad \} \\
 & \langle \forall\, a',b' :: a'{\circ}R{\circ}b{\circ}\top\!\top{\circ}a{\circ}R{\circ}b' \subseteq a'{\circ}R{\circ}b' \rangle \\
= & \{ \quad \text{all-or-nothing} \quad \} \\
 & \langle \forall\, a',b' : a'{\circ}\top\!\top{\circ}b \subseteq R \wedge a{\circ}\top\!\top{\circ}b' \subseteq R : a'{\circ}\top\!\top{\circ}b{\circ}\top\!\top{\circ}a{\circ}\top\!\top{\circ}b' \subseteq R \rangle \\
= & \{ \quad \text{cone rule, } a \text{ and } b \text{ are points} \quad \} \\
 & \langle \forall\, a',b' : a'{\circ}\top\!\top{\circ}b \subseteq R \wedge a{\circ}\top\!\top{\circ}b' \subseteq R : a'{\circ}\top\!\top{\circ}b' \subseteq R \rangle \quad .
\end{array}
$$

□

3.3 Squares, Coreflexives and Domains

Within relation algebra, there are various ways that sets can be represented as relations. Schmidt and Ströhlein [SS93] use "conditions" (relations of the form $R{\circ}\top\!\top$ or $\top\!\top{\circ}R$ — called "vectors" by Schmidt and Ströhlein), Freyd and Ščedrov [Fv90] use coreflexives. A third possibility is to use "squares" (as suggested by Voermans [Voe99]).

Definition 41 A (homogeneous) relation R is a *square* iff $R = R \circ \top\!\top \circ R^{\cup}$.
□

Points are squares. Also if a and b are points (of appropriate type), the relations $R^{\cup} \circ a \circ R$ and $R \circ b \circ R^{\cup}$ are squares. (This is an easy consequence of the properties (13) and (15).) We see later (lemma 45) that $R^{\cup} \circ a \circ R$ represents the set of all points b such that a and b are related by R, and similarly for $R \circ b \circ R^{\cup}$.

Formally, coreflexives, conditionals and squares are isomorphic representations of sets. Nevertheless, choosing which to use can make a considerable difference to concise calculation. Squares have the disadvantage that they are not closed under union (although squares are closed under intersection); coreflexives and conditionals are both closed under union and intersection. The only advantage of using conditionals over coreflexives and squares is that they are closed under negation but the advantage is not significant. (Schmidt and Ströhlein [SS93] make extensive use of negation but most can be eliminated by the use of factors.) The overwhelming advantage of using coreflexives is their convenience in expressing restrictions on the left and right domain of relations, in combination with the associativity of composition. So, if p is a coreflexive, $R{\circ}p{\circ}S$ simultaneously restricts the right domain of R and the left domain of S to elements in the set represented by p . If conditions are used, one must choose between using a right condition to restrict the right domain of R and a left condition to restrict the left domain of S . Squares can also be used to restrict the left or right domain of a relation

—there are several instances in section 6.3.1— but cannot be used to simultaneously restrict the right and left domains of two relations. For this reason, we generally prefer to use coreflexives to represent sets, except in very special circumstances.

Definition 42 (Domain Operators) Given relation R of type $A{\sim}B$, the *coreflexive representation* $R{<}$ of the *left domain* of R is defined by the equation

$$R{<} \;=\; I \cap R \circ R^{\cup}$$

and the *coreflexive representation* $R{>}$ of the *right domain* of R is defined by the equation

$$R{>} \;=\; I \cap R^{\cup} \circ R \;\; .$$

□

Aside Freyd and Ščedrov [Fv90] call $R{<}$ the "domain" of R; they do not appear to give a name to $R{>}$. Like us, they also use the names "source" and "target". In their account a relation of type $A{\sim}B$ has source A and target B; we reverse the names. (See the warning above.) Bird and De Moor [BdM97] call $R{>}$ the "domain" of R and $R{<}$ the "range" of R. **End of Aside**

In our earlier work on relation algebra, the domain operators play a very significant rôle, and the same is true here. We regard knowledge of their properties as so fundamental that we often explain steps making use of domain calculus with the simple hint "domains". The most fundamental property of the domain operators —monotonicity— we use silently. Sometimes (for example in the proof of lemma 43) we state the properties within everywhere brackets.

Below we document some properties not given elsewhere which combine properties of points and domains.

Lemma 43 For all relations R and points a and b (of appropriate type),

$$a \subseteq R{<} \;\equiv\; (a{\circ}R){>} \neq \perp\!\!\perp \quad \text{, and}$$

$$b \subseteq R{>} \;\equiv\; (R{\circ}b){<} \neq \perp\!\!\perp \;\; .$$

Proof We prove the second equation.

$$(R{\circ}b){<} \neq \perp\!\!\perp$$

$= \quad \{ \quad \text{cone rule: (5)} \quad \}$

$$\top\!\!\top \circ (R{\circ}b){<} \circ \top\!\!\top \;=\; \top\!\!\top$$

$= \quad \{ \quad [\; R{<} \circ \top\!\!\top \;=\; R{\circ}\top\!\!\top \;] \text{ with } R{:=}R{\circ}b \quad \}$

$$
\begin{array}{ll}
 & \top\!\top \circ R \circ b \circ \top\!\top = \top\!\top \\
= & \{ \quad [\ \top\!\top \circ R_{>} = \top\!\top \circ R\ ] \quad \} \\
 & \top\!\top \circ R_{>} \circ b \circ \top\!\top = \top\!\top \\
= & \{ \quad \text{cone rule: (5)} \quad \} \\
 & R_{>} \circ b \neq \bot\!\bot \\
= & \{ \quad R_{>} \circ b \subseteq b\ ; \\
 & \quad\ \text{so, by atomicity of } b,\ R_{>} \circ b = b \vee R_{>} \circ b = \bot\!\bot\ ; \\
 & \quad\ \text{also, } b \neq \bot\!\bot \quad \} \\
 & R_{>} \circ b = b \\
= & \{ \quad R_{>} \circ b = R_{>} \cap b \quad \} \\
 & b \subseteq R_{>}\ .
\end{array}
$$

□

For a point b the square $R \circ b \circ R^{\cup}$ represents the set of all points a such that a and b are related by R. This is made precise in lemma 44 and its corollary, lemma 45.

Lemma 44 For all relations R of type $A{\sim}B$, all coreflexives p of type $A{\sim}A$ and all points b of type B,

$$p \subseteq R \circ b \circ R^{\cup} \equiv p \circ \top\!\top \circ b \subseteq R\ .$$

Symmetrically, for all relations R of type $A{\sim}B$, all coreflexives q of type $B{\sim}B$ and all points a of type A,

$$q \subseteq R^{\cup} \circ a \circ R \equiv a \circ \top\!\top \circ q \subseteq R\ .$$

Proof By mutual implication:

$$
\begin{array}{ll}
 & p \subseteq R \circ b \circ R^{\cup} \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & p \circ \top\!\top \circ b \subseteq R \circ b \circ R^{\cup} \circ \top\!\top \circ b \\
\Rightarrow & \{ \quad R^{\cup} \circ \top\!\top \subseteq \top\!\top\ ;\ b \text{ is a point: (15) and } b \subseteq I \quad \} \\
 & p \circ \top\!\top \circ b \subseteq R \\
\Rightarrow & \{ \quad \text{converse and monotonicity} \quad \} \\
 & p \circ \top\!\top \circ b \circ b \circ \top\!\top \circ p^{\cup} \subseteq R \circ b \circ R^{\cup} \\
\Rightarrow & \{ \quad b \text{ is a point: (13) and (14)}
\end{array}
$$

p is a coreflexive, so $p^{\cup} = p$; monotonicity $\}$

$$p \circ \top\!\!\top \circ p \;\subseteq\; R \circ b \circ R^{\cup}$$

$\Rightarrow \quad \{ \quad I \subseteq \top\!\!\top$ and $p \circ p = p \quad \}$

$$p \;\subseteq\; R \circ b \circ R^{\cup} \;\; .$$

□

Property (18) is the most basic formulation of membership of pairs in a relation. It can also be formulated in terms of squares and in terms of domains:

Lemma 45 For all relations R and points a and b (of appropriate type),

$$(a \;\subseteq\; R \circ b \circ R^{\cup}) \;\;=\;\; (a \circ \top\!\!\top \circ b \subseteq R) \;\;=\;\; (b \;\subseteq\; R^{\cup} \circ a \circ R) \;\; .$$

Proof Straightforward instantiation of lemma 44:

$$a \;\subseteq\; R \circ b \circ R^{\cup}$$

$= \quad \{ \quad$ lemma 44 with $p := a \quad \}$

$$a \circ \top\!\!\top \circ b \subseteq R$$

$= \quad \{ \quad$ lemma 44 with $p := b \quad \}$

$$b \;\subseteq\; R^{\cup} \circ b \circ R \;\; .$$

□

Lemma 46 For all relations R and points a and b (of appropriate type),

$$(a \subseteq (R \circ b)_{<}) \;\;=\;\; (a \circ \top\!\!\top \circ b \subseteq R) \;\;=\;\; (b \subseteq (a \circ R)_{>}) \;\; .$$

Proof

$$a \circ \top\!\!\top \circ b \subseteq R$$

$\Rightarrow \quad \{ \quad$ monotonicity and a is a coreflexive, so $a \circ a = a \quad \}$

$$a \circ \top\!\!\top \circ b \subseteq a \circ R$$

$\Rightarrow \quad \{ \quad$ monotonicity $\quad \}$

$$(a \circ \top\!\!\top \circ b)_{>} \subseteq (a \circ R)_{>}$$

$= \quad \{ \quad$ domains: definition 42, a and b are points: (14) and (15) $\quad \}$

$$b \subseteq (a \circ R)_{>}$$

$\Rightarrow \quad \{ \quad$ monotonicity $\quad \}$

$$a \circ \top\!\!\top \circ b \;\subseteq\; a \circ \top\!\!\top \circ (a \circ R)_{>}$$

$$
\begin{array}{ll}
= & \{ \quad \text{domains: } [\ \top\!\!\top \circ R_{>} \ = \ \top\!\!\top \circ R\ ] \text{ with } R := a \circ R \quad \} \\
& a \circ \top\!\!\top \circ b \ \subseteq \ a \circ \top\!\!\top \circ a \circ R \\
= & \{ \quad a \text{ is a point, so } a \circ \top\!\!\top \circ a = a \quad \} \\
& a \circ \top\!\!\top \circ b \subseteq a \circ R \\
\Rightarrow & \{ \quad a \text{ is a coreflexive, monotonicity} \quad \} \\
& a \circ \top\!\!\top \circ b \subseteq R \ \ .
\end{array}
$$

That is, we have shown by mutual implication that

$$a \circ \top\!\!\top \circ b \subseteq R \ \equiv \ b \subseteq (a \circ R)_{>} \ \ .$$

A symmetric calculation establishes that

$$a \circ \top\!\!\top \circ b \subseteq R \ \equiv \ a \subseteq (R \circ b)_{<} \ \ .$$

□

Combined with property (18), lemmas 45 and 46 give six alternative ways of formulating the membership relation $a \circ \top\!\!\top \circ b \subseteq R$. All are useful.

Lemma 47 For all relations R and points a (of appropriate type),

$$a \subseteq R_{<} \ \equiv \ \langle \exists b : b \subseteq R_{>} : a \circ \top\!\!\top \circ b \subseteq R \rangle \ \ .$$

Also, for all relations R and points b (of appropriate type),

$$b \subseteq R_{>} \ \equiv \ \langle \exists a : a \subseteq R_{<} : a \circ \top\!\!\top \circ b \subseteq R \rangle \ \ .$$

Proof We prove the first equation:

$$
\begin{array}{ll}
& a \subseteq R_{<} \\
= & \{ \quad \text{lemma 43} \quad \} \\
& (a \circ R)_{>} \neq \bot\!\!\bot \\
= & \{ \quad \text{lemma 23} \quad \} \\
& \langle \exists b :: b \subseteq (a \circ R)_{>} \rangle \\
= & \{ \quad \text{lemma 46} \quad \} \\
& \langle \exists b :: a \circ \top\!\!\top \circ b \subseteq R \rangle \\
= & \{ \quad \text{domains (specifically, } a \circ \top\!\!\top \circ b \subseteq R \Rightarrow b \subseteq R_{>}) \quad \} \\
& \langle \exists b : b \subseteq R_{>} : a \circ \top\!\!\top \circ b \subseteq R \rangle \ \ .
\end{array}
$$

□

Lemma 48 gives a pointwise interpretations of the factor operators. Although we typically try to avoid pointwise reasoning, the lemma is sometimes indispensable.

Lemma 48 For all relations R of type $A{\sim}C$ and S of type $B{\sim}C$ (for some A, B and C) and all points a and b,

$$a{\circ}\top\!\!\top{\circ}b \subseteq R/S \quad\equiv\quad (b{\circ}S)_{>} \subseteq (a{\circ}R)_{>} \ .$$

Dually, for all relations R of type $C{\sim}A$ and S of type $C{\sim}B$, and all points a and b,

$$a{\circ}\top\!\!\top{\circ}b \subseteq R\backslash S \quad\equiv\quad (R{\circ}a)_{<} \subseteq (S{\circ}b)_{<} \ .$$

Proof By mutual implication:

$$
\begin{array}{ll}
 & a{\circ}\top\!\!\top{\circ}b \subseteq R/S \\
= & \{ \quad \text{definition of factor} \quad \} \\
 & a{\circ}\top\!\!\top{\circ}b{\circ}S \subseteq R \\
\Rightarrow & \{ \quad a \text{ and } b \text{ are points, monotonicity and domains} \\
 & \quad\ \ \text{(see initial steps in proof of lemma 46)} \quad \} \\
 & (b{\circ}S)_{>} \subseteq (a{\circ}R)_{>} \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & a \circ \top\!\!\top \circ (b{\circ}S)_{>} \subseteq a \circ \top\!\!\top \circ (a{\circ}R)_{>} \\
= & \{ \quad \text{domains} \quad \} \\
 & a{\circ}\top\!\!\top{\circ}b{\circ}S \subseteq a{\circ}\top\!\!\top{\circ}a{\circ}R \\
= & \{ \quad a \text{ is a point (so } a{\circ}\top\!\!\top{\circ}a = a\text{)} \quad \} \\
 & a{\circ}\top\!\!\top{\circ}b{\circ}S \subseteq a{\circ}R \\
\Rightarrow & \{ \quad a \text{ is a coreflexive} \quad \} \\
 & a{\circ}\top\!\!\top{\circ}b{\circ}S \subseteq R \\
= & \{ \quad \text{definition of factor} \quad \} \\
 & a{\circ}\top\!\!\top{\circ}b \subseteq R/S \ .
\end{array}
$$

The second equivalence is proved similarly.

$$a{\circ}\top\!\!\top{\circ}b \subseteq R\backslash S$$

$=\quad\{\quad$ definition of factor $\quad\}$

$$R{\circ}a{\circ}\top\!\!\top{\circ}b \subseteq S$$

$\Rightarrow\quad\{\quad$ monotonicity and coreflexives

(see initial steps in proof of lemma 46) $\quad\}$

$$(R{\circ}a)_< \subseteq (S{\circ}b)_<$$

$\Rightarrow\quad\{\quad$ (as in above calculation) $\quad\}$

$$a{\circ}\top\!\!\top{\circ}b \subseteq R\backslash S \quad .$$

□

For relations R and S with the same source, the relation $R/S \cap (S/R)^\cup$ is the "symmetric left division" of R and S. Dually, for relations R and S with the same target, the relation $R\backslash S \cap (S\backslash R)^\cup$ is their "symmetric right division". (See the discussion at the beginning of section 3.5.) The following corollary of lemma 48 gives a pointwise interpretation of these "division" operators.

Corollary 49 For all relations R and S with the same source, and all points a and b (of appropriate type),

$$a{\circ}\top\!\!\top{\circ}b \subseteq R/S \cap (S/R)^\cup \equiv (a{\circ}R)_> = (b{\circ}S)_> \quad .$$

Dually, for all relations R and S with the same target, and all points a and b (of appropriate type),

$$a{\circ}\top\!\!\top{\circ}b \subseteq R\backslash S \cap (S\backslash R)^\cup \equiv (R{\circ}a)_< = (S{\circ}b)_< \quad .$$

Proof Straightforward application of lemma 48 and anti-symmetry:

$$a{\circ}\top\!\!\top{\circ}b \subseteq R/S \cap (S/R)^\cup$$

$=\quad\{\quad$ infima and converse $\quad\}$

$$a{\circ}\top\!\!\top{\circ}b \subseteq R/S \;\wedge\; b{\circ}\top\!\!\top{\circ}a \subseteq S/R$$

$=\quad\{\quad$ lemma 48 $\quad\}$

$$(b{\circ}S)_> \subseteq (a{\circ}R)_> \;\wedge\; (a{\circ}R)_> \subseteq (b{\circ}S)_>$$

$=\quad\{\quad$ anti-symmetry $\quad\}$

$$(a{\circ}R)_> = (b{\circ}S)_> \quad .$$

□

3.4 Functionality

A relation R of type $A{\sim}B$ is said to be *functional* if $R\circ R^{\cup} \subseteq I_A$. A relation R of type $A{\sim}B$ is said to be *surjective* if $R\circ R^{\cup} \supseteq I_A$. Equivalently, a relation R of type $A{\sim}B$ is surjective if $R{<} = I_A$. A relation R of type $A{\sim}B$ that is both functional and surjective is thus defined by the property $R\circ R^{\cup} = I_A$.

(Other words used for functional are "quasi-fonctionelle" [Rig48], "simple" [Fv90, BdM97] and "univalent" [SS93].)

Dual to functionality and surjectivity are the notions of injectivity and totality, respectively. A relation R of type $A{\sim}B$ is said to be *injective* if $R^{\cup}\circ R \subseteq I_B$. A relation R of type $A{\sim}B$ is said to be *total* if $R^{\cup}\circ R \supseteq I_B$. Equivalently, a relation R of type $A{\sim}B$ is surjective if $R{>} = I_B$.

Typically, we use lowercase letters f, g, h to denote functional relations. As the terminology suggests, these point-free definitions correspond to notions that are more usually defined in terms of points. The pointwise interpretations are explained below, beginning with the interpretation of a functional relation as what others might call a "partial function".

The standard notion of a partial function is a relation that defines a unique output value for each input value in its domain. In our axiom system we formulate this as follows.

Suppose R of type $A{\sim}B$ is functional and suppose b is a point of type B such that $b \subseteq R{>}$. We assert that the equation

(50) $\quad a{:}\ \ a{\in}A{:}\ \ a\circ\top\!\!\top\circ b \subseteq R$

has exactly one solution. Conversely, we assert that if equation (50) has exactly one solution for all points b such that $b \subseteq R{>}$, the relation R is functional. (In (50) the expression "$a{\in}A$" limits the range of the dummy a to points of type A; this notation will be used later where the range of a dummy cannot be deduced from other considerations.)

Equation (51) is an example of the sort of indirect specification anticipated in section 2.5. (See in particular theorem 25.) More formally, for functional relation f and point b such that $b \subseteq f{>}$, equation (51) defines $f.b$ as the unique solution of the equation:

$$a{::}\ \ \mathrm{point}.a\ \wedge\ a\circ\top\!\!\top\circ b \subseteq f\quad .$$

Suppose we denote this unique solution by $f.b$. The defining property of $f.b$ is thus

(51) $\quad \langle\forall\, a,b\ :\ b \subseteq f{>}\ :\ a\circ\top\!\!\top\circ b \subseteq f\ \equiv\ a = f.b\rangle\quad .$

But it is not immediately obvious that $f.b$ is well-defined in our axiom system. Theorem 52 provides a formal justification.

Theorem 52 Suppose relation R has type $A{\sim}B$. Then

(53) $R \circ R^{\cup} \subseteq I_A \;\equiv\; \langle \forall b : b \subseteq R{>} : \text{point}.(R \circ b \circ R^{\cup}) \rangle$.

Moreover, if f is a relation of type $A{\sim}B$ and $f \circ f^{\cup} \subseteq I_A$, the relation $f \circ b \circ f^{\cup}$ is a point of type A and

(54) $\langle \forall a,b \;:\; b \subseteq f{>} \;:\; a \circ \top\!\!\top \circ b \subseteq f \;\equiv\; a = f \circ b \circ f^{\cup} \rangle$.

In words, f is functional iff, for all points b in the right domain of f , the relation $f \circ b \circ f^{\cup}$ defines a unique point of type A , which point we denote by $f.b$.

Proof We prove (53) by mutual implication. First,

$R \circ R^{\cup} \subseteq I_A$

$=$ { domains }

$R \circ R{>} \circ R^{\cup} \subseteq I_A$

$=$ { saturation axiom: (16) }

$R \circ \langle \cup b \;:\; b \subseteq R{>} \;:\; b \rangle \circ R^{\cup} \subseteq I_A$

$=$ { distributivity }

$\langle \forall b \;:\; b \subseteq R{>} \;:\; R \circ b \circ R^{\cup} \subseteq I_A \rangle$

$\Leftarrow$ { definition 12 of a point }

$\langle \forall b : b \subseteq R{>} : \text{point}.(R \circ b \circ R^{\cup}) \rangle$.

Thus we have established the "if" part of the equivalence. Now, for the "only-if", assume $R \circ R^{\cup} \subseteq I_A$.

We first note that, for all b such that $b \subseteq R{>}$, equation (50) has *at most one* solution since, for all points a and a' of type A ,

$a \circ \top\!\!\top \circ b \subseteq R \;\wedge\; a' \circ \top\!\!\top \circ b \subseteq R$

$\Rightarrow$ { converse and monotonicity }

$a \circ \top\!\!\top \circ b \circ b \circ \top\!\!\top \circ a' \subseteq R \circ R^{\cup}$

$=$ { b is a point, so $\top\!\!\top \circ b \circ b \circ \top\!\!\top = \top\!\!\top$ }

$a \circ \top\!\!\top \circ a' \subseteq R \circ R^{\cup}$

$\Rightarrow$ { assumption: $R \circ R^{\cup} \subseteq I_A$, transitivity of the subset relation }

$a \circ \top\!\!\top \circ a' \subseteq I_A$

$\Rightarrow$ { a and a' are points: (21) }

$a = a'$.

That is,

(55) $\langle \forall b : b \subseteq R_> : \langle \forall a,a' : a \circ \top\!\!\top \circ b \subseteq R \ \wedge\ a' \circ \top\!\!\top \circ b \subseteq R : a = a' \rangle\rangle$.

By lemma 43, equation (50) has *at least one* solution for all points b such that $b \subseteq R_>$. That is,

(56) $\langle \forall b : b \subseteq R_> : \langle \exists a :: a \circ \top\!\!\top \circ b \subseteq R \rangle\rangle$.

Thus equation (50) has *exactly one* solution for all points b such that $b \subseteq f_>$. So:

$\langle \forall b : b \subseteq R_> : \mathsf{point}.(R \circ b \circ R^{\cup})\rangle$

$=$ { $R \circ b \circ R^{\cup}$

  $\subseteq$ { assumption: $b \subseteq R_>$, monotonicity }

  $R \circ R_> \circ R^{\cup}$

  $=$ { domains }

  $R \circ R^{\cup}$

  $\subseteq$ { assumption: $R \circ R^{\cup} \subseteq I_A$ }

  I_A ,

 theorem 25 with $p := R \circ b \circ R^{\cup}$ }

$\langle \forall b : b \subseteq R_> : \langle \exists a :: a \subseteq R \circ b \circ R^{\cup} \rangle\rangle$

$\wedge$ $\langle \forall b : b \subseteq R_> : \langle \forall a,a' : a \subseteq R \circ b \circ R^{\cup} \ \wedge\ a' \subseteq R \circ b \circ R^{\cup} : a = a' \rangle\rangle$

$=$ { lemma 45 }

$\langle \forall b : b \subseteq R_> : \langle \exists a :: a \circ \top\!\!\top \circ b \subseteq R \rangle\rangle$

$\wedge$ $\langle \forall b : b \subseteq R_> : \langle \forall a,a' : a \circ \top\!\!\top \circ b \subseteq R \ \wedge\ a' \circ \top\!\!\top \circ b \subseteq R : a = a' \rangle\rangle$

$=$ { (55) and (56) }

true .

This concludes the proof of (53).

Now, assuming that $f \circ f^{\cup} \subseteq I$, it follows from (53) (with $R := f$) that $f \circ b \circ f^{\cup}$ is a point. Also, for all points a and b (of types A and B, respectively),

$b \subseteq f_> \ \wedge\ a \circ \top\!\!\top \circ b \subseteq f$

$=$ { lemma 46 (aiming to eliminate first conjunct) }

$b \subseteq f_> \ \wedge\ b \subseteq (a \circ f)_> \ \wedge\ a \circ \top\!\!\top \circ b \subseteq f$

$=$ { monotonicity and lemma 46 }

$$a\circ\top\!\top\circ b \subseteq f$$
$= \quad \{ \quad$ lemma 45 $\quad \}$
$$a \subseteq f\circ b\circ f^{\cup}$$
$= \quad \{ \quad$ $f\circ b\circ f^{\cup}$ is a point, definitions 12 and 8 $\quad \}$
$$a = f\circ b\circ f^{\cup} \quad .$$

□

Occasionally we need to define a functional relation. Sometimes we specify the relation by means of an equation: "we define f of type ... by $f.b = \ldots$". More often, we use the notation $\langle b :: \ldots\rangle$ to denote a total function, or $\langle b : \ldots : \ldots\rangle$ to denote a (non-total) functional, the range part being used to specify a restriction on the domain. This is consistent with our notation for suprema and infima (such as in universal and existential quantifications).

A consequence of the unicity property expressed by (51) is the property that, for all functional relations f of type $C{\sim}A$ and g of type $C{\sim}B$,

(57) $\quad a\circ\top\!\top\circ b \subseteq f^{\cup}\circ g \;\equiv\; a\subseteq f{>} \wedge f.a = g.b \wedge b\subseteq g{>} \quad .$

The proof exploits the irreducibility of points:

$$a\circ\top\!\top\circ b \subseteq f^{\cup}\circ g$$
$= \quad \{ \quad$ domains, saturation axiom: (16) and distributivity $\quad \}$
$$a\circ\top\!\top\circ b \subseteq \langle\cup c : c\in C : f^{\cup}\circ c\circ g\rangle$$
$= \quad \{ \quad$ points are irreducible: (20) $\quad \}$
$$\langle\exists c : c\in C : a\circ\top\!\top\circ b \subseteq f^{\cup}\circ c\circ g\rangle$$
$= \quad \{ \quad$ (22) $\quad \}$
$$\langle\exists c : c\in C : a\circ\top\!\top\circ c \subseteq f^{\cup} \wedge c\circ\top\!\top\circ b \subseteq g\rangle$$
$= \quad \{ \quad$ converse, lemma 46 and (51) $\quad \}$
$$\langle\exists c : c\in C : a\subseteq f{>} \wedge c = f.a \wedge b\subseteq g{>} \wedge c = g.b\rangle$$
$= \quad \{ \quad$ Leibniz and predicate calculus $\quad \}$
$$a\subseteq f{>} \wedge f.a = g.b \wedge b\subseteq g{>} \quad .$$

Now suppose R is a surjective relation of type $A{\sim}B$. In this case, for all points a of type A, the equation

(58) $\quad b:\;\; b\in B:\;\; a\circ\top\!\top\circ b \subseteq R$

has *at least one* solution since:

$$I_B \subseteq R^\cup \circ R$$

$=\quad \{\quad$ saturation axiom: (16) and supremum $\quad\}$

$$\langle \forall b : b{\in}B : b \subseteq R^\cup \circ R \rangle$$

$=\quad \{\quad$ saturation axiom: (16) and distributivity $\quad\}$

$$\langle \forall b : b{\in}B : b \subseteq \langle \cup a : a{\in}A : R^\cup \circ a \circ R \rangle \rangle$$

$=\quad \{\quad$ points are irreducible: (20) $\quad\}$

$$\langle \forall b : b{\in}B : \langle \exists a : a{\in}A : b \subseteq R^\cup \circ a \circ R \rangle \rangle$$

$=\quad \{\quad$ lemma 45 $\quad\}$

$$\langle \forall b : b{\in}B : \langle \exists a : a{\in}A : a \circ \top\!\top \circ b \subseteq R \rangle \rangle \quad .$$

In the same way, pointwise formulations of the dual notions of injectivity and totality can be derived. Our terminology reflects a bias in the interpretation of relations as having output on the left and input on the right. A more neutral terminology such as "left-functional", "right-functional", "left-total" and "right-total" would be preferable.

Care must be taken when using the above pointwise definitions in our axiom system. The problem is the overloading of the symbol $\top\!\top$: sometimes the type information is essential. For example, the left-domain operator (which we denote by the postfix symbol $<$) defines a total function of type $Cor.A \leftarrow (A{\sim}B)$, for all types A and B, where $Cor.A$ denotes the set of coreflexives of type A. Denoting this function by $Ldom$, we must be careful to distinguish between $Ldom.R$ and $R<$. This is because, according to (50),

(59) $\quad Ldom.R \circ \top\!\top \circ R \subseteq Ldom \quad ;$

on the other hand,

(60) $\quad R< \circ \top\!\top \circ R = R \circ \top\!\top \circ R$

and it doesn't make sense to write

$$R< \circ \top\!\top \circ R \;\subseteq\; < \;\; !$$

In equation (59), both R and $Ldom.R$ are *points* of type $A{\sim}B$ and $Cor.A$, respectively, and the symbol "$\top\!\top$" has type $Cor.A{\sim}(A{\sim}B)$ whereas in equation (60) $R<$ is not a point, the leftmost occurrence of the symbol "$\top\!\top$" has type $A{\sim}A$ and its rightmost occurrence has type $A{\sim}B$.

We conclude this section with the observation that functionality can be defined via a Galois connection. Specifically, the relation f is functional iff, for all relations R and S (of appropriate type),

(61) $\quad f \circ R \subseteq S \quad \equiv \quad f> \circ R \subseteq f^\cup \circ S \quad .$

It is a simple exercise to show that (61) is equivalent to the property $f \circ f^{\cup} \subseteq I$. (Although (61) doesn't immediately fit the standard definition of a Galois connection, it can be turned into standard form by restricting the range of the dummy R to relations that satisfy $f{>} \circ R = R$, i.e. relations R such that $R{<} \subseteq f{>}$.)

The converse-dual of (61) is also used frequently: g is functional iff, for all R and S,

(62) $\quad R \circ g^{\cup} \subseteq S \quad \equiv \quad R \circ g{>} \subseteq S \circ g \quad .$

Comparing the Galois connections defining the over and under operators (see section 3.2) with the Galois connection defining functionality (see (61)) suggests a formal relationship between "division" by a functional relation and composition with the relation's converse. The precise form of this relationship is given by the following lemma. (The lemma is complicated by the fact that it has five free variables. Simpler, possibly better known, instances can be obtained by instantiating one or more of f, g, U and W to the identity relation.) The lemma is crucial to fully understanding Riguet's "analogie frappante"; see lemma 178 in section 8.2.

Lemma 63 Suppose f and g are functional. Then for all U, V and W,

$$\begin{array}{ll} & f^{\cup} \circ (g{<} \circ U) \backslash V / (W \circ f{<}) \circ g \\ = & f{>} \circ (g^{\cup} \circ U \circ f) \backslash (g^{\cup} \circ V \circ f) / (g^{\cup} \circ W \circ f) \circ g{>} \quad . \end{array}$$

Proof Guided by the assumed functionality of f and g, we use the rule of indirect equality. Specifically, we have, for all R, U, V and W,

$$\begin{array}{ll} & f{>} \circ R \circ g{>} \;\subseteq\; f^{\cup} \circ (g{<} \circ U) \backslash V / (W \circ f{<}) \circ g \\ = & \{ \quad \text{assumption: } f \text{ and } g \text{ are functional, (61) and (62)} \quad \} \\ & f \circ R \circ g^{\cup} \;\subseteq\; (g{<} \circ U) \backslash V / (W \circ f{<}) \\ = & \{ \quad \text{factors} \quad \} \\ & g{<} \circ U \circ f \circ R \circ g^{\cup} \circ W \circ f{<} \;\subseteq\; V \\ = & \{ \quad \text{assumption: } f \text{ and } g \text{ are functional} \\ & \quad\quad \text{i.e.} \quad f \circ f^{\cup} = f{<} \wedge g \circ g^{\cup} = g{<} \quad \} \\ & g \circ g^{\cup} \circ U \circ f \circ R \circ g^{\cup} \circ W \circ f \circ f^{\cup} \;\subseteq\; V \\ = & \{ \quad \text{assumption: } f \text{ and } g \text{ are functional,} \\ & \quad\quad [\,(R^{\cup}){>} = R{<}\,] \text{ with } R{:=}f \text{ and } R{:=}g \quad \} \\ & g^{\cup} \circ U \circ f \circ R \circ g^{\cup} \circ W \circ f \;\subseteq\; g^{\cup} \circ V \circ f \end{array}$$

$$
\begin{array}{ll}
= & \{ \quad \text{domains} \quad \} \\
& g^{\cup} \circ U \circ f \circ f{>} \circ R \circ g{>} \circ g^{\cup} \circ W \circ f \ \subseteq \ g^{\cup} \circ V \circ f \\
= & \{ \quad \text{factors} \quad \} \\
& f{>} \circ R \circ g{>} \ \subseteq \ (g^{\cup} \circ U \circ f)\backslash(g^{\cup} \circ V \circ f)/(g^{\cup} \circ W \circ f) \\
= & \{ \quad f{>} \text{ and } g{>} \text{ are coreflexives} \quad \} \\
& f{>} \circ R \circ g{>} \ \subseteq \ f{>} \circ (g^{\cup} \circ U \circ f)\backslash(g^{\cup} \circ V \circ f)/(g^{\cup} \circ W \circ f) \circ g{>}
\end{array}
$$

The lemma follows by instantiating R to the left and right sides of the claimed equation, simplifying using domain calculus, and then applying the reflexivity and anti-symmetry of the subset relation.
□

3.5 Formulations of Power Transpose

Warning This section makes use of the notion of "symmetric division" as defined in [BdM97, Oli18] but *not* as defined in [Fv90]. "Symmetric division" can be defined in two non-equivalent ways that we call *symmetric left-division* and *symmetric right-division*. Given relations R of type $A{\sim}B$ and S of type $A{\sim}C$, the symmetric *right-division* is a relation of type $B{\sim}C$ defined in terms of *right* factors as

$$R\backslash S \ \cap \ (S\backslash R)^{\cup} \quad .$$

Dually, given relations R of type $B{\sim}A$ and S of type $C{\sim}A$, the symmetric left-division is a relation of type $B{\sim}C$ defined in terms of left factors as

$$R/S \ \cap \ (S/R)^{\cup} \quad .$$

Clearly, just from their types, neither the "symmetric" left-division nor the "symmetric" right-division is a symmetric relation. Possibly the justification for the use of the word "symmetric" is that, for homogeneous relation R, $R \cap R^{\cup}$ is a symmetric relation (indeed the largest symmetric relation included in R). Both [BdM97, Oli18] and [Fv90] use the notation $\frac{R}{S}$ (in the case of [BdM97, Oli18] to denote symmetric right-division and in the case of [Fv90] to denote symmetric left-division). The motivation for this is that the notation suggests a number of cancellation rules similar to the ones used in ordinary arithmetic. Great care must be taken, however, because —unlike in ordinary arithmetic— the cancellation rules are one-sided. For example, for symmetric right-division, we have the rule

$$R \ = \ R \circ \frac{R}{R}$$

but this is *not* valid if $\frac{R}{S}$ is defined to be symmetric left-division. Even worse, the expression $\frac{R}{R}\circ R$ does not even make sense (if $\frac{R}{S}$ is defined to be symmetric right-division) if R is a truly heterogeneous relation —with unequal source and target— purely on type grounds! For this reason, the notation $R\backslash\!\backslash S$ will be used here to denote the symmetric right-division. The reader should take great care when comparing formulae with those in [Fv90]. **End of Warning**

Given a relation R of type $A{\sim}B$, the *(left) power transpose* [Fv90, BdM97] of R is a total function, denoted in this paper[2] by ΓR, of type $2^A{\leftarrow}B$. A pointwise definition of the (left) power transpose (using traditional set notation) is

$$\Gamma R.b \;=\; \{a \mid a\ R\ b\} \;\; .$$

As remarked in section 4, there are three different but isomorphic mechanisms for representing sets in relation algebra: as coreflexives, (left or right) conditionals and squares. Using coreflexives, the power transpose ΓR of R is represented by the function

$$\langle b :: (R{\circ}b)_{<}\rangle \;\; .$$

It has type $\mathsf{Cor}.A \leftarrow B$ where $\mathsf{Cor}.A$ denotes the type of coreflexives of type $A{\sim}A$.

Rather than use coreflexives to define power transpose, Freyd and Ščedrov [Fv90] postulate a number of axioms that define ΓR in terms of set membership. Their approach is followed by Bird and De Moor [BdM97]. For our purposes, only two properties are needed. The first is that ΓR is a total function. That is, for all R, S and T of appropriate type,

$$(64) \qquad \Gamma R\circ S \subseteq T \quad\equiv\quad S \subseteq (\Gamma R)^{\cup}\circ T \;\; .$$

This is the Galois connection (61) with $f := \Gamma R$ and specialised to the case that $f_{>} = I$ (i.e. f is total); in line with our common policy when using well-known Galois connections, we refer to the rule as a "shunting rule". The second property of ΓR that we use is

$$(65) \qquad (\Gamma R)^{\cup}\circ\Gamma S \;=\; R\backslash S \,\cap\, (S\backslash R)^{\cup} \;\; .$$

From a calculational viewpoint, the two rules together enable reasoning about power transpose on the smaller and larger side of a set inclusion, respectively.

The property (65) can be derived from the definition of ΓR in our axiom system. Here is the proof.

Lemma 66 For all relations R and S,

$$(\Gamma R)^{\cup}\circ\Gamma S \;=\; R\backslash S \cap (S\backslash R)^{\cup} \;\; .$$

[2] Freyd and Ščedrov [Fv90] use the symbol "Λ" rather than "Γ". In just the same way that we prefer the symbols "$\backslash\!\backslash$" and "$/\!/$" for asymmetric, but dual, operators, we prefer to use an asymmetric symbol for left power transpose, thus opening the possibility of using its mirror image for right power transpose.

Proof We use indirect equality. For all relations X, R and S, we have

$$
\begin{array}{ll}
 & X \subseteq (\Gamma R)^{\cup} \circ \Gamma S \\
= & \quad \{ \quad \text{saturation property: (19)} \quad \} \\
 & \langle \forall a,b : a\circ\top\top\circ b \subseteq X : a\circ\top\top\circ b \subseteq (\Gamma R)^{\cup} \circ \Gamma S \rangle \\
= & \quad \{ \quad \text{(57) with } f,g := \Gamma R,\Gamma S \text{ and definition of } \Gamma \quad \} \\
 & \langle \forall a,b : a\circ\top\top\circ b \subseteq X : (R\circ a)_{<} = (S\circ b)_{<} \rangle \\
= & \quad \{ \quad \text{corollary 49} \quad \} \\
 & \langle \forall a,b : a\circ\top\top\circ b \subseteq X : a\circ\top\top\circ b \subseteq R\backslash S \cap (S\backslash R)^{\cup} \rangle \\
= & \quad \{ \quad \text{saturation property: (19)} \quad \} \\
 & X \subseteq R\backslash S \cap (S\backslash R)^{\cup} \quad .
\end{array}
$$

Summarising, for all X, R and S,

$$X \subseteq (\Gamma R)^{\cup} \circ \Gamma S \ \equiv \ X \subseteq R\backslash S \cap (S\backslash R)^{\cup} \quad .$$

That is, by indirect equality,

$$(\Gamma R)^{\cup} \circ \Gamma S \ = \ R\backslash S \cap (S\backslash R)^{\cup} \quad .$$

□

Abbreviating the right side of lemma 66 to $R\backslash\!\backslash S$, the lemma becomes, for all R and S,

(67) $\quad (\Gamma R)^{\cup} \circ \Gamma S \ = \ R\backslash\!\backslash S \quad .$

We use both forms of the lemma below.

3.6 Per Domains

The relation $R\backslash\!\backslash R$ is an equivalence relation. Voermans [Voe99] calls it the "greatest domain" of R. Voermans emphasises the importance of the relation $R_{>} \circ R\backslash\!\backslash R$, which is a partial equivalence relation that better reflects the right (per-)domain of R. (In acccordance with his thesis, "domains" are pers rather than coreflexives.)

Definition 68 (Partial Equivalence Relations (per)) A relation is a *partial equivalence relation* iff it is symmetric and transitive. That is, R is a partial equivalence relation iff

$$R = R^{\cup} \ \wedge \ R\circ R \subseteq R \quad .$$

Henceforth we abbreviate partial equivalence relation to *per*.

□

Pers are studied in detail in section 5. In this section the focus is on the particular per $R> \circ R\setminus\!\!\setminus R$.

As remarked elsewhere [Oli18], the *symmetric left division* inherits a number of (left) cancellation properties from the properties of factorisation in terms of which it is defined. For our purposes, the only cancellation property we use is the following (inherited from the property $R \circ R\backslash R = R$).

Lemma 69

$$R \circ R\setminus\!\!\setminus R = R \ .$$

Proof By mutual inclusion:

$$
\begin{array}{cl}
 & R \circ R\setminus\!\!\setminus R \\
= & \{ \quad \text{definition} \quad \} \\
 & R \circ (R\backslash R \cap (R\backslash R)^{\cup}) \\
\subseteq & \{ \quad \text{monotonicity} \quad \} \\
 & R \circ R\backslash R \\
= & \{ \quad \text{cancellation } [\ R \circ R\backslash S \subseteq S\ ] \text{ (with } R,S := R,R \text{) and } [\ I \subseteq R\backslash R\ ] \quad \} \\
 & R \\
\subseteq & \{ \quad [\ I \subseteq S\setminus\!\!\setminus S\ ] \text{ with } S := R \quad \} \\
 & R \circ R\setminus\!\!\setminus R \ .
\end{array}
$$

□

By way of preparation, it is useful to record the left and right domains of the relation $R\setminus\!\!\setminus R \circ R>$:

Lemma 70

$$(R\setminus\!\!\setminus R \circ R>)> \ = \ R> \ = \ (R> \circ R\setminus\!\!\setminus R)< \ ,$$

$$(R\setminus\!\!\setminus R \circ R>)< \ = \ R> \ = \ (R> \circ R\setminus\!\!\setminus R)> \ ,$$

$$R\setminus\!\!\setminus R \circ R> \ = \ R> \circ R\setminus\!\!\setminus R \circ R> \ = \ R> \circ R\setminus\!\!\setminus R \ .$$

Proof The first two equations follow from the fact that

$$(R\setminus\!\!\setminus R)< \ = \ I \ = \ (R\setminus\!\!\setminus R)>$$

(because $I \subseteq R\backslash R$ and $R\setminus\!\!\setminus R$ is the symmetric closure of $R\backslash R$). For example:

$$
\begin{array}{ll}
 & (R_{>} \circ R \backslash\backslash R)_{<} \\
= & \{ \quad \text{domains} \quad \} \\
 & (R_{>} \circ (R \backslash\backslash R)_{<})_{<} \\
= & \{ \quad (R \backslash\backslash R)_{<} = I \quad \} \\
 & (R_{>})_{<} \\
= & \{ \quad R_{>} \text{ is a coreflexive, domains} \quad \} \\
 & R_{>} \quad .
\end{array}
$$

The second two equations follow from lemma 69.

$$
\begin{array}{ll}
 & (R \backslash\backslash R \circ R_{>})_{<} \\
= & \{ \quad \text{domains} \quad \} \\
 & (R \circ (R \backslash\backslash R)^{\cup})_{>} \\
= & \{ \quad R \backslash\backslash R \text{ is symmetric} \quad \} \\
 & (R \circ R \backslash\backslash R)_{>} \\
= & \{ \quad \text{lemma 69} \quad \} \\
 & R_{>} \quad ,
\end{array}
$$

and

$$
\begin{array}{ll}
 & (R_{>} \circ R \backslash\backslash R)_{>} \\
= & \{ \quad \text{domains} \quad \} \\
 & (R \circ R \backslash\backslash R)_{>} \\
= & \{ \quad \text{lemma 69} \quad \} \\
 & R_{>} \quad .
\end{array}
$$

Combining the domain equations, we have

$$
\begin{array}{ll}
 & R \backslash\backslash R \circ R_{>} \\
= & \{ \quad (R_{>} \circ R \backslash\backslash R)_{<} = R_{>} \text{, domains} \quad \} \\
 & R_{>} \circ R \backslash\backslash R \circ R_{>} \\
= & \{ \quad (R_{>} \circ R \backslash\backslash R)_{>} = R_{>} \text{, domains} \quad \} \\
 & R_{>} \circ R \backslash\backslash R \quad .
\end{array}
$$

□

Following Voermans [Voe99], we define:

$$(71)\qquad R{\succ} \;=\; R_{>}\circ R\backslash\!\backslash R\ \ .$$

The following theorem extends [Rig48, Corollaire, p.134] from equivalence relations to pers. (As pointed out in [Oli18], Riguet calls $R\backslash\!\backslash R$ the "noyau" of R.)

Theorem 72 A relation R is a per iff $R = R{\succ}$.

Proof The proof is by mutual implication.

For the "if" part we show that $R{\succ}$ is a per (for all relations R). Separate calculations establish that $R{\succ}$ is symmetric and transitive.

$$
\begin{array}{ll}
 & R{\succ}\ \text{ is symmetric} \\
= & \{\qquad \text{definition} \qquad\} \\
 & R\backslash\!\backslash R\circ R_{>} = (R\backslash\!\backslash R\circ R_{>})^{\cup} \\
= & \{\qquad \text{converse and } R\backslash\!\backslash R = (R\backslash\!\backslash R)^{\cup} \qquad\} \\
 & R\backslash\!\backslash R\circ R_{>} = R_{>}\circ R\backslash\!\backslash R \\
= & \{\qquad \text{lemma 70} \qquad\} \\
 & \mathsf{true}\ \ .
\end{array}
$$

Next,

$$
\begin{array}{ll}
 & R{\succ}\ \text{ is transitive} \\
= & \{\qquad \text{definition} \qquad\} \\
 & R\backslash\!\backslash R\circ R_{>}\circ R\backslash\!\backslash R\circ R_{>} \subseteq R\backslash\!\backslash R\circ R_{>} \\
\Leftarrow & \{\qquad R_{>}\subseteq I \text{ and monotonicity} \qquad\} \\
 & R\backslash\!\backslash R \circ R\backslash\!\backslash R \subseteq R\backslash\!\backslash R \\
= & \{\qquad \text{definition and factors} \qquad\} \\
 & R \circ R\backslash\!\backslash R \circ R\backslash\!\backslash R \subseteq R \quad\wedge\quad R\circ(R\backslash\!\backslash R \circ R\backslash\!\backslash R)^{\cup} \subseteq R \\
= & \{\qquad \text{lemma 69, converse and } R\backslash\!\backslash R = (R\backslash\!\backslash R)^{\cup} \qquad\} \\
 & \mathsf{true}\ \ .
\end{array}
$$

Now, for the only-if part, suppose R is a per. Then

$R \subseteq R\backslash\backslash R \circ R_{>}$.

$=$ { domains: $R \circ R_{>} = R$ }

$R \circ R_{>} \subseteq R\backslash\backslash R \circ R_{>}$

$\Leftarrow$ { monotonicity }

$R \subseteq R\backslash\backslash R$

$=$ { definition }

$R \subseteq R\backslash R \cap (R\backslash R)^{\cup}$

$=$ { R is symmetric, i.e. $R = R^{\cup}$; intersection }

$R \subseteq R\backslash R \;\wedge\; R^{\cup} \subseteq (R\backslash R)^{\cup}$

$=$ { converse }

$R \subseteq R\backslash R$

$=$ { factors }

$R \circ R \subseteq R$

$=$ { assumption: R is transitive }

true .

Also,

$R\backslash\backslash R \circ R_{>} \subseteq R$

$=$ { lemma 70 }

$R_{>} \circ R\backslash\backslash R \subseteq R$

$=$ { lemma 69 }

$R_{>} \circ R\backslash\backslash R \subseteq R \circ R\backslash\backslash R$

$\Leftarrow$ { monotonicity }

$R_{>} \subseteq R$

$=$ { lemma 101 }

true .

The equality $R = R{\succ}$ follows by mutual inclusion and (71).

□

Lemma 73 For all points b and b' such that $b \subseteq R_{>}$ and $b' \subseteq R_{>}$,

$$(R \circ b)_{<} = (R \circ b')_{<} \;\equiv\; (b \circ R\backslash\backslash R)_{>} = (b' \circ R\backslash\backslash R)_{>}$$

Proof By mutual implication:

$$
\begin{array}{ll}
 & (R\circ b)_< \; = \; (R\circ b')_< \\
= & \{ \quad \text{corollary 49 with } R,S := R,R \text{ and definition of } R\backslash\!\backslash R \quad \} \\
 & b\circ\top\!\top\circ b' \subseteq R\backslash\!\backslash R \\
= & \{ \quad R\backslash\!\backslash R = (R\backslash\!\backslash R)^{\cup} \quad \} \\
 & b\circ\top\!\top\circ b' \subseteq R\backslash\!\backslash R \;\wedge\; b'\circ\top\!\top\circ b \subseteq R\backslash\!\backslash R \\
\Rightarrow & \{ \quad \text{domains, monotonicity and } R\backslash\!\backslash R \text{ is transitive} \quad \} \\
 & (b'\circ R\backslash\!\backslash R)_> \subseteq (b\circ R\backslash\!\backslash R)_> \;\wedge\; (b\circ R\backslash\!\backslash R)_> \subseteq (b'\circ R\backslash\!\backslash R)_> \\
= & \{ \quad \text{anti-symmetry} \quad \} \\
 & (b\circ R\backslash\!\backslash R)_> \; = \; (b'\circ R\backslash\!\backslash R)_> \\
\Rightarrow & \{ \quad R\backslash\!\backslash R \text{ is an equivalence relation} \\
 & \quad\quad \text{so } b \subseteq (b\circ R\backslash\!\backslash R)_> \text{ and } b' \subseteq (b'\circ R\backslash\!\backslash R)_> \quad \} \\
 & b' \subseteq (b\circ R\backslash\!\backslash R)_> \;\wedge\; b \subseteq (b'\circ R\backslash\!\backslash R)_> \\
= & \{ \quad \text{lemma 46 with } a,b,R := b, b', R\backslash\!\backslash R \quad \} \\
 & b\circ\top\!\top\circ b' \subseteq R\backslash\!\backslash R \;\wedge\; b'\circ\top\!\top\circ b \subseteq R\backslash\!\backslash R \\
= & \{ \quad \text{as in first step} \quad \} \\
 & (R\circ b)_< \; = \; (R\circ b')_< \;\; .
\end{array}
$$

□

3.7 Orderings

There are various well-known notions of ordering: preorder, partial and linear (aka total) ordering. For our purposes all of these are too strict. So, in this section, we introduce the notion of a “provisional ordering”. The adjective “provisional” has been chosen because the notion provides just what we need.

Definition 74 Suppose p and T are both homogeneous relations of type $A{\sim}A$, for some A. Then T is said to be a *preorder over the domain* p if

$$
p \subseteq I_A \cap T \;\;\wedge\;\; T = p\circ T\circ p \;\;\wedge\;\; T\circ T \subseteq T \;\; .
$$

□

A trivial property that is nevertheless used frequently:

Lemma 75 T is a preorder over the domain p equivales $T^{\cup}$ is a preorder over the domain p.

Proof Immediate from the definition and properties of converse.
□

Lemma 76 Suppose T is a preorder over the domain p. Then

$$T \;=\; p\circ T\backslash T \;=\; T/T\circ p \;=\; p\circ T\backslash T/T\circ p \;.$$

Proof The proof of the leftmost equality is by mutual inclusion. First

$T \subseteq p\circ T\backslash T$

$=$ { assumptions: $T=p\circ T$ and p is coreflexive }

$T \subseteq T\backslash T$

$=$ { factors }

$T\circ T\subseteq T$

$=$ { assumption: T is transitive }

true .

For the opposite inclusion, we first prove that

(77) $p\circ p\backslash T \;=\; T$.

The proof is by indirect equality. Suppose X is a relation such that $X=p\circ X$. Then

$X \subseteq p\circ p\backslash T$

$=$ { assumptions: $X=p\circ X$ and p is coreflexive }

$X \subseteq p\backslash T$

$=$ { factors }

$p\circ X\subseteq T$

$=$ { assumption: $X=p\circ X$ }

$X\subseteq T$.

Property (77) follows by instantiating X to T (using the assumption that $T=p\circ T\circ p$) and to $p\circ p\backslash T$ and then using the anti-symmetry of the subset relation. Now we can proceed further:

$$p \circ T\backslash T \subseteq T$$
$= \quad \{ \quad (77) \quad \}$
$$p \circ T\backslash T \subseteq p \circ p\backslash T$$
$\Leftarrow \quad \{ \quad$ monotonicity $\quad \}$
$$T\backslash T \subseteq p\backslash T$$
$\Leftarrow \quad \{ \quad$ factors $\quad \}$
$$p \subseteq T$$
$= \quad \{ \quad$ assumption: $p \subseteq I_A \cap T \quad \}$

true .

That $T = T/T \circ p$ follows from lemma 75 and the properties of converse. Finally,

$$T$$
$= \quad \{ \quad$ assumption: $T = p \circ T \circ p \quad \}$
$$T \circ p$$
$= \quad \{ \quad T = p \circ T\backslash T$ (proved above) $\quad \}$
$$p \circ T\backslash T \circ p$$
$= \quad \{ \quad T = T/T \circ p$ (proved above) $\quad \}$
$$p \circ T\backslash (T/T \circ p) \circ p$$
$= \quad \{$

$$T\backslash (T/T \circ p) \circ p$$
$\subseteq \quad \{ \quad$ assumption: $p \subseteq I$, monotonicity $\quad \}$
$$T\backslash T/T \circ p \quad .$$

Also,

$$T\backslash T/T \circ p \subseteq T\backslash (T/T \circ p) \circ p$$
$= \quad \{ \quad$ assumption: $p \subseteq I \quad \}$
$$T\backslash T/T \circ p \subseteq T\backslash (T/T \circ p)$$
$= \quad \{ \quad$ factors $\quad \}$
$$T \circ T\backslash T/T \circ p \subseteq T/T \circ p$$
$= \quad \{ \quad$ cancellation $\quad \}$

true .

That is, by mutual inclusion,

$$T\backslash(T/T\circ p)\circ p = T\backslash T/T\circ p \quad \}$$

$$p\circ T\backslash T/T\circ p \ .$$

□

We assume the reader is familiar with the notions of an ordering and a linear (or total) ordering. We now define extend these notions to provisional orderings. (The at-most relation on the integers is both anti-symmetric and linear. The at-most relation restricted to some arbitrary subset of the integers is an example of a linear provisional ordering according to the definition below.)

Definition 78 Suppose T is a homogeneous relation of type $A{\sim}A$, for some A. Then T is said to be *provisionally anti-symmetric* if

$$T\cap T^{\cup} \subseteq I_A \ .$$

Also, T is said to be a *provisional ordering* if T is provisionally anti-symmetric and T is a preorder over the domain $T\cap T^{\cup}$. Finally, T is said to be a *linear provisional ordering* if T is a provisional ordering and

$$T\cup T^{\cup} \ = \ (T\cap T^{\cup})\circ\top\!\top\circ(T\cap T^{\cup}) \ .$$

□

Definition 78 weakens the equality in the standard notion of anti-symmetry to an inclusion. The standard definition of a partial ordering —an anti-symmetric preorder— is weakened accordingly (as mentioned earlier, in order to "provide" for our needs).

Lemma 79 Suppose T is a provisional ordering. Then

$$T_< \ = \ T\cap T^{\cup} \ = \ T_> \ .$$

Proof For the first equality, we have

$$T\cap T^{\cup} \subseteq T_<$$

$= \quad \{ \quad$ definition of $T_<$ and assumption: $T\cap T^{\cup} \subseteq I_A \quad \}$

$$T\cap T^{\cup} \subseteq T\circ T^{\cup}$$

$\Leftarrow \quad \{ \quad$ transitivity $\quad \}$

$$T\cap T^{\cup} \ \subseteq \ (T\circ T\cap I_A)\circ T^{\cup} \ \subseteq \ T\circ T^{\cup}$$

$= \quad \{ \quad$ modular law; $T\circ T\subseteq T \quad \}$

$$\mathsf{true} \ .$$

Also,

$$T_{<} \subseteq T \cap T^{\cup}$$

$=$ { assumption: $T \cap T^{\cup} \subseteq I_A$; domains }

$$(T \cap T^{\cup}) \circ T = T$$

$=$ { assumption: T is a provisional ordering over $T \cap T^{\cup}$

i.e. $(T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) = T \wedge T \cap T^{\cup} \subseteq I_A$ }

true .

The second equality is obtained by instantiating T to $T^{\cup}$.

□

4 Squares and Rectangles

Squares are by definition homogeneous relations. We now introduce the notion of a "rectangle"; rectangles are typically heterogeneous. Squares are rectangles; properties of squares are typically obtained by specialising properties of rectangles. (For example, lemma 83 shows that the intersection of two rectangles is a rectangle by giving an explicit construction; the same construction applies to squares from which it is easily shown that the intersection of two squares is a square.)

Definition 80 (Rectangle) A relation R is a *rectangle* iff $R = R \circ \top\top \circ R$.

□

An example of a rectangle is the "pair" $a \circ \top\top \circ b$ where a and b are points. More generally, we have:

Lemma 81 For all relations R and S, $R \circ \top\top \circ S$ is a rectangle. In particular, if R has type $A \sim B$, S has type $B \sim C$, and b is a point of type B , the relation $R \circ b \circ S$ is a rectangle.

Proof Because the proof is based on the cone rule, a case analysis is necessary. In the case that either R or S is the empty relation, the lemma clearly holds (because $R \circ \top\top \circ S$ is the empty relation, and the empty relation is a rectangle). Suppose now that both R and S are non-empty. Then

$$R \circ \top\top \circ S \circ \top\top \circ R \circ \top\top \circ S$$

$=$ { cone rule: (5) (applied twice), assumption: $R \neq \perp\perp$ and $S \neq \perp\perp$ }

$$R \circ \top\top \circ S$$.

That $R\circ b\circ S$ is a rectangle is an instance since, by (15), $R\circ b\circ S = R\circ b\circ \top\!\top\circ b\circ S$.
□

The type information in the statement of lemma 81 provides a useful guide when introducing definitions of particular rectangles.

4.1 Inclusion and Intersection

Using colloquial terminology, the left and right domain of a rectangle are the "sides" of the rectangle. In general, a rectangle is defined by its two sides. More precisely:

Lemma 82 Suppose R and S are rectangles of the same type. Then

$$R\subseteq S \quad\equiv\quad R_{<}\subseteq S_{<} \;\wedge\; R_{>}\subseteq S_{>} \quad .$$

It follows that

$$R=S \quad\equiv\quad R_{<}=S_{<} \;\wedge\; R_{>}=S_{>} \quad .$$

Proof By mutual implication:

$$\begin{array}{ll}
 & R\subseteq S \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R_{<}\subseteq S_{<} \;\wedge\; R_{>}\subseteq S_{>} \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R_{<}\circ\top\!\top\circ R_{>} \;\subseteq\; S_{<}\circ\top\!\top\circ S_{>} \\
= & \{ \quad \text{domains} \quad \} \\
 & R\circ\top\!\top\circ R\subseteq S\circ\top\!\top\circ S \\
= & \{ \quad \text{assumption: } R \text{ and } S \text{ are rectangles, definition 80} \quad \} \\
 & R\subseteq S \quad .
\end{array}$$

The second property follows straightforwardly from the anti-symmetry of the subset relation.
□

Lemma 83 The intersection of two rectangles is a rectangle. Specifically, for all rectangles R and S,

$$R\cap S \;=\; (R_{<}\cap S_{<})\circ\top\!\top\circ(R_{>}\cap S_{>}) \quad .$$

Proof We have, for all R, S, T and U,

$$
\begin{array}{ll}
& R\circ\top\!\!\top\circ S \;\cap\; T\circ\top\!\!\top\circ U \\
= & \{ \quad \text{property of conditionals} \quad \} \\
& R\circ\top\!\!\top \;\cap\; \top\!\!\top\circ S \;\cap\; T\circ\top\!\!\top \;\cap\; \top\!\!\top\circ U \\
= & \{ \quad \text{property of conditionals} \quad \} \\
& (R\cap T)\circ\top\!\!\top \;\cap\; \top\!\!\top\circ(S\cap U) \\
= & \{ \quad \text{property of conditionals} \quad \} \\
& (R\cap T)\circ\top\!\!\top\circ(S\cap U) \quad .
\end{array}
$$

(The properties of conditionals used above are not shown in this paper but easily proven. Hint: use the modularity rule.) Also, for all R and S, $R\circ\top\!\!\top\circ S = R_{<}\circ\top\!\!\top\circ S_{>}$. So

$$
\begin{array}{ll}
& R\cap S \\
= & \{ \quad \text{assumption: } R \text{ and } S \text{ are rectangles} \quad \} \\
& R\circ\top\!\!\top\circ R \cap S\circ\top\!\!\top\circ S \\
= & \{ \quad [\; R\circ\top\!\!\top\circ S = R_{<}\circ\top\!\!\top\circ S_{>} \;] \text{ with } R,S := R,R \text{ and } R,S := S,S \quad \} \\
& R_{<}\circ\top\!\!\top\circ R_{>} \;\cap\; S_{<}\circ\top\!\!\top\circ S_{>} \\
= & \{ \quad \text{above with } R,S,T,U := R_{<},R_{>},S_{<},S_{>} \quad \} \\
& (R_{<}\cap S_{<})\circ\top\!\!\top\circ(R_{>}\cap S_{>}) \quad .
\end{array}
$$

□

Lemma 84 If U is a rectangle then, for all points b (of appropriate type)

$$
(U\circ b)_{<} = U_{<} \quad\vee\quad (U\circ b)_{<} = \bot\!\!\bot \quad .
$$

Proof

$$
\begin{array}{ll}
& (U\circ b)_{<} \\
= & \{ \quad \text{assumption: } U \text{ is a rectangle} \quad \} \\
& (U\circ\top\!\!\top\circ U\circ b)_{<} \\
= & \{ \quad \text{domains} \quad \} \\
& (U\circ\top\!\!\top\circ U_{>}\circ b)_{<} \\
= & \{ \quad \text{assumption: } b \text{ is a point. So } U_{>}\circ b = b \vee U_{>}\circ b = \bot\!\!\bot \quad \} \\
& \textsf{if } U_{>}\circ b = b \to (U\circ\top\!\!\top\circ b)_{<} \;\square\; U_{>}\circ b = \bot\!\!\bot \to \bot\!\!\bot \textsf{ fi} \\
= & \{ \quad \text{assumption: } b \text{ is a point. So } (\top\!\!\top\circ b)_{<} = I \quad \} \\
& \textsf{if } U_{>}\circ b = b \to U_{<} \;\square\; U_{>}\circ b = \bot\!\!\bot \to \bot\!\!\bot \textsf{ fi} \quad .
\end{array}
$$

□

4.2 Completely Disjoint Rectangles

As is well-known, an equivalence relation *partitions* its domain into a set of disjoint *classes*. Also well-known is that the existence of such a partitioning is precisely formulated by the function that maps an element of the domain to its *equivalence class*: two elements are equivalent if and only if their equivalence classes are equal. When represented by relations, equivalence classes are squares. The theory of difunctional relations generalises this partitioning property to "completely disjoint" rectangles. This section lays the foundations for this theory. Specifically, theorem 100 formulates a correspondence between pairs of functional relations and sets of completely disjoint rectangles. (See section 5.5 for further discussion of functional relations.)

Definition 85 (Indexed Set) Suppose A, B and K are types and $\mathcal{R}$ is a function with source K and target rectangles of type $A{\sim}B$. Then $\mathcal{R}$ is said to be an *indexed set of rectangles*; the type K is said to be the *index set* of $\mathcal{R}$ and rectangles $\mathcal{R}.k$, where k ranges over K, are said to be the *elements* of $\mathcal{R}$.
□

Two relations R and S are *disjoint* if $R{\cap}S = \perp\!\!\perp$. One can show that, for all rectangles R and S,

$$R{\cap}S = \perp\!\!\perp \quad\equiv\quad R{<}\cap S{<} = \perp\!\!\perp \;\vee\; R{>}\cap S{>} = \perp\!\!\perp \quad .$$

(This is a consequence of lemma 83.) The definition of "completely" disjoint strengthens the disjunction to a conjunction. Note that we don't use continued equality because the symbol "$\perp\!\!\perp$" is overloaded.

Definition 86 (Completely Disjoint) Two rectangles R and S are said to be *completely disjoint* iff

$$R{<}\cap S{<} = \perp\!\!\perp \quad\wedge\quad R{>}\cap S{>} = \perp\!\!\perp \quad .$$

Suppose $\mathcal{R}$ is an indexed set of non-empty rectangles. Then $\mathcal{R}$ is said to be a *completely disjoint set of rectangles* iff, for all j and k in the index set of $\mathcal{R}$,

$$\mathcal{R}.j = \mathcal{R}.k \quad\not\equiv\quad (\mathcal{R}.j){<}\cap(\mathcal{R}.k){<} = \perp\!\!\perp \;\wedge\; (\mathcal{R}.j){>}\cap(\mathcal{R}.k){>} = \perp\!\!\perp \quad .$$

□

It is straightforward to show that the elements of a completely disjoint set of rectangles are non-empty.

Because of the central rôle that composition and converse play in relation algebra, it is often better to exploit the cone rule to reformulate the above definition. Specifically:

Lemma 87 For all rectangles R and S,

(88) $R< \cap S< \;=\; \perp\!\!\perp \quad\equiv\quad \top\!\!\top \circ R^{\cup} \circ S \circ \top\!\!\top \;=\; \perp\!\!\perp$

and

(89) $R> \cap S> \;=\; \perp\!\!\perp \quad\equiv\quad \top\!\!\top \circ R \circ S^{\cup} \circ \top\!\!\top \;=\; \perp\!\!\perp$.

Also, suppose $\mathcal{R}$ is a set of *non-empty* rectangles indexed by K. Then $\mathcal{R}$ is a completely disjoint set iff, for all j and k in K,

(90) $\mathcal{R}.j = \mathcal{R}.k \quad\Leftarrow\quad \top\!\!\top \circ (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \circ \top\!\!\top = \top\!\!\top \;\vee\; \top\!\!\top \circ \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} \circ \top\!\!\top = \top\!\!\top$.

Proof Property (88) is proved as follows.

$$
\begin{array}{ll}
 & R< \cap S< \;=\; \perp\!\!\perp \\
= & \{ \quad \text{for coreflexives p and q, } p \cap q = p \circ q \quad \} \\
 & R< \circ S< \;=\; \perp\!\!\perp \\
= & \{ \quad \text{cone rule: (5)} \quad \} \\
 & \top\!\!\top \circ R< \circ S< \circ \top\!\!\top \;=\; \perp\!\!\perp \\
= & \{ \quad \text{domains (specifically, } [\; S< \circ \top\!\!\top = S \circ \top\!\!\top \;] \text{ and its dual)} \quad \} \\
 & \top\!\!\top \circ R^{\cup} \circ S \circ \top\!\!\top \;=\; \perp\!\!\perp \quad .
\end{array}
$$

Its dual (89) is proved similarly. Now suppose $\mathcal{R}$ is a set of non-empty rectangles indexed by K. Then, for all j and k in K,

$$
\begin{array}{ll}
 & \mathcal{R}.j = \mathcal{R}.k \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & \top\!\!\top \circ (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \circ \top\!\!\top \quad = \quad \top\!\!\top \circ (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.j \circ \top\!\!\top \\
= & \{ \quad \text{for all R such that } R \neq \perp\!\!\perp , \\
 & \qquad \top\!\!\top \circ R^{\cup} \circ R \circ \top\!\!\top \\
 & \quad = \quad \{ \quad \text{domains} \quad \} \\
 & \qquad \top\!\!\top \circ R< \circ R< \circ \top\!\!\top \\
 & \quad = \quad \{ \quad R< \text{ is a coreflexive, so } R< \circ R< = R< \text{, domains} \quad \} \\
 & \qquad \top\!\!\top \circ R \circ \top\!\!\top \\
 & \quad = \quad \{ \quad \text{assumption: } R \neq \perp\!\!\perp \text{, cone rule: (5)} \quad \} \\
 & \qquad \top\!\!\top
\end{array}
$$

$$\begin{array}{ll}
& \{ \quad \text{assumption: } \mathcal{R}.j \neq \perp\!\!\perp \quad \} \\
& \top\!\!\top \circ (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \circ \top\!\!\top \;=\; \top\!\!\top \\
= & \{ \quad \text{cone rule: (5)} \quad \} \\
& \top\!\!\top \circ (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \circ \top\!\!\top \;\neq\; \perp\!\!\perp \;.
\end{array}$$

That is,

$$\mathcal{R}.j = \mathcal{R}.k \;\Rightarrow\; \top\!\!\top \circ (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \circ \top\!\!\top \neq \perp\!\!\perp \;.$$

Dually

$$\mathcal{R}.j = \mathcal{R}.k \;\Rightarrow\; \top\!\!\top \circ \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} \circ \top\!\!\top \neq \perp\!\!\perp \;.$$

It follows that

$$\mathcal{R}.j = \mathcal{R}.k \;\Rightarrow\; \top\!\!\top \circ (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \circ \top\!\!\top \neq \perp\!\!\perp \;\vee\; \top\!\!\top \circ \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} \circ \top\!\!\top \neq \perp\!\!\perp \;.$$

(Note the disjunction. This is weaker than a conjunction but is all we need.) So, for all j and k in K,

$$\begin{array}{ll}
& \mathcal{R}.j = \mathcal{R}.k \;\not\equiv\; (\mathcal{R}.j)< \cap (\mathcal{R}.k)< \,=\, \perp\!\!\perp \;\wedge\; (\mathcal{R}.j)> \cap (\mathcal{R}.k)> \,=\, \perp\!\!\perp \\
= & \{ \quad \text{(88) and (89) with } R,S := \mathcal{R}.j\,,\mathcal{R}.k \text{ and boolean algebra} \quad \} \\
& \mathcal{R}.j = \mathcal{R}.k \;\equiv\; \top\!\!\top \circ (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \circ \top\!\!\top \neq \perp\!\!\perp \;\vee\; \top\!\!\top \circ \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} \circ \top\!\!\top \neq \perp\!\!\perp \\
= & \{ \quad \text{mutual implication} \quad \} \\
& \quad (\mathcal{R}.j = \mathcal{R}.k \;\Rightarrow\; \top\!\!\top \circ (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \circ \top\!\!\top \neq \perp\!\!\perp \;\vee\; \top\!\!\top \circ \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} \circ \top\!\!\top \neq \perp\!\!\perp) \\
& \wedge\; (\mathcal{R}.j = \mathcal{R}.k \;\Leftarrow\; \top\!\!\top \circ (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \circ \top\!\!\top \neq \perp\!\!\perp \;\vee\; \top\!\!\top \circ \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} \circ \top\!\!\top \neq \perp\!\!\perp) \\
= & \{ \quad \text{assumption: } \mathcal{R}.j \neq \perp\!\!\perp\text{, above} \quad \} \\
& \mathcal{R}.j = \mathcal{R}.k \;\Leftarrow\; \top\!\!\top \circ (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k \circ \top\!\!\top \neq \perp\!\!\perp \;\vee\; \top\!\!\top \circ \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} \circ \top\!\!\top \neq \perp\!\!\perp \;.
\end{array}$$

□

Lemma 91 Suppose f and g are relations with common target C such that

$$f \circ f^{\cup} \;=\; f< \;=\; g \circ g^{\cup} \;=\; g< \;.$$

Then the relation $f^{\cup} \circ g$ is the supremum of a completely disjoint set of rectangles. Specifically, with dummy c ranging over points of type C,

$$f^{\cup} \circ g \;=\; \langle \cup c : c \subseteq g< : f^{\cup} \circ c \circ g \rangle \;.$$

Moreover, for all points c such that $c \subseteq g<$, $f^{\cup} \circ c \circ g$ is a non-empty rectangle and, for all points c and c',

$$c \subseteq g< \;\wedge\; c' \subseteq g< \;\wedge\; c \neq c'$$
$$\Rightarrow\;(f^{\cup}\circ c\circ g)< \cap (f^{\cup}\circ c'\circ g)< \;=\; \perp\!\!\perp \;\wedge\; (f^{\cup}\circ c\circ g)> \cap (f^{\cup}\circ c'\circ g)> \;=\; \perp\!\!\perp \;.$$

That is, elements of the set of rectangles $f^{\cup}\circ c\circ g$ indexed by points c in $g<$ are completely disjoint.

Proof As remarked in lemma 81, the relation $R\circ c\circ S$ is a rectangle, for all points c and all relations R and S; so this is also true of $f^{\cup}\circ c\circ g$. If $c \subseteq g<$, the rectangle is non-empty since

$$\top\!\!\top \circ f^{\cup}\circ c\circ g\circ \top\!\!\top$$
$$= \quad \{ \quad \text{domains} \quad \}$$
$$\top\!\!\top \circ f<\circ c\circ g<\circ \top\!\!\top$$
$$= \quad \{ \quad \text{assumption: } c \subseteq g< \text{ and } f< = g< \quad \}$$
$$\top\!\!\top\circ c\circ \top\!\!\top$$
$$= \quad \{ \quad c \text{ is a point} \quad \}$$
$$\top\!\!\top$$

and $\top\!\!\top$ is non-empty by the cone rule. Also

$$f^{\cup}\circ g$$
$$= \quad \{ \quad g = g<\circ g \text{ and saturation axiom: (16)} \quad \}$$
$$f^{\cup}\circ \langle \cup c : c \subseteq g< : c\rangle \circ g$$
$$= \quad \{ \quad \text{distributivity} \quad \}$$
$$\langle \cup c : c \subseteq g< : f^{\cup}\circ c\circ g\rangle \;.$$

For the final property, we aim to apply (90) with $\mathcal{R}$ instantiated to

$$\langle c : c \subseteq g< : f^{\cup}\circ c\circ g\rangle \;.$$

Suppose $c \subseteq g<$ and $c' \subseteq g<$. (These replace the dummies j and k in (90).) Then

$$(f^{\cup}\circ c\circ g)<$$
$$= \quad \{ \quad [\,(R\circ S)< \;=\; (R\circ S<)<\,] \text{ with } R,S := f^{\cup}\circ c\,,\, g \quad \}$$
$$(f^{\cup}\circ c\circ g<)<$$
$$= \quad \{ \quad c \subseteq g< \;\equiv\; c\circ g< = c \quad \}$$
$$(f^{\cup}\circ c)<$$
$$= \quad \{ \quad \text{domains} \quad \}$$
$$(c\circ f)> \;.$$

That is,

(92) $\quad (f^{\cup}\circ c\circ g)_{<} \;=\; (c{\circ}f)_{>} \quad .$

Similarly, exploiting in addition the assumption that $f_{<} = g_{<}$,

(93) $\quad (f^{\cup}\circ c\circ g)_{>} \;=\; (c{\circ}g)_{>} \quad .$

Bearing the right side of (90) in mind, we calculate:

$$
\begin{array}{cl}
 & \top\!\!\top\circ(f^{\cup}\circ c\circ g)^{\cup}\circ f^{\cup}\circ c'\circ g\circ\top\!\!\top \\
= & \{ \quad \text{domains} \quad \} \\
 & \top\!\!\top\circ(f^{\cup}\circ c\circ g)_{<}\circ(f^{\cup}\circ c'\circ g)_{<}\circ\top\!\!\top \\
= & \{ \quad \text{(92) with } c{:=}c \text{ and } c{:=}c'\text{, domains} \quad \} \\
 & \top\!\!\top\circ c\circ f\circ f^{\cup}\circ c'\circ\top\!\!\top \\
= & \{ \quad f\circ f^{\cup} \;=\; f_{<} \;=\; g_{<} \text{ and } c\subseteq g_{<} \\
 & \quad\;\; c \text{ and } g_{<} \text{ are coreflexives, so } c\circ f\circ f^{\cup} = c \quad \} \\
 & \top\!\!\top\circ c\circ c'\circ\top\!\!\top \quad .
\end{array}
$$

That is,

(94) $\quad \top\!\!\top\circ(f^{\cup}\circ c\circ g)^{\cup}\circ f^{\cup}\circ c'\circ g\circ\top\!\!\top \;=\; \top\!\!\top\circ c\circ c'\circ\top\!\!\top \quad .$

A similar calculation exploiting (93) shows that

(95) $\quad \top\!\!\top\circ f^{\cup}\circ c'\circ g\circ(f^{\cup}\circ c\circ g)^{\cup}\circ\top\!\!\top \;=\; \top\!\!\top\circ c'\circ c\circ\top\!\!\top \quad .$

So

$$
\begin{array}{cl}
 & \top\!\!\top\circ(f^{\cup}\circ c\circ g)^{\cup}\circ f^{\cup}\circ c'\circ g\circ\top\!\!\top \;=\; \top\!\!\top \\
= & \{ \quad (94) \quad \} \\
 & \top\!\!\top\circ c'\circ c\circ\top\!\!\top \;=\; \top\!\!\top \\
= & \{ \quad c \text{ and } c' \text{ are points: (17)} \quad \} \\
 & c = c' \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & f^{\cup}\circ c\circ g \;=\; f^{\cup}\circ c'\circ g \quad .
\end{array}
$$

Similarly, by exploiting (95),

$$\top\!\!\top \circ f^{\cup} \circ c' \circ g \circ (f^{\cup} \circ c \circ g)^{\cup} \circ \top\!\!\top \;=\; \top\!\!\top \;\Rightarrow\; f^{\cup} \circ c \circ g \;=\; f^{\cup} \circ c' \circ g \quad .$$

The lemma follows by instantiating $\mathcal{R}$ in lemma 87 as announced earlier.
□

We now establish the converse of lemma 91. (The proof is quite long because of all the details that need to be checked.)

Lemma 96 Suppose relation R is the supremum of a completely disjoint set of rectangles. Then $R = f^{\cup} \circ g$ for some pair of relations f and g such that

$$f \circ f^{\cup} \;=\; f{<} \;=\; g \circ g^{\cup} \;=\; g{<} \quad .$$

Proof Suppose $\mathcal{R}$ is a completely disjoint set of rectangles indexed by the set K. Suppose also that $R = \cup\mathcal{R}$. Define the relations f and g by, for all k in K and all points a such that $a \subseteq R{<}$,

(97) $\quad k \circ \top\!\!\top \circ a \subseteq f \;\;\equiv\;\; a \circ (\mathcal{R}.k){<} = a \quad ,$

and, for all k in K and all points b such that $b \subseteq R{>}$

(98) $\quad k \circ \top\!\!\top \circ b \subseteq g \;\;\equiv\;\; (\mathcal{R}.k){>} \circ b = b \quad .$

Both f and g are functional. For example, here is the proof that f is functional: for all j and k in K,

$$
\begin{array}{ll}
& j \circ \top\!\!\top \circ k \;\subseteq\; f \circ f^{\cup} \\
= & \{ \quad \text{saturation axiom: (16) and irreducibility: (20)} \quad \} \\
& \langle \exists a :: j \circ \top\!\!\top \circ a \subseteq f \wedge a \circ \top\!\!\top \circ j \subseteq f^{\cup} \rangle \\
= & \{ \quad \text{(97) and converse} \quad \} \\
& \langle \exists a \;::\; a \circ (\mathcal{R}.j){<} = a \;\wedge\; a \circ (\mathcal{R}.k){<} = a \rangle \\
\Rightarrow & \{ \quad \text{coreflexives} \quad \} \\
& (\mathcal{R}.j){<} \cap (\mathcal{R}.k){<} \;\neq\; \bot\!\!\bot \\
\Rightarrow & \{ \quad \mathcal{R} \text{ is completely disjoint, definition 86} \quad \} \\
& j = k \quad .
\end{array}
$$

That is, by the saturation axiom and the definition of I_K, $f \circ f^{\cup} \subseteq I_K$.

Both of f and g are also surjective. For suppose k is in K. Then

$$\begin{array}{ll}
 & \text{true} \\
= & \{ \quad \text{definition 86 with } j:=k \quad \} \\
 & \mathcal{R}.k \neq \bot\!\!\bot \\
= & \{ \quad \text{saturation axiom: (16)} \quad \} \\
 & \langle \exists a :: a \circ (\mathcal{R}.k)_< = a \rangle \\
= & \{ \quad (97) \quad \} \\
 & \langle \exists a :: k \circ \top\!\!\top \circ a \subseteq f \rangle \\
\Rightarrow & \{ \quad a \text{ and } k \text{ are points, so } k = k \circ \top\!\!\top \circ k = k \circ \top\!\!\top \circ a \circ \top\!\!\top \circ k \quad \} \\
 & k \subseteq f \circ f^{\cup} \quad .
\end{array}$$

That is, by the saturation axiom, $I_K \subseteq f \circ f^{\cup}$.

Combining the functionality of f with its surjectivity, we conclude that $f \circ f^{\cup} = I_K$. Similarly, $g \circ g^{\cup} = I_K$. So we have constructed relations f and g such that

(99) $\quad f \circ f^{\cup} = f_< = I_K = g \circ g^{\cup} = g_< \quad .$

We now have to show that $R = f^{\cup} \circ g$.

A first step is to show that $f_> = R_<$ and $g_> = R_>$. We have, for all points a,

$$\begin{array}{ll}
 & a \subseteq R_< \\
= & \{ \quad R = \cup \mathcal{R} \quad \} \\
 & a \subseteq (\cup \mathcal{R})_< \\
= & \{ \quad \text{distributivity} \quad \} \\
 & a \subseteq \langle \cup k :: (\mathcal{R}.k)_< \rangle \\
= & \{ \quad \text{irreducibility of points} \quad \} \\
 & \langle \exists k :: a \subseteq (\mathcal{R}.k)_< \rangle \\
= & \{ \quad \text{coreflexives} \quad \} \\
 & \langle \exists k :: a \circ (\mathcal{R}.k)_< = a \rangle \\
= & \{ \quad (97) \quad \} \\
 & \langle \exists k :: k \circ \top\!\!\top \circ a \subseteq f \rangle \\
= & \{ \quad \text{domains} \quad \} \\
 & a \subseteq f_< \quad .
\end{array}$$

We conclude by the saturation axiom (16) that $f_> = R_<$. Again, the property $g_> = R_>$ is proved similarly. It follows that

$(f^{\cup} \circ g)_{>}$

$=$ { domains }

$(f_{<} \circ g)_{>}$

$=$ { (99) (specifically, $f_{<} = g_{<}$) }

$g_{>}$

$=$ { above }

$R_{>}$.

Similarly, $(f^{\cup} \circ g)_{<} = R_{<}$. So, for all points a and b such that $a \subseteq R_{<}$ and $b \subseteq R_{>}$,

$a \circ f^{\cup} \circ g \circ b$

$=$ { $f_{>} = R_{<}$ and $g_{>} = R_{>}$, distributivity and saturation axiom: (16) }

$\langle \cup k : k \subseteq f_{<} \wedge k \subseteq g_{<} : a \circ f^{\cup} \circ k \circ g \circ b \rangle$

$=$ { (99) }

$\langle \cup k : k \in K : a \circ f^{\cup} \circ k \circ g \circ b \rangle$

$=$ { all-or-nothing }

$\langle \cup k : a \circ \top\!\!\top \circ k \subseteq f^{\cup} \wedge k \circ \top\!\!\top \circ b \subseteq g : a \circ \top\!\!\top \circ k \circ k \circ \top\!\!\top \circ b \rangle$

$=$ { (97) and (98), and k is a point }

$\langle \cup k : a \circ (\mathcal{R}.k)_{<} = a \wedge (\mathcal{R}.k)_{>} \circ b = b : a \circ \top\!\!\top \circ b \rangle$

$=$ { a is a point, so $a \circ (\mathcal{R}.k)_{<} = a \vee a \circ (\mathcal{R}.k)_{<} = \bot\!\!\bot$

b is a point, so $(\mathcal{R}.k)_{>} \circ b = b \vee (\mathcal{R}.k)_{>} \circ b = \bot\!\!\bot$

range disjunction and $\bot\!\!\bot$ is least }

$\langle \cup k :: a \circ (\mathcal{R}.k)_{<} \circ \top\!\!\top \circ (\mathcal{R}.k)_{>} \circ b \rangle$

$=$ { $\mathcal{R}.k$ is a rectangle }

$\langle \cup k :: a \circ \mathcal{R}.k \circ b \rangle$

$=$ { $R = \langle \cup k :: \mathcal{R}.k \rangle$ and distributivity }

$a \circ R \circ b$

We conclude that $R = f^{\cup} \circ g$ by the extensionality property.

□

Theorem 100 A relation R is the supremum of a set of completely disjoint rectangles if and only if $R = f^{\cup} \circ g$ for some pair of relations f and g such that

$$f \circ f^{\cup} \;=\; f_{<} \;=\; g \circ g^{\cup} \;=\; g_{<} \quad .$$

Proof "If" is lemma 91 and "only-if" is lemma 96.
□

5 Partial Equivalence Relations

An equivalence relation is a reflexive, symmetric and transitive (homogeneous) relation. Reflexivity means that the left domain, the right domain, the source and the target of the relation are all the same. A *partial* equivalence relation —recall definition 68— is symmetric and transitive but not necessarily reflexive; however, symmetry means that its left and right domains are equal. Working with partial equivalence relations is slightly more complicated than working with equivalence relations but fits in better with later sections.

The theorem we prove in this section is that every partial equivalence relation is covered by a set of disjoint squares. Other properties of partial equivalence relations are documented on the way. Particularly important is theorem 115 which characterises partial equivalence relations very succinctly.

5.1 Characterisation Theorem

Obviously from the definitions, a square is a rectangle. As we shall see, the equivalence classes defined by a per are (represented by) squares, which are themselves pers.

Compared to equivalence relations, the absence of the reflexivity property is of no consequence. Its rôle is taken by the following lemma. (An equivalence relation is a per such that $R_{<} = R_{>} = I$ where I is the identity relation on the source/target of the relation.)

Lemma 101 Suppose R is a per. Then

$$R_{<} \;=\; R_{>} \;\subseteq\; R \quad .$$

Proof The equality $R_{<} = R_{>}$ is immediate from the definition of the domain operators and the fact that a per is symmetric. Also,

$$R_{>} \;\subseteq\; R$$

$$\Leftarrow \qquad \{ \qquad R_{>} \;=\; I \cap R^{\cup} \circ R \text{, transitivity of subset relation} \qquad \}$$

$$R^{\cup} \circ R \;\subseteq\; R$$

$=$ { assumption: R is a per, definition 68 and Leibniz }

true .

□

Because the left and right domain of a per are equal, we refer to its *domain*, omitting the adjective left or right.

Corollary 102 Suppose R is a per. Then

$$R \neq \perp\!\!\perp \;\equiv\; \langle \exists a : \mathrm{point}.a : a \subseteq R \rangle \;.$$

Proof By mutual implication. First note that, for all R,

(103) $R \neq \perp\!\!\perp \;\equiv\; \langle \exists\, a,b : \mathrm{point}.a \wedge \mathrm{point}.b : a \circ \top\!\!\top \circ b \subseteq R \rangle$.

(Formally, this is a consequence of (18) and the properties of $\perp\!\!\perp$.) Thus

$\langle \exists a : \mathrm{point}.a : a \subseteq R \rangle$

$=$ { (15) }

$\langle \exists a : \mathrm{point}.a : a \circ \top\!\!\top \circ a \subseteq R \rangle$

$\Rightarrow$ { weakening with $b := a$ }

$\langle \exists\, a,b : \mathrm{point}.a \wedge \mathrm{point}.b : a \circ \top\!\!\top \circ b \subseteq R \rangle$

$\Rightarrow$ { monotonicity }

$\langle \exists\, a,b : \mathrm{point}.a \wedge \mathrm{point}.b : a \circ \top\!\!\top \circ b \circ b \circ \top\!\!\top \circ a \subseteq R \circ R^{\cup} \rangle$

$\Rightarrow$ { one-point rule with $b := a$ }

$\langle \exists a : \mathrm{point}.a : a \subseteq R \circ R^{\cup} \rangle$

$\Rightarrow$ { definition 68, Leibniz and transitivity }

$\langle \exists a : \mathrm{point}.a : a \subseteq R \rangle$.

That is, by mutual implication,

(104) $\langle \exists a : \mathrm{point}.a : a \subseteq R \rangle \;\equiv\; \langle \exists\, a,b : \mathrm{point}.a \wedge \mathrm{point}.b : a \circ \top\!\!\top \circ b \subseteq R \rangle$.

The corollary follows by combining (103) and (104).

□

The goal of this section is the proof of the following characterisation of pers:

Theorem 105 For all relations R, the following statements are equivalent:

(i) R is a per (i.e. $R = R^{\cup} \;\wedge\; R \circ R \subseteq R$),

(ii) $R = R \circ R^{\cup}$,

(iii) $R = R{>} \circ R \backslash\!\backslash R$,

(iv) R is the supremum of a set of disjoint squares,

(v) $\langle \exists f \;:\; f \circ f^{\cup} = f{<} \;:\; R = f^{\cup} \circ f \rangle$.

□

An informal understanding of theorem 105 is that a per partitions its domain into disjoint sets — commonly called *equivalence classes*. Two ways of representing the equivalence classes are given by 105(iv) and 105(v): either (iv) by disjoint squares or (v) by a functional relation f whereby two points in the domain of a per are in the same equivalence class iff they are mapped to the same value by f. (There are, of course, other ways of representing the classes.)

We present two distinct proofs of theorem 105. Both proofs have several parts; both have in common proofs that 105(iv) and 105(v) are equivalent (using an if-and-only-if argument) and that 105(v) implies 105(i). The proofs differ in that in one we prove that 105(i) is equivalent to 105(ii) which implies 105(iv) and in the second we prove that 105(i) is equivalent to 105(iii) which implies 105(v). In section 5.2, we present the common parts. Sections 5.3 and 5.4 establish that 105(i) implies 105(ii) and 105(i) implies 105(iii), respectively.

5.2 Common Parts of the Proof

The easiest part is that 105(v) implies 105(i):

Lemma 106 Suppose f is a functional relation (i.e. $f \circ f^{\cup} = f{<}$). Then $f^{\cup} \circ f$ is a per.

Proof For arbitrary relation S, $S^{\cup} \circ S$ is symmetric. So $f^{\cup} \circ f$ is symmetric. Establishing transitivity is where we need the assumption that f is functional:

$(f^{\cup} \circ f) \circ (f^{\cup} \circ f)$

$=$ { associativity }

$f^{\cup} \circ (f \circ f^{\cup}) \circ f$

$=$ { f is functional and monotonicity }

$f^{\cup} \circ f{<} \circ f$

$=$ { domains }

$f^{\cup} \circ f$.

□

The proof that 105(iv) and 105(v) are equivalent is an instance of theorem 100. Recall that theorem 100 establishes an equivalence between a set of completely disjoint *rectangles* and a *pair* of functional relations. (See the theorem for precise details.) That 105(iv) and 105(v) are equivalent is the special case where the rectangles are squares and the pair of functional relations are equal (i.e. there is just one of them).

Note that the notions of disjoint and completely disjoint are the same for squares (because the left and right domains of a square are equal).

Instantiating lemma 91 in the case that f and g are equal, we get:

Lemma 107 Suppose f of type $A{\sim}B$ is such that

$$f \circ f^{\cup} \;=\; f{<} \;\; .$$

Then the relation $f^{\cup} \circ f$ is the supremum of a set of disjoint squares. Specifically, with dummy a ranging over points of type A,

(108) $f^{\cup} \circ f \;=\; \langle \cup a : a \subseteq f{<} : f^{\cup} \circ a \circ f \rangle \;\; .$

Moreover, for all points a such that $a \subseteq f{<}$, $f^{\cup} \circ a \circ f$ is a non-empty square and, for all points a and a', such that $a \subseteq f{<}$ and $a' \subseteq f{<}$,

(109) $f^{\cup} \circ a \circ f \;=\; f^{\cup} \circ a' \circ f \;\;\equiv\;\; a = a'$

and

(110) $f^{\cup} \circ a \circ f \,\cap\, f^{\cup} \circ a' \circ f = \bot\!\!\bot \;\;\equiv\;\; a \neq a' \;\; .$

It follows that , for all points a and a',

(111) $f^{\cup} \circ a \circ f \;=\; f^{\cup} \circ a' \circ f \quad \vee \quad f^{\cup} \circ a \circ f \,\cap\, f^{\cup} \circ a' \circ f \;=\; \bot\!\!\bot \;\; .$

In words, a functional relation “divides” its right domain (the points on which the relation is defined) into a collection of disjoint “equivalence classes” (represented by squares) whereby two “inputs” are equivalent iff they are mapped to the same “output”.

Proof As mentioned above, the lemma is an instance of lemma 91. However, because the details of lemma 91 are more complicated, it is easier to give a separate proof.

Property (108) follows from the saturation axiom:

$$\begin{array}{ll} & f^{\cup} \circ f \\ = & \{ \quad \text{domains} \quad \} \\ & f^{\cup} \circ f{<} \circ f \end{array}$$

$$
\begin{array}{ll}
= & \{ \quad \text{saturation: (16)} \quad \} \\
& f^{\cup} \circ \langle \cup a : a \subseteq f{<} : a \rangle \circ f \\
= & \{ \quad \text{distributivity} \quad \} \\
& \langle \cup a : a \subseteq f{<} : f^{\cup} \circ a \circ f \rangle \ .
\end{array}
$$

That $f^{\cup} \circ a \circ f$ is a square follows from the fact that it is a rectangle (because a is a point) and is symmetric. Also.

$$
\begin{array}{ll}
& f^{\cup} \circ a \circ f \neq \perp\!\!\perp \\
= & \{ \quad \text{cone rule (5)} \quad \} \\
& \top\!\!\top \circ f^{\cup} \circ a \circ f \circ \top\!\!\top \ = \ \top\!\!\top \\
= & \{ \quad \text{domains} \quad \} \\
& \top\!\!\top \circ f{<} \circ a \circ f{<} \circ \top\!\!\top \ = \ \top\!\!\top \\
= & \{ \quad \text{assumption: } a \subseteq f{<} \quad \} \\
& \top\!\!\top \circ a \circ \top\!\!\top \ = \ \top\!\!\top \\
= & \{ \quad a \text{ is a point: (14)} \quad \} \\
& \text{true} \ .
\end{array}
$$

The proof of (109) is straightforward:

$$
\begin{array}{ll}
& f^{\cup} \circ a \circ f \ = \ f^{\cup} \circ a' \circ f \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
& f \circ f^{\cup} \circ a \circ f \circ f^{\cup} \ = \ f \circ f^{\cup} \circ a' \circ f \circ f^{\cup} \\
= & \{ \quad \text{assumptions: } a \subseteq f{<} = f \circ f^{\cup} \text{ and } a' \subseteq f{<} = f \circ f^{\cup} \quad \} \\
& a = a' \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
& f^{\cup} \circ a \circ f \ = \ f^{\cup} \circ a' \circ f \ .
\end{array}
$$

Now we prove (110). Assuming a and a' are points such that $a \neq a'$,

$$
\begin{array}{ll}
& f^{\cup} \circ a \circ f \ \cap \ f^{\cup} \circ a' \circ f \\
= & \{ \quad \text{distributivity (consequence of Galois connection (61))} \quad \} \\
& f^{\cup} \circ (a \cap a') \circ f \\
= & \{ \quad \text{assumption: } a \neq a' \text{ and } a \text{ and } a' \text{ are atoms} \quad \}
\end{array}
$$

$$f^{\cup} \circ \perp\!\!\perp \circ f$$

$= \quad \{ \quad \perp\!\!\perp$ is zero of composition $\}$

$$\perp\!\!\perp \ .$$

□

The next step is to specialise lemma 96 in the case that R is the supremum of a disjoint set of squares (rather than rectangles). Again, it is easier to revise the proof rather than explain how the instantiation is made. The following lemma is needed both here and later.

Lemma 112 Suppose R is a per and p is coreflexive. Then

$$(p \subseteq R \circ p \circ R^{\cup}) \ = \ (p \subseteq R \circ R^{\cup}) \ = \ (p \subseteq R) \ .$$

(As always, continued equalities are read conjunctionally.) Hence, if R is a per, for all coreflexives p ,

$$p \subseteq R_{<} \ \equiv \ p \subseteq R \ .$$

Proof By mutual implication

$$p \ \subseteq \ R \circ p \circ R^{\cup}$$

$\Rightarrow \quad \{ \quad$ assumption: p is coreflexive (so $p \subseteq I$) $\}$

$$p \subseteq R \circ R^{\cup}$$

$\Rightarrow \quad \{ \quad$ R is a per, i.e. symmetric ($R = R^{\cup}$) and transitive ($R \circ R \subseteq R$) $\}$

$$p \subseteq R$$

$\Rightarrow \quad \{ \quad$ assumption: p is coreflexive (so $p \circ p = p$) and monotonicity $\}$

$$p \subseteq R \circ p$$

$\Rightarrow \quad \{ \quad$ converse and monotonicity $\}$

$$p \circ p \ \subseteq \ R \circ p \circ p \circ R^{\cup}$$

$= \quad \{ \quad$ p is coreflexive (so $p \circ p = p$) $\}$

$$p \ \subseteq \ R \circ p \circ R^{\cup} \ .$$

We have thus proved that the properties $p \subseteq R \circ p \circ R^{\cup}$, $p \subseteq R \circ R^{\cup}$ and $p \subseteq R$ are all equivalent. The final property is immediate from the assumption that p is coreflexive (i.e. $p \subseteq I$) and the identity $R_{<} \ = \ I \cap R \circ R^{\cup}$.

□

(Let $\mathcal{E}R$ denote the function $\langle p :: R \circ p \circ R^{\cup} \rangle$. This function is essentially the same as what Bird and De Moor [BdM97] call the *existential image* of R. Lemma 112 states that, if R is a per, the greatest fixed point of $\mathcal{E}R$ is $R_{<}$.)

For several theorems (for example lemma 116 below), we introduce the assumption that a is a point such that $a \subseteq R$. Recalling that, for point a, $a = a \circ \top\!\!\top \circ a$ and $a \circ \top\!\!\top \circ a$ is interpreted as the pair (a, a), the assumption that $a \subseteq R$ is interpreted as a is related to itself by R. In the case that R is a per, lemma 112 states that this is equivalent to $a \subseteq R_{<}$ and (by virtue of lemma 101) to $a \subseteq R_{>}$.

Now we can state and prove the special case of lemma lemma 96:

Lemma 113 Suppose relation R is covered by a disjoint set of squares. Then

$$\langle \exists f : f \circ f^{\cup} = f_{<} : R = f^{\cup} \circ f \rangle \ .$$

Proof Suppose $\mathcal{R}$ is a disjoint set of squares indexed by the set K. Suppose also that $R = \cup \mathcal{R}$. For points a such that $a \subseteq R$ define $f.a$ to be the unique solution of the equation

$$k{::}\ \ a \subseteq \mathcal{R}.k \ \ .$$

The function f is well-defined: the equation has at most one solution because $\mathcal{R}$ is a disjoint set of squares and it has at least one solution because $R = \cup \mathcal{R}$ and $a \subseteq R$ and a is a point. (Points are irreducible.)

We now show that $R = f^{\cup} \circ f$. Suppose $a \subseteq R$ and $b \subseteq R$. Then

$$\begin{aligned}
& a \circ R \circ b \\
= \quad & \{ \quad R = \langle \cup k :: \mathcal{R}.k \rangle \text{ and distributivity} \quad \} \\
& \langle \cup k :: a \circ \mathcal{R}.k \circ b \rangle \\
= \quad & \{ \quad \mathcal{R}.k \text{ is a square, domains} \quad \} \\
& \langle \cup k :: a \circ (\mathcal{R}.k)_{<} \circ \top\!\!\top \circ (\mathcal{R}.k)_{>} \circ b \rangle \\
= \quad & \{ \quad a \text{ is a point, so } a \circ (\mathcal{R}.k)_{<} = a \ \vee \ a \circ (\mathcal{R}.k)_{<} = \bot\!\!\bot \\
& \quad\;\; b \text{ is a point, so } (\mathcal{R}.k)_{>} \circ b = b \ \vee \ (\mathcal{R}.k)_{>} \circ b = \bot\!\!\bot \\
& \quad\;\; \text{range disjunction and } \bot\!\!\bot \text{ is least} \quad \} \\
& \langle \cup k \ : \ a \circ (\mathcal{R}.k)_{<} = a \ \wedge \ (\mathcal{R}.k)_{>} \circ b = b \ : \ a \circ \top\!\!\top \circ b \rangle \\
= \quad & \{ \quad \text{lemma 112 (with } p,R := a, \mathcal{R}.k \text{ and } p,R := b, \mathcal{R}.k) \\
& \quad\;\; \text{(applicable because a square is a per)} \quad \} \\
& \langle \cup k \ : \ a \subseteq \mathcal{R}.k \wedge b \subseteq \mathcal{R}.k \ : \ a \circ \top\!\!\top \circ b \rangle
\end{aligned}$$

$$= \quad \{ \quad \text{assumption: } a \subseteq R \text{ and } b \subseteq R \text{, definition of } f \quad \}$$

$$\langle \cup k : k = f.a \wedge k = f.b : a \circ \top\!\!\top \circ b \rangle$$

$$= \quad \{ \quad \text{saturation property: (19)} \quad \}$$

$$a \circ f^{\cup} \circ f \circ b \quad .$$

We conclude that $R = f^{\cup} \circ f$ by the saturation axiom.
□

To conclude this section, we have established the equivalence of 105(iv) and 105(v):

Theorem 114 A relation R is the supremum of a set of disjoint squares if and only if $R = f^{\cup} \circ f$ for some relation f such that $f \circ f^{\cup} = f<$.

Proof "If" is lemma 107 and "only-if" is lemma 113.
□

5.3 Covering by Disjoint Classes

We now make a start on proving theorem 105 by showing that 105(i) implies 105(iv). The key property is that 105(i) and 105(ii) are equivalent:

Theorem 115 For all relations R, R is a per equivales $R = R \circ R^{\cup}$. Symmetrically, for all relations R, R is a per equivales $R = R^{\cup} \circ R$.

Proof By mutual implication. First, suppose R is a per. Then

$$R \circ R$$

$$\subseteq \quad \{ \quad \text{assumption: } R \text{ is transitive} \quad \}$$

$$R$$

$$= \quad \{ \quad \text{domains} \quad \}$$

$$R \circ R>$$

$$\subseteq \quad \{ \quad \text{assumption: } R \text{ is a per, lemma 101} \quad \}$$

$$R \circ R \quad .$$

That is, by the anti-symmetry of the subset relation, $R = R \circ R$. But R is symmetric. That is, $R = R^{\cup}$. So, by Leibniz's rule, $R = R \circ R^{\cup}$.

For the follows-from, we have:

$$R = R \circ R^{\cup}$$

$$= \quad \{ \quad (R \circ R^{\cup})^{\cup} = R \circ R^{\cup} \quad \}$$

$$R = R\circ R^{\cup} = R^{\cup}$$

$\Rightarrow$ { subset relation is reflexive, Leibniz }

$$R\circ R \subseteq R \;\wedge\; R = R^{\cup}$$

$=$ { definition }

$$\mathsf{per}.R \ .$$

□

Lemma 116 If R is a per, then for all points a and b such that $a\subseteq R$ and $b\subseteq R$,

$$a\circ\top\!\top\circ b \subseteq R \;\equiv\; R\circ a\circ R = R\circ b\circ R \ .$$

Proof We have

$$a\circ\top\!\top\circ b \subseteq R$$

$=$ { $R = R^{\cup}$ and lemma 45 }

$$a \subseteq R\circ b\circ R$$

$\Rightarrow$ { monotonicity }

$$R\circ a\circ R \subseteq R\circ R\circ b\circ R\circ R$$

$=$ { R is a per, theorem 115 and $R = R^{\cup}$ }

$$R\circ a\circ R \subseteq R\circ b\circ R$$

That is,

(117) $a\circ\top\!\top\circ b \subseteq R \;\Rightarrow\; R\circ a\circ R \subseteq R\circ b\circ R$.

Symmetrically (since $R = R^{\cup}$)

(118) $a\circ\top\!\top\circ b \subseteq R \;\Rightarrow\; R\circ b\circ R \subseteq R\circ a\circ R$.

Combining (117) and (118) using the anti-symmetry of the subset relation, we get:

(119) $a\circ\top\!\top\circ b \subseteq R \;\Rightarrow\; R\circ a\circ R = R\circ b\circ R$.

For the converse implication,

$$R\circ a\circ R = R\circ b\circ R$$

$\Rightarrow$ { assumption: $a\subseteq R$; a is a point }

$$a\subseteq R\circ b\circ R$$

$=$ { R is a per, so $R = R^{\cup}$ }

$$
\begin{array}{ll}
 & a \subseteq R \circ b \circ R^{\cup} \\
= & \{ \quad \text{lemma 45} \quad \} \\
 & a \circ \top\!\top \circ b \subseteq R \quad .
\end{array}
$$

□

Lemma 120 Suppose R is a per and a and b are points such that $a \subseteq R$ and $b \subseteq R$. Then

$$(121) \quad R \circ a \circ R^{\cup} = R \circ b \circ R^{\cup} \quad \vee \quad R \circ a \circ R^{\cup} \cap R \circ b \circ R^{\cup} = \bot\!\bot \quad .$$

Moreover,

$$(122) \quad R = \langle \cup a : a \subseteq R : R \circ a \circ R^{\cup} \rangle \quad .$$

In words, a per is covered by the set of disjoint squares $R \circ a \circ R^{\cup}$ where a ranges over points at most R.

Proof The dummy c in the following calculation ranges over points.

$$
\begin{array}{ll}
 & R \circ a \circ R \cap R \circ b \circ R \neq \bot\!\bot \\
= & \{ \quad \text{assumption: } R \text{ is a per and } a \text{ and } b \text{ are points,} \\
 & \quad\ \ \text{so } R \circ a \circ R \cap R \circ b \circ R \text{ is a per (easy proof omitted);} \\
 & \quad\ \ \text{corollary 102} \quad \} \\
 & \langle \exists c :: c \subseteq R \circ a \circ R \cap R \circ b \circ R \rangle \\
= & \{ \quad \text{lemma 45} \quad \} \\
 & \langle \exists c :: c \circ \top\!\top \circ a \subseteq R \wedge c \circ \top\!\top \circ b \subseteq R \rangle \\
\Rightarrow & \{ \quad \text{assumption: } R \text{ is a per (i.e. symmetric and transitive)} \quad \} \\
 & \langle \exists c :: a \circ \top\!\top \circ c \circ \top\!\top \circ b \subseteq R \rangle \\
= & \{ \quad c \text{ is a point, so } \top\!\top \circ c \circ \top\!\top = \top\!\top \quad \} \\
 & \langle \exists c :: a \circ \top\!\top \circ b \subseteq R \rangle \\
= & \{ \quad \text{range of quantification is non-empty} \quad \} \\
 & a \circ \top\!\top \circ b \subseteq R \\
= & \{ \quad \text{assumption: } R \text{ is a per; lemma 116} \quad \} \\
 & R \circ a \circ R = R \circ b \circ R \quad .
\end{array}
$$

We have thus proved that, if R is a per and a and b are points such that $a \subseteq R>$ and $b \subseteq R>$,

$$R \circ a \circ R \cap R \circ b \circ R \neq \bot\!\!\bot \quad \Rightarrow \quad R \circ a \circ R \;=\; R \circ b \circ R$$

which is equivalent to (121). The property (122) is straightforward:

$$\begin{array}{ll}
 & R \\
= & \{ \quad \text{assumption: } R \text{ is a per, lemma 115} \quad \} \\
 & R \circ R^{\cup} \\
= & \{ \quad \text{domains} \quad \} \\
 & R \circ R> \circ R^{\cup} \\
= & \{ \quad \text{saturation axiom: (16)} \quad \} \\
 & R \circ \langle \cup a : a \subseteq R> : a \rangle \circ R^{\cup} \\
= & \{ \quad \text{distributivity and, by lemma 112, } a \subseteq R> \equiv a \subseteq R \quad \} \\
 & \langle \cup a : a \subseteq R : R \circ a \circ R^{\cup} \rangle \quad .
\end{array}$$

□

5.4 Characterisation by Greatest Extension

Now for the second proof of theorem 105, we exploit theorem 72 to show that 105(i) implies 105(v).

Theorem 123 A relation R is a per iff $R = (\Gamma R \circ R>)^{\cup} \circ (\Gamma R \circ R>)$. (Note that $\Gamma R \circ R>$ is a functional relation.)

Proof By mutual implication. First, assume that R is a per. Then

$$\begin{array}{ll}
 & R \\
= & \{ \quad R = R^{\cup}\text{, domains} \quad \} \\
 & R> \circ R \\
= & \{ \quad \text{theorem 72} \quad \} \\
 & R> \circ R \backslash\!\backslash R \circ R> \\
= & \{ \quad (\Gamma R)^{\cup} \circ \Gamma S = R \backslash\!\backslash S \ \text{(theorem 66)} \quad \} \\
 & R> \circ (\Gamma R)^{\cup} \circ \Gamma R \circ R> \\
= & \{ \quad \text{converse} \quad \} \\
 & (\Gamma R \circ R>)^{\cup} \circ (\Gamma R \circ R>) \quad .
\end{array}$$

The converse is immediate from lemma 106.
□

Theorem 123 also gives a characterisation of a per as the supremum of a disjoint set of squares:

Theorem 124 Suppose relation R is a per. Then, for all points a and b such that $a \subseteq R$ and $b \subseteq R$,

$$(125) \quad \Gamma R.a = \Gamma R.b \quad \not\equiv \quad \Gamma R.a \cap \Gamma R.b = \perp\!\!\perp \ .$$

Moreover,

$$(126) \quad R = \langle \cup a : a \subseteq R : \Gamma R.a \circ \top\!\!\top \circ (\Gamma R.a)^{\cup} \rangle \ .$$

Proof Assume that a and b are points such that $a \subseteq R$ and $b \subseteq R$. Note that, by lemma 101, this is equivalent to $a \subseteq R_{>}$ and $b \subseteq R_{>}$.

$$\Gamma R.a \cap \Gamma R.b \neq \perp\!\!\perp$$
$$= \quad \{ \quad \text{definition of } \Gamma R \quad \}$$
$$(R \circ a)_{<} \cap (R \circ b)_{<} \neq \perp\!\!\perp$$
$$= \quad \{ \quad \text{domains are coreflexives, cone rule} \quad \}$$
$$\top\!\!\top \circ (R \circ a)_{<} \circ (R \circ b)_{<} \circ \top\!\!\top = \top\!\!\top$$
$$= \quad \{ \quad \text{domains} \quad \}$$
$$\top\!\!\top \circ a \circ R^{\cup} \circ R \circ b \circ \top\!\!\top = \top\!\!\top$$
$$= \quad \{ \quad R \text{ is a per, theorem 115} \quad \}$$
$$\top\!\!\top \circ a \circ R \circ b \circ \top\!\!\top = \top\!\!\top$$
$$= \quad \{ \quad \text{cone rule and all-or-nothing rule} \quad \}$$
$$a \circ \top\!\!\top \circ b \subseteq R$$
$$= \quad \{ \quad R \text{ is a per and hence symmetric} \quad \}$$
$$a \circ \top\!\!\top \circ b \subseteq R \ \wedge \ b \circ \top\!\!\top \circ a \subseteq R$$
$$\Rightarrow \quad \{ \quad a \circ \top\!\!\top \circ b \subseteq R$$
$$\Rightarrow \quad \{ \quad \text{domains} \quad \}$$
$$a \subseteq (R \circ b)_{<}$$
$$\Rightarrow \quad \{ \quad \text{montonicity} \quad \}$$
$$(R \circ a)_{<} \subseteq (R \circ (R \circ b)_{<})_{<}$$

$=\qquad \{\quad \text{domains} \quad\}$

$(R\circ a)_< \ \subseteq\ (R\circ R\circ b)_<$

$\Rightarrow\qquad \{\quad \text{assumption: } R \text{ is a per and so is transitive} \quad\}$

$(R\circ a)_< \ \subseteq\ (R\circ b)_<$

$=\qquad \{\quad \text{assumption: } a\subseteq R_> \text{ and } b\subseteq R_>\text{, definition of } \Gamma R \quad\}$

$\Gamma R.a \ \subseteq\ \Gamma R.b \quad\}$

$\Gamma R.a \subseteq \Gamma R.b \ \wedge\ \Gamma R.b \subseteq \Gamma R.a$

$=\qquad \{\quad \text{anti-symmetry} \quad\}$

$\Gamma R.a \ =\ \Gamma R.b$

$=\qquad \{\quad \text{assumptions: } a\subseteq R \text{ and } b\subseteq R\text{, (57)} \quad\}$

$a\circ\top\!\!\top\circ b \ \subseteq\ (\Gamma R)^\cup\circ\Gamma R$

$=\qquad \{\quad \text{lemma 66 and (71)} \quad\}$

$a\circ\top\!\!\top\circ b \ \subseteq\ R_\succ$

$=\qquad \{\quad \text{assumption: } R \text{ is a per, theorem 72} \quad\}$

$a\circ\top\!\!\top\circ b \ \subseteq\ R$

$\Rightarrow\qquad \{\quad \text{assumptions: } a\subseteq R \text{ and } R \text{ is a per; domains} \quad\}$

$a \ \subseteq\ (R\circ a)_< \ \wedge\ a \ \subseteq\ (R\circ b)_<$

$=\qquad \{\quad \text{infimum and definition of } \Gamma R \quad\}$

$a \ \subseteq\ \Gamma R.a\cap\Gamma R.b$

$\Rightarrow\qquad \{\quad a \text{ is a point, so } a\neq\perp\!\!\perp \quad\}$

$\Gamma R.a\cap\Gamma R.b \ \neq\ \perp\!\!\perp\ .$

Property (125) follows by mutual implication and the definition of exclusive-or.

Now we prove (126)

$\langle\cup a : a\subseteq R : \Gamma R.a\circ\top\!\!\top\circ(\Gamma R.a)^\cup\rangle$

$=\qquad \{\quad \text{definition of } \Gamma R \quad\}$

$\langle\cup a : a\subseteq R : (R\circ a)_<\circ\top\!\!\top\circ(R\circ a)_<\rangle$

$=\qquad \{\quad \text{domains} \quad\}$

$\langle\cup a : a\subseteq R : R\circ a\circ\top\!\!\top\circ a\circ R^\cup\rangle$

$=\qquad \{\quad a \text{ is a point} \quad\}$

$$\langle \cup a : a \subseteq R : R \circ a \circ R^{\cup} \rangle$$

$=$ { distributivity and saturation axiom: (16) }

$$R \circ R^{\cup}$$

$=$ { assumption: R is a per, theorem 115 }

$$R \ .$$

□

From a calculational perspective, the second proof of theorem 105 is less attractive than the first (because the second proof is based on theorem 72 which involves the more complicated symmetric left-division operator whilst the first proof is based on theorem 115 which involves only elementary operators). The second proof is more appealing when R is an equivalence relation: in that case $R{>} = I$ and thus one obtains the property that R is an equivalence relation iff $R = R \backslash\backslash R$. In this form, theorem 123 corresponds to the traditional way that theorem 105 is proved: the pointwise interpretation of ΓR is the function that maps point b to the set of points a related to b by R; in the case that R is an equivalence relation, $\Gamma R.b$ is the equivalence class containing b. Theorem 124 is also a very well-known theorem. Property (125) is interpreted as the theorem that two points either have equal equivalence classes or their equivalence classes are disjoint. (The "or" in this statement is an exclusive-or, not an inclusive-or.) Property (126) is interpreted as the theorem that a per is equal to the union of the squares defined by its individual equivalence classes.

5.5 Partitioning by Functional Relations

As is well-known, a functional relation defines a partial equivalence relation on its source: points in the right domain of the relation are equivalent iff they are mapped to the same value. This is the property formulated in lemma 107.

Our formalism admits a different formulation. Specifically, lemma 106 states that, for any functional relation f, the relation $f^{\cup} \circ f$ is a partial equivalence relation. Applying lemma 120, we deduce that $f^{\cup} \circ f$ is covered by the set of disjoint squares

$$f^{\cup} \circ f \circ b \circ f^{\cup} \circ f$$

where b ranges over points at most $f^{\cup} \circ f$. Noting that points are coreflexives, the dummy b ranges over elements of $f{>}$. So, we have:

$$(127) \quad f^{\cup} \circ f \;=\; \langle \cup b : b \subseteq f{>} : f^{\cup} \circ f \circ b \circ f^{\cup} \circ f \rangle \ .$$

At first sight, this is not the same as lemma 107since it doesn't define equivalence classes (disjoint squares) as a function of values (points in the left domain of f) but, instead, as functions of inputs (points in the right domain of f). In order to exploit the lemmas and theorems in earlier sections, we need to establish a precise relationship between these two apparently distinct ways of expressing the equivalence classes.

Recalling (108), we have:

(128) $f^{\cup}\circ f \;=\; \langle \cup a : a \subseteq f{<} : f^{\cup}\circ a\circ f \rangle$.

The equivalence between the covering (127) and (128) is established as follows:

$$
\begin{array}{ll}
 & \langle \cup a : a \subseteq f{<} : f^{\cup}\circ a\circ f \rangle \\
= & \{ \quad \text{lemma 47} \quad \} \\
 & \langle \cup a : \langle \exists b : b \subseteq f{>} : a\circ\top\!\top\circ b \subseteq f \rangle : f^{\cup}\circ a\circ f \rangle \\
= & \{ \quad \text{theorem 52} \quad \} \\
 & \langle \cup a : \langle \exists b : b \subseteq f{>} : a = f\circ b\circ f^{\cup} \rangle : f^{\cup}\circ a\circ f \rangle \\
= & \{ \quad \text{range disjunction} \quad \} \\
 & \langle \cup b : b \subseteq f{>} : \langle \cup a : a = f\circ b\circ f^{\cup} : f^{\cup}\circ a\circ f \rangle\rangle \\
= & \{ \quad \text{by theorem 52, } f\circ b\circ f^{\cup} \text{ is a point;} \\
 & \qquad \text{one-point rule} \quad \} \\
 & \langle \cup b : b \subseteq f{>} : f^{\cup}\circ f\circ b\circ f^{\cup}\circ f \rangle \ .
\end{array}
$$

In words, (127) and (128) are different but nevertheless equivalent ways of expressing the division of the right domain of a functional relation f into disjoint squares (effectively, equivalence classes). Formula (128) corresponds to the conventional way of defining an equivalence class as the set of points b that are mapped to the same value a.

(Note the use of theorem 52 in the last calculation. The dummy a ranges over points; so the use of the one-point rule is only valid if $f\circ b\circ f^{\cup}$ is a point. Note also that the word "point" in "one-point rule" is possibly misleading; "one-instance rule" might be a better name in this context.)

The characterisation theorem for difunctionals (theorem 132) has the consequence that a difunctional relation divides its left and right domains into classes that are in (1–1) correspondence.

Lemma 129 Suppose f and g are relations with common target C such that

$$f\circ f^{\cup} \;=\; f{<} \;=\; g\circ g^{\cup} \;=\; g{<} \ .$$

Then the functions $\langle X :: g^{\cup}\circ f\circ X\circ f^{\cup}\circ g \rangle$ and $\langle Y :: f^{\cup}\circ g\circ Y\circ g^{\cup}\circ f \rangle$ define a (1–1) correspondence between the classes of the partial equivalence relations $f^{\cup}\circ f$ and $g^{\cup}\circ g$. That is, for all c,

$$\langle X :: g^{\cup}\circ f\circ X\circ f^{\cup}\circ g \rangle . (f^{\cup}\circ c\circ f) \;=\; g^{\cup}\circ c\circ g$$

and

$$\langle Y :: f^{\cup} \circ g \circ Y \circ g^{\cup} \circ f \rangle . (g^{\cup} \circ c \circ g) \;=\; f^{\cup} \circ c \circ f \;\; .$$

Proof The verification of the first equality is as follows.

$$
\begin{array}{ll}
 & \langle X :: g^{\cup} \circ f \circ X \circ f^{\cup} \circ g \rangle . (f^{\cup} \circ c \circ f) \\
= & \{ \quad \text{definition of function application} \quad \} \\
 & g^{\cup} \circ f \circ f^{\cup} \circ c \circ f \circ f^{\cup} \circ g \\
= & \{ \quad \text{assumption: } f \circ f^{\cup} = f< = g \circ g^{\cup} = g< \quad \} \\
 & g^{\cup} \circ g< \circ c \circ g< \circ g \\
= & \{ \quad \text{domains} \quad \} \\
 & \mathsf{true} \;\; .
\end{array}
$$

The second equality is verified in the same way.
□

6 Difunctional Relations

This section is where our study of difunctional relations and block-ordered relations begins.

As Riguet remarked, difunctional relations generalise both functional relations [Rig48] and pers [Rig50, "quasi-equivalences"] in the sense that a difunctional is characterised by a pair of functional relations whilst a per is characterised by a single functional relation (theorem 105); equivalently, a difunctional is a union of completely disjoint rectangles whilst a per is the union of disjoint squares (theorem 114). See theorems 132 and 134. We present several different calculational proofs of theorem 132 in section 6.3 using both point-free and pointwise calculations, with a view to gaining insight into the efficacy and aesthetics of the calculational method. Note that, although the proofs are quite different, the constructed characterisations are essentially the same, as is made precise in section 6.2. Theorem 134 is a straightforward combination of theorem 132 and the (already-proven) theorem 100.

The "difunctional closure" of a relation is the smallest difunctional relation that is a superset of a given relation. Its definition and properties, given in section 6.4, involve the application of standard techniques of Galois connections and fixed-point calculus; as such, it is included here for completeness.

Whereas the "difunctional closure" of a relation is a superset of the relation, the "diagonal" of a relation is a subset of the relation. The "diagonal" of a relation is introduced in section 7. (Recall the mental picture, depicted in fig. 2, of the "diagonal" of the "staircase" relation depicted in fig. 1.)

Both the "diagonal" and the "difunctional closure" ("fermeture difonctionelle") are due to Riguet [Rig50, Rig51]; our contribution is partly historical —giving true credit to the original publications— , partly to make the constructions more accessible to modern readers, but primarily as an application of the calculational method.

6.1 Formal Definition and Characterisation

In this subsection we give the formal definition of a "difunctional relation" and state the theorem (theorem 132) that we prove in subsection 6.3. Theorem 132 uses the notion of a "characterisation" of a difunctional relation; this notion is also introduced in this subsection.

Formally, relation R is *difunctional* equivales

(130) $R \circ R^{\cup} \circ R \subseteq R$.

Equivalently, relation R is *difunctional* equivales

(131) $R \circ R^{\cup} \circ R = R$.

(The equivalence of these two definitions is a simple consequence of the properties of the domain operators. Definition (130) is more useful when it is required to establish that a particular relation is difunctional, whereas definition (131) is more useful when it is required to exploit the fact that a particular relation is difunctional.)

In order to relate this formal definition to the informal mental picture, an important step on the way is to characterise difunctional relations via a pair of functional relations. Recall that a relation R is said to be *functional* iff $R \circ R^{\cup} = R{<}$ (where $R{<}$ denotes the left domain of R: see definition 42). We use lower case letters f, g, h and k to denote functional relations. The theorem is the following.

Theorem 132 (Characterisation Theorem) For all relations R,

$$R \text{ is difunctional } \equiv \langle \exists f,g : f \circ f^{\cup} = f{<} = g \circ g^{\cup} = g{<} : R = f^{\cup} \circ g \rangle .$$

□

Theorem 132 —which is due to Riguet [Rig50]— is key to establishing the property that difunctional relations are exactly the relations that fit the mental picture shown in fig. 2 of a collection of completely disjoint rectangles. Later, we say that difunctional relations are "characterised" by a pair of functional relations. The formal definition is as follows.

Definition 133 A *characterisation (of a difunctional)* is a pair of functional relations with the same target (but possibly different sources). A *minimal characterisation (of a difunctional)* is a pair of relations f and g with the same target such that

$$f \circ f^{\cup} \;=\; f{<} \;=\; g \circ g^{\cup} \;=\; g{<} \quad .$$

That is, a minimal characterisation is a pair of functional relations with equal left domains.
□

The mental picture of a difunctional relation (fig. 2) is a set of completely disjoint rectangles. We can now make the picture precise.

Recall the definition of minimal characterisations, definition 133. Theorem 100 expresses the equivalence of minimal characterisations with sets of completely disjoint rectangles. So, by combining theorems 132 and 100, we have:

Theorem 134 A relation R is difunctional if and only if it is the supremum of a set of completely disjoint rectangles.
□

The "minimality" requirement —the domain restrictions on f and g— may be omitted ("without loss of generality" in mathematical jargon). It is necessary, however, to establishing the uniqueness of the characterisation. (See theorem 136.) Formally we have:

Lemma 135 Suppose f and g are functional relations with the same target. Then

$$f^{\cup} \circ g \quad = \quad (g{<} \circ f)^{\cup} \circ (f{<} \circ g) \quad .$$

Moreover, $g{<} \circ f$ and $f{<} \circ g$ are functional relations and

$$(g{<} \circ f) \circ (g{<} \circ f)^{\cup} \quad = \quad (g{<} \circ f){<} \quad = \quad (f{<} \circ g) \circ (f{<} \circ g)^{\cup} \quad = \quad (f{<} \circ g){<} \quad .$$

That is, the pair $g{<} \circ f$ and $f{<} \circ g$ is a minimal characterisation.

Proof We show that $g{<} \circ f$ is functional as follows.

$$\begin{array}{ll}
 & (g{<} \circ f) \circ (g{<} \circ f)^{\cup} \\
= & \{ \quad \text{associativity and converse} \quad \} \\
 & g{<} \circ f \circ f^{\cup} \circ g{<} \\
= & \{ \quad f \text{ is functional, so } f \circ f^{\cup} = f{<} \quad \} \\
 & g{<} \circ f{<} \circ g{<} \\
= & \{ \quad \text{coreflexives commute and are idempotent} \quad \} \\
 & f{<} \circ g{<} \quad .
\end{array}$$

Symmetrically,

$$(f< \circ g) \circ (f< \circ g)^{\cup} \;=\; g< \circ f< \;\;.$$

That is, $f< \circ g$ is functional. The lemma follows immediately from the fact that composition of coreflexives is symmetric and yields a coreflexive.

□

Warning Symmetry places a major rôle in reasoning about difunctional relations. (Obviously, R is difunctional equivales $R^{\cup}$ is difunctional.) But our definition of "functional" is asymmetric and reflects a right-to-left bias in our interpretation of relations as having inputs and outputs. Jaoua et al [JMBD91] choose a left-to-right interpretation: they use the term "deterministic" to mean $R^{\cup} \circ R \subseteq I$. Their formulation of theorem 132 is correspondingly different. See also our earlier warning on "symmetric division". **End of Warning**

The name "difunctional" is suggestive of theorem 132; Riguet's 1948 paper [Rig48, Proposition 11] introduces the notion and gives a (natural-language-based) proof. Riguet's 1950 paper [Rig50] states that it is a generalisation of the theorem that a relation R is a partial equivalence relation equivales $R = f^{\cup} \circ f$ for some functional relation f. Since then it appears to have become a folklore theorem. Hutton and Voermans [GE92, lemma 39], for example, state the theorem but do not provide a proof nor an attribution. The English text of [SS93, p.75] suggests that Schmidt and Ströhlein may be aware of the theorem but they also do not provide a proof. (They prove the easy "if" part of the theorem but not the converse; [SS93, Proposition 4.4.10] states that the characterisation "may be achieved *in essentially one* fashion" (their emphasis) but the accompanying proof actually establishes that the characterisation can be achieved in *at most* one fashion. That is, if such a characterisation exists, it is unique "up to a bijection".)

A theme of this section is how to formalise different proofs of theorem 132. One issue is whether or not the so-called "power transpose" of a relation, espoused by Freyd and Ščedrov [Fv90] and Bird and De Moor [BdM97], is sufficiently expressive. A second issue is the extent to which pointwise (as opposed to point-free) reasoning is desirable.

Section 6.2 sets the scene. The proof of theorem 132 is an "if-and-only-if" proof and the section begins with the (trivial) proof of the "if" part. The main task is thus to give an explicit construction of a characterisation of a given difunction (the "only-if" part). A formal theorem —theorem 136— states that although the details of the proof may be different, the constructed characterisations are formally equivalent (in a way made precise by the theorem). A very informal outline of several different ways of making the construction is then given.

The informal account in section 6.2 is made precise in sections 6.3.1 and 6.3.2; the former proves theorem 132 by showing how to construct a set of "rectangles" that "covers" a given difunctional relation whilst the latter presents a construction in terms of the "power transpose" of the given relation. Section 6.3.3 gives a third method of proving theorem 132 that exploits theorem 105. As already remarked —see theorem 134— no matter how a characterisation is constructed, it defines a "completely disjoint covering" of the given difunction.

6.2 Different Proofs, Identical Characterisations

The proof of theorem 132 is by mutual implication. Follows-from is straightforward. Assume

$$\langle \exists f,g : f \circ f^{\cup} = f_{<} = g \circ g^{\cup} = g_{<} : R = f^{\cup} \circ g \rangle \ .$$

Then

$$\begin{array}{ll} & R \circ R^{\cup} \circ R \\ = & \{ \quad \text{assumption and converse} \quad \} \\ & f^{\cup} \circ g \circ g^{\cup} \circ f \circ f^{\cup} \circ g \\ = & \{ \quad \text{assumption: } f \circ f^{\cup} = g_{<} = g \circ g^{\cup} \quad \} \\ & f^{\cup} \circ g_{<} \circ g_{<} \circ g \\ = & \{ \quad g_{<} \circ g = g \text{, and } R = f^{\cup} \circ g \quad \} \\ & R \ . \end{array}$$

The much more demanding task —which occupies all of subsection 6.3— is to establish the existence of a (minimal) characterisation of a given difunction. The theorem that there is "essentially *at most one*" is the following.

Theorem 136 Suppose f and g are relations such that

$$f \circ f^{\cup} = f_{<} = g \circ g^{\cup} = g_{<} \ .$$

Suppose also that h and k are relations such that

$$h \circ h^{\cup} = h_{<} = k \circ k^{\cup} = k_{<} \ .$$

Suppose further that

$$f^{\cup} \circ g = h^{\cup} \circ k \ .$$

Then, if ϕ is defined by $\phi = f \circ h^{\cup}$,

$$\phi \circ \phi^{\cup} = f_{<} \ \wedge \ \phi^{\cup} \circ \phi = h_{<} \ .$$

(In words, ϕ is a bijection with left domain the common left domain of f and g, and right domain the common left domain of h and k.) Moreover,

$$f = \phi \circ h \ \wedge \ g = \phi \circ k \ .$$

Proof

$$
\begin{array}{ll}
 & \phi \circ \phi^{\cup} \\
= & \{ \quad \text{definition, converse} \quad \} \\
 & f \circ h^{\cup} \circ h \circ f^{\cup} \\
= & \{ \quad \text{assumption:}\ \ h< \ = \ k \circ k^{\cup} \quad \} \\
 & f \circ h^{\cup} \circ k \circ k^{\cup} \circ h \circ f^{\cup} \\
= & \{ \quad \text{assumption:}\ \ f^{\cup} \circ g \ = \ h^{\cup} \circ k \quad \} \\
 & f \circ f^{\cup} \circ g \circ g^{\cup} \circ f \circ f^{\cup} \\
= & \{ \quad \text{assumption:}\ \ f \circ f^{\cup} \ = \ f< \ = \ g \circ g^{\cup} \quad \} \\
 & f<
\end{array}
$$

and

$$
\begin{array}{ll}
 & \phi^{\cup} \circ \phi \\
= & \{ \quad \text{definition, converse} \quad \} \\
 & h \circ f^{\cup} \circ f \circ h^{\cup} \\
= & \{ \quad \text{assumption:}\ \ f< \ = \ g \circ g^{\cup} \quad \} \\
 & h \circ f^{\cup} \circ g \circ g^{\cup} \circ f \circ h^{\cup} \\
= & \{ \quad \text{assumption:}\ \ f^{\cup} \circ g \ = \ h^{\cup} \circ k \ \ \text{(used twice)} \quad \} \\
 & h \circ h^{\cup} \circ k \circ k^{\cup} \circ h \circ h^{\cup} \\
= & \{ \quad \text{assumption:}\ \ h \circ h^{\cup} \ = \ h< \ = \ k \circ k^{\cup} \quad \} \\
 & h< \ \ .
\end{array}
$$

Finally,

$$
\begin{array}{ll}
 & \phi \circ h \\
= & \{ \quad \text{definition} \quad \} \\
 & f \circ h^{\cup} \circ h \\
= & \{ \quad \text{assumption:}\ \ h< \ = \ k \circ k^{\cup} \quad \} \\
 & f \circ h^{\cup} \circ k \circ k^{\cup} \circ h \\
= & \{ \quad \text{assumption:}\ \ f^{\cup} \circ g \ = \ h^{\cup} \circ k \ \ \text{(used twice)} \quad \} \\
 & f \circ f^{\cup} \circ g \circ g^{\cup} \circ f
\end{array}
$$

$=$ { assumption: $f \circ f^{\cup} = f< = g \circ g^{\cup}$ }

f .

The property $g = \phi \circ k$ is proved similarly.

□

As the name "functional" suggests, the only-if part of theorem 132 is established by defining a type C, for each a in the left domain of R, a point $f.a$ in C, and, for each point b in the right domain of R, a point $g.b$ in C. The requirement is that, $f.a$ and $g.b$ are equal exactly when a and b are related by R. Fig. 4 shows three different but isomorphic (in the sense of theorem 136) characterisations of the relation depicted in fig. 2.

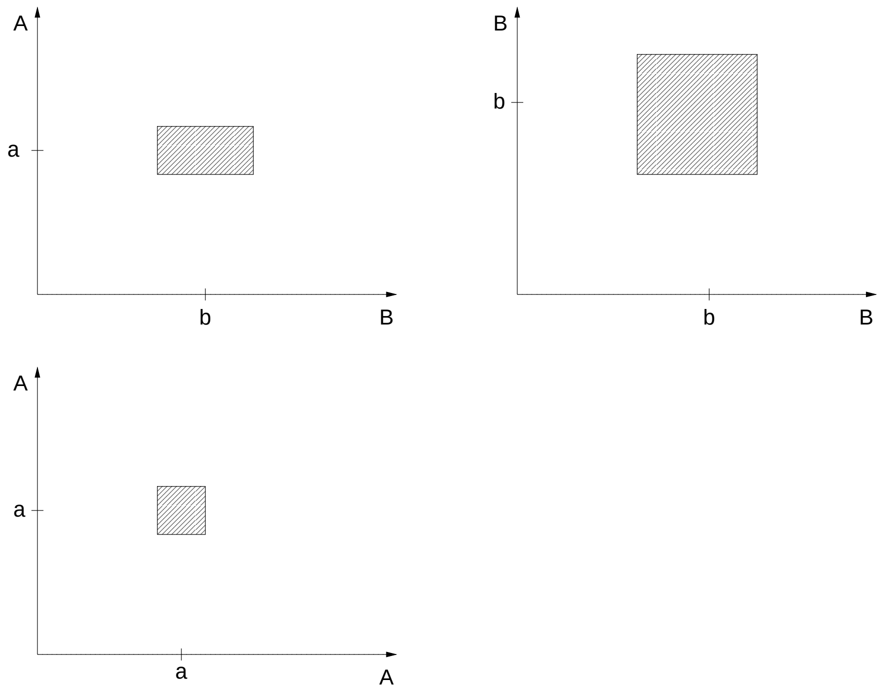


Figure 4: Three Different (but Isomorphic) Characterisations

In the top-left figure, the type C is the set of rectangles (relations of type $A{\sim}B$) defined by the relation R: the functional relation f maps a point a in the left domain of R to the rectangle defined by a and, similarly, the functional relation g maps a point b in the right domain of R to the rectangle defined by b. If a and b are points related by R, the rectangles $f.a$ and $g.b$ are equal; if a and b are not related by R, the rectangles $f.a$ and $g.b$ are not equal (and, in fact, they are "completely disjoint" in the sense that there are no points common to their sides).

In the top-right figure, the type C is a set of squares of type $B{\sim}B$ and, in the bottom-left figure the type C is a set of squares of type $A{\sim}A$. In the case of the top-right figure, the functional relation g maps point b to the square defined by b. The definition of f is more complicated: for a point a in the left domain of R, the value of $f.a$ is the square defined by some point b such that a and b are points related by R. The definitions of f and g are similar in the case of the bottom-left figure. (Just interchange the rôles of a and b.)

Of course, a "square" is defined by a "side" of the square. So there is a fourth and a fifth way of representing a difunctional relation as a pair of functional relations: the type C can be defined to be the set of subsets of the left domain of R or the set of subsets of the right domain of R and, in each case, appropriate definitions of f and g must be constructed.

As mentioned earlier, all of these characterisations are the same — in the sense made precise by theorem 136.

6.3 The Characterisation Theorem

As illustrated by fig. 4, there are three different ways to approach the proof[3] of theorem 132. The top-right and bottom-left figures are "dual" in the sense that one depicts a homogeneous relation on the target of the given relation whilst the other depicts a homogeneous relation on the source of the given relation. The top-left figure is more attractive because it does not exhibit any bias towards the source or target of the given relation. Section 6.3.1 presents such an unbiased proof of theorem 132 whilst section 6.3.2 presents the dual proofs. Section 6.3.3 gives yet another proof based on exploiting theorem 105.

6.3.1 The Rectangle Proof

A relation R is a partial equivalence relation exactly when $R\circ R^{\cup} = R$; the "classes" of R are the squares $R\circ a\circ R^{\cup}$ where a is a point such that $a\subseteq R$. (See section 5 and, in particular, theorem 115.) A relation R is a difunction exactly when $R\circ R^{\cup}\circ R = R$. By analogy and type considerations, this suggests that, if $a\subseteq R{<}$, the rectangle defined by a is given by $R\circ R^{\cup}\circ a\circ R$; similarly, if $b\subseteq R{>}$, the rectangle defined by b is given by $R\circ b\circ R^{\cup}\circ R$. This is the key to the proof.

Lemma 137 Suppose R of type $A{\sim}B$ is difunctional. Then, for all points a and b,

$$a\circ\top\!\top\circ b \subseteq R \;\;\Rightarrow\;\; R\circ R^{\cup}\circ a\circ R \;=\; R\circ b\circ R^{\cup}\circ a\circ R \;=\; R\circ b\circ R^{\cup}\circ R \;\; .$$

Proof Assume R is difunctional. Assume also that $a\circ\top\!\top\circ b \subseteq R$. Then

[3] Strictly, the "only-if" part of the proof. Recall from section 6.2 that the "if" part is trivial.

$$\begin{array}{ll} & R\circ b\circ R^{\cup}\circ R \\ = & \{\quad b \text{ is a point} \quad\} \\ & R\circ b\circ b\circ R^{\cup}\circ R \\ \subseteq & \{\quad \text{assumption: } a\circ\top\!\!\top\circ b \subseteq R \text{ , lemma 45} \quad\} \\ & R\circ b\circ R^{\cup}\circ a\circ R\circ R^{\cup}\circ R \\ \subseteq & \{\quad R \text{ is difunctional} \quad\} \\ & R\circ b\circ R^{\cup}\circ a\circ R \ . \end{array}$$

That is,

$$R\circ b\circ R^{\cup}\circ R \ \subseteq\ R\circ b\circ R^{\cup}\circ a\circ R \ .$$

By a symmetric argument

$$R\circ R^{\cup}\circ a\circ R \ \subseteq\ R\circ b\circ R^{\cup}\circ a\circ R \ .$$

But, since a is a point (and thus coreflexive),

$$R\circ b\circ R^{\cup}\circ a\circ R \ \subseteq\ R\circ b\circ R^{\cup}\circ R \ .$$

Symmetrically,

$$R\circ b\circ R^{\cup}\circ a\circ R \ \subseteq\ R\circ R^{\cup}\circ a\circ R \ .$$

The lemma follows by the anti-symmetry of equality.
□

The “only-if” part of theorem 132 is a consequence of lemma 137. Specifically, suppose R is difunctional. Let C be the set of subsets of the relation R defined as follows:

$$C \ = \ \{a : a\subseteq R< : R\circ R^{\cup}\circ a\circ R\} \ .$$

(The dummy a ranges over points.) Note that $C = C'$ where

$$C' \ = \ \{b : b\subseteq R> : R\circ b\circ R^{\cup}\circ R\}$$

since

$$\begin{array}{ll} & \{a : a\subseteq R< : R\circ R^{\cup}\circ a\circ R\} \\ = & \{\quad \text{domains} \quad\} \\ & \{a : \langle\exists b :: a\circ R\circ b = a\circ\top\!\!\top\circ b\rangle : R\circ R^{\cup}\circ a\circ R\} \\ = & \{\quad \text{range disjunction} \quad\} \end{array}$$

$\{a,b : a{\circ}R{\circ}b = a{\circ}\top\!\top{\circ}b : R \circ R^{\cup} \circ a \circ R\}$

$=$ { assumption: R is difunctional; lemma 137 }

$\{a,b : a{\circ}R{\circ}b = a{\circ}\top\!\top{\circ}b : R \circ b \circ R^{\cup} \circ R\}$

$=$ { range disjunction and domains (as in first two steps) }

$\{b : b \subseteq R> : R \circ b \circ R^{\cup} \circ R\}$.

Define f and g by, for all points a such that $a \subseteq R<$ and all points b such that $b \subseteq R>$,

(138) $f.a \;=\; R \circ R^{\cup} \circ a \circ R \quad\wedge\quad g.b \;=\; R \circ b \circ R^{\cup} \circ R$.

Then, by definition, f and g are both functional, and surjective onto C and C', respectively. That is —exploiting the fact that C and C' are equal—

$f \circ f^{\cup} = I_C = g \circ g^{\cup}$.

We must now show that $R = f^{\cup} \circ g$. Guided by the definitions of f and g, we calculate that:

$R \circ R^{\cup} \circ a \circ R \;=\; R \circ b \circ R^{\cup} \circ R$

$\Rightarrow$ { Leibniz }

$R \circ R^{\cup} \circ a \circ R \circ R^{\cup} \;=\; R \circ b \circ R^{\cup} \circ R \circ R^{\cup}$

$\Rightarrow$ { assumption: R is difunctional (thus so too is $R^{\cup}$),
$R< \subseteq R \circ R^{\cup}$ }

$R< \circ a \circ R< \subseteq R \circ b \circ R^{\cup}$

$=$ { assumption: $a \subseteq R<$ }

$a \subseteq R \circ b \circ R^{\cup}$

$=$ { lemma 45 }

$a{\circ}\top\!\top{\circ}b \subseteq R$

$\Rightarrow$ { assumption: R is difunctional; lemma 137 }

$R \circ R^{\cup} \circ a \circ R \;=\; R \circ b \circ R^{\cup} \circ R$.

We conclude (by mutual implication) that

$R \circ R^{\cup} \circ a \circ R \;=\; R \circ b \circ R^{\cup} \circ R \quad\equiv\quad a{\circ}\top\!\top{\circ}b \subseteq R$.

But, by the definitions of f and g and the definition of function application,

$R \circ R^{\cup} \circ a \circ R \;=\; R \circ b \circ R^{\cup} \circ R \quad\equiv\quad a{\circ}\top\!\top{\circ}b \;\subseteq\; f^{\cup} \circ g$.

Thus $R = f^{\cup} \circ g$ by the saturation axiom: (16).

6.3.2 The Power-Transpose Construction

Recalling fig. 4 once again, two alternative —but dual— ways of proving theorem 132 are to construct functional relations that return square relations. Equivalently, one can construct functional relations that return the "side" of such a square, i.e. a subset of the source or, dually, a subset of the target of the given difunctional relation. In this section, we present such a construction using the power transpose function. The proof was obtained by revising the proof given by Jaoua et al [JMBD91] in a way that eliminated the unnecessary assumption that R is homogeneous. One component of the characterisation is the relation $\Gamma R \circ R^{\cup}$. Since this is not obviously functional, we need a lemma to show that it is.

Lemma 139 For all relations R,

$$R \text{ is difunctional} \quad \equiv \quad \Gamma R \circ R^{\cup} \ \subseteq \ \Gamma(R \circ R^{\cup}) \circ R_{<} \ .$$

Proof

$$\begin{array}{ll}
 & \Gamma R \circ R^{\cup} \ \subseteq \ \Gamma(R \circ R^{\cup}) \circ R_{<} \\
= & \{ \quad \text{domains (specifically, } R^{\cup} \circ R_{<} = R^{\cup}) \quad \} \\
 & \Gamma R \circ R^{\cup} \ \subseteq \ \Gamma(R \circ R^{\cup}) \\
= & \{ \quad \Gamma R \text{ is a total function; shunting rule} \quad \} \\
 & R^{\cup} \ \subseteq \ (\Gamma R)^{\cup} \circ \Gamma(R \circ R^{\cup}) \\
= & \{ \quad \text{lemma 66} \quad \} \\
 & R^{\cup} \ \subseteq \ R \backslash (R \circ R^{\cup}) \ \cap \ ((R \circ R^{\cup}) \backslash R)^{\cup} \\
= & \{ \quad \text{converse is an order isomorphism, factors} \quad \} \\
 & R \circ R^{\cup} \subseteq R \circ R^{\cup} \quad \wedge \quad R \circ R^{\cup} \circ R \subseteq R \\
= & \{ \quad \text{definition} \quad \} \\
 & R \text{ is difunctional} \ .
\end{array}$$

□

Corollary 140 For all difunctional relations R,

$$(\Gamma R \circ R^{\cup}) \ \circ \ (\Gamma R \circ R^{\cup})^{\cup} \ = \ \Gamma R \circ R_{>} \circ (\Gamma R)^{\cup} \ .$$

In particular, if R is difunctional, $\Gamma R \circ R^{\cup}$ is functional.

Proof The proof is by mutual inclusion. First, for all relations R,

$$(\Gamma R \circ R^\cup) \circ (\Gamma R \circ R^\cup)^\cup$$
$= \quad \{ \quad$ converse $\quad \}$
$$\Gamma R \circ R^\cup \circ R \circ (\Gamma R)^\cup$$
$\supseteq \quad \{ \quad R^\cup \circ R \supseteq R_> \;$, monotonicity $\quad \}$
$$\Gamma R \circ R_> \circ (\Gamma R)^\cup \;\; .$$

Second, for all difunctional relations R,

$$\Gamma R \circ R^\cup \circ R \circ (\Gamma R)^\cup \;\subseteq\; \Gamma R \circ R_> \circ (\Gamma R)^\cup$$
$\Leftarrow \quad \{ \quad$ assumption: R is difunctional; lemma 139 $\quad \}$
$$\Gamma(R \circ R^\cup) \circ R_< \circ (\Gamma(R \circ R^\cup))^\cup \;\subseteq\; \Gamma R \circ R_> \circ (\Gamma R)^\cup$$
$= \quad \{ \quad \Gamma(R \circ R^\cup)$ is a total function, shunting : (64), and (67) $\quad \}$
$$R_< \;\subseteq\; (R \circ R^\cup)\backslash\!\!\backslash R \circ R_> \circ ((R \circ R^\cup)\backslash\!\!\backslash R)^\cup$$
$\Leftarrow \quad \{ \quad$ domains (specifically $R_< \subseteq R \circ R^\cup$ and $R = R \circ R_>$) $\quad \}$
$$R \circ R_> \circ R^\cup \;\subseteq\; (R \circ R^\cup)\backslash\!\!\backslash R \circ R_> \circ ((R \circ R^\cup)\backslash\!\!\backslash R)^\cup$$
$\Leftarrow \quad \{ \quad$ monotonicity and converse $\quad \}$
$$R \;\subseteq\; (R \circ R^\cup)\backslash\!\!\backslash R$$
$= \quad \{ \quad$ assumption: R is difunctional

as in last two steps of proof of lemma 139 $\quad \}$

true .

□

Theorem 141 Suppose R is a difunctional relation. Then the relations $\Gamma R \circ R^\cup$ and $\Gamma R \circ R_>$ are both functional. Moreover,

$$(\Gamma R \circ R^\cup) \circ (\Gamma R \circ R^\cup)^\cup \quad = \quad (\Gamma R \circ R_>) \circ (\Gamma R \circ R_>)^\cup$$

and

$$R \quad = \quad (\Gamma R \circ R^\cup)^\cup \circ (\Gamma R \circ R_>) \;\; .$$

That is, these two relations fulfill the requirements of f and g in theorem 132.

Dually, the relations $\Gamma(R^\cup) \circ R$ and $\Gamma(R^\cup) \circ R_<$ are both functional. Moroever,

$$(\Gamma(R^\cup) \circ R_<) \circ (\Gamma(R^\cup) \circ R_<)^\cup \quad = \quad (\Gamma(R^\cup) \circ R) \circ (\Gamma(R^\cup) \circ R)^\cup$$

and

$$R \quad = \quad (\Gamma(R^\cup) \circ R_<)^\cup \circ (\Gamma(R^\cup) \circ R) \;\; .$$

That is, these two functions also fulfill the requirements of f and g theorem 132.

Proof That $\Gamma R \circ R_>$ is functional is immediate from the fact that ΓR is a total function (by definition) and $R_>$ is a subset of the identity relation. That $\Gamma R \circ R^\cup$ is functional was shown in corollary 140. It remains to prove the final equation.

$$
\begin{array}{ll}
 & (\Gamma R \circ R^\cup)^\cup \circ (\Gamma R \circ R_>) \\
= & \{ \quad \text{converse} \quad \} \\
 & R \circ (\Gamma R)^\cup \circ \Gamma R \circ R_> \\
= & \{ \quad (67) \quad \} \\
 & R \circ R \backslash\!\backslash R \circ R_> \\
= & \{ \quad \text{lemma 69} \quad \} \\
 & R \circ R_> \\
= & \{ \quad \text{domains} \quad \} \\
 & R \quad .
\end{array}
$$

The dual theorem is obtained by instantiating R to $R^\cup$ (and noting that R is difunctional equivales $R^\cup$ is difunctional) and simplifying.
□

Theorem 123 is an instance of theorem 141. In order to show that this is the case, it is necessary to prove that, for a per R,

$$\Gamma R \circ R^\cup \;=\; \Gamma R \circ R_> \quad .$$

This is done as follows:

$$
\begin{array}{ll}
 & \Gamma R \circ R^\cup \;=\; \Gamma R \circ R_> \\
= & \{ \quad R \text{ is a per, so } R^\cup = R\text{; lemma 101} \quad \} \\
 & \Gamma R \circ R \;\subseteq\; \Gamma R \circ R_> \\
\Leftarrow & \{ \quad \Gamma R \text{ is functional} \quad \} \\
 & R \;\subseteq\; (\Gamma R)^\cup \circ \Gamma R \circ R_> \\
= & \{ \quad \text{lemma 66} \quad \} \\
 & R \;\subseteq\; R \backslash\!\backslash R \circ R_> \\
= & \{ \quad \text{theorem 72} \quad \} \\
 & \text{true} \quad .
\end{array}
$$

6.3.3 The Per Construction

The third method of proving theorem 132 exploits theorem 105. So far as we are aware, the construction was first documented by Winter [Win04] but it was probably already known to Riguet [Rig50]. (See section 10.)

The basis for the construction is the construction of a per from a difunctional:

Lemma 142 For all relations R, $R\circ R^{\cup}$ is a per if R is difunctional.

Proof Suppose R is difunctional. We exploit theorem 115:

$$
\begin{array}{ll}
 & R\circ R^{\cup} \text{ is a per} \\
= & \{ \quad \text{theorem 115 with } R := R\circ R^{\cup} \text{ and converse} \quad \} \\
 & R\circ R^{\cup} \;=\; R\circ R^{\cup}\circ R\circ R^{\cup} \\
\Leftarrow & \{ \quad \text{Leibniz} \quad \} \\
 & R \;=\; R\circ R^{\cup}\circ R \\
= & \{ \quad (131) \quad \} \\
 & R \text{ is difunctional.}
\end{array}
$$

□

Suppose now that R is difunctional. Exploiting lemma 142 combined with theorem 105,

(143) $\langle \exists f : f\circ f^{\cup} \;=\; f< \;:\; R\circ R^{\cup} = f^{\cup}\circ f \rangle$.

Suppose therefore that $f\circ f^{\cup} \;=\; f<$ and $R\circ R^{\cup} = f^{\cup}\circ f$. Define the relation g by

(144) $g \;=\; f\circ R$.

Then

$$
\begin{array}{ll}
 & g\circ g^{\cup} \\
= & \{ \quad (144) \text{ and converse} \quad \} \\
 & f\circ R\circ R^{\cup}\circ f^{\cup} \\
= & \{ \quad (143) \quad \} \\
 & f\circ f^{\cup}\circ f\circ f^{\cup} \\
= & \{ \quad (143) \quad \} \\
 & f<\circ f< \\
= & \{ \quad f< \text{ is a coreflexive} \quad \} \\
 & f< \;.
\end{array}
$$

It follows that $g{<} = g \circ g^{\cup}$. Thus

(145) $\quad f \circ f^{\cup} \;=\; f{<} = g{<} = g \circ g^{\cup} \quad .$

Moreover,

$$
\begin{array}{ll}
 & f^{\cup} \circ g \\
= & \{ \qquad (144) \quad \} \\
 & f^{\cup} \circ f \circ R \\
= & \{ \qquad R \circ R^{\cup} = f^{\cup} \circ f \quad \} \\
 & R \circ R^{\cup} \circ R \\
= & \{ \qquad R \text{ is difunctional: } (131) \quad \} \\
 & R \quad .
\end{array}
$$

Combined with (145), we have thus shown that

(146) $\quad \langle \exists f,g \;\; : \;\; f \circ f^{\cup} \;=\; f{<} \;=\; g \circ g^{\cup} \;=\; g{<} \;\; : \;\; R = f^{\cup} \circ g \rangle$

as required to prove the only-if part of theorem 132.

Winter does not prove theorem 105; instead he assumes the theorem is valid. So, although his proof of theorem 132 is very short and elegant, the real effort goes into proving theorem 105. It is interesting to compare the details of Winter's construction with the functionals constructed in theorem 141. Applying the instantiation $R := R \circ R^{\cup}$ in theorem 105 and simplifying, Winter's construction yields

$$R \quad = \quad (\Gamma(R \circ R^{\cup}) \circ R{<})^{\cup} \;\circ\; (\Gamma(R \circ R^{\cup}) \circ R) \quad .$$

This is, of course, an isomorphic characterisation of R in the sense of theorem 136. Recalling our earlier informal account of how to prove the theorem, the construction corresponds in essence to the bottom-left figure of fig. 4.

6.4 Difunctional Closure

The diagonal of a relation R is a subset of R. Because a difunctional relation is a prefix point of a monotonic function (the function $\langle X :: X \circ X^{\cup} \circ X \rangle$) fixed-point calculus predicts that the least prefix point

$$\langle \mu X \;::\; R \cup X \circ X^{\cup} \circ X \rangle$$

is the least difunctional relation that includes R — the *difunctional closure* of R. More precisely,

$$\langle \mu X \;::\; R \cup X \circ X^{\cup} \circ X \rangle \text{ is difunctional}$$

and

$$\langle \forall S : S \circ S^{\cup} \circ S \subseteq S : R \subseteq S \equiv \langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle \subseteq S \rangle \ .$$

(The general theorem is that, if f is a monotonic endofunction on a complete lattice, the function $f^{\star}$ defined by

$$f^{\star}.x = \langle \mu y :: x \sqcup f.y \rangle$$

has the property that

$$\langle \forall y \ : \ f.y \sqsubseteq y \ : \ x \sqsubseteq y \ \equiv \ f^{\star}.x \sqsubseteq y \rangle \ .$$

The straightforward proof is left to the reader. Examples include the transitive closure and the reflexive-transitive closure of a relation. See [Bac02] for an exposition of the techniques involved.)

In this section, we explore simplifications of the definition of difunctional closure.

The following theorem expresses the same result but in more familiar terms (specifically in terms of the reflexive-transitive closure operator).

Theorem 147 (Difunctional Closure) For all relations R,

$$\langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle \ = \ \langle \mu X :: R \cup X \circ R^{\cup} \circ X \rangle \ .$$

Hence,

$$\langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle \ = \ R \circ (R^{\cup} \circ R)^{*} \ .$$

Also,

$R \circ (R^{\cup} \circ R)^{*}$ is difunctional

and

$$\langle \forall S \ : \ S \circ S^{\cup} \circ S \subseteq S \ : \ R \subseteq S \ \equiv \ R \circ (R^{\cup} \circ R)^{*} \subseteq S \rangle \ .$$

(Thus $\langle R :: R \circ (R^{\cup} \circ R)^{*} \rangle$ is the upper adjoint in a Galois connection (of the relations of a given type and the difunctional relations of the same type) of the function that “forgets” that a difunctional relation is indeed difunctional.)

Proof We establish the equality by mutual inclusion. We begin by noting that the equality

$$\langle \mu X :: R \cup X \circ R^{\cup} \circ X \rangle \ = \ R \circ (R^{\cup} \circ R)^{*}$$

is an instance of (the possibly little known) exercise 67(c) in [Bac02]. Also

$$\langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle$$
$$= \quad \{ \quad \text{diagonal rule of fixed-point calculus} \quad \}$$
$$\langle \mu X :: \langle \mu Y :: R \cup Y \circ X^{\cup} \circ Y \rangle \rangle$$
$$= \quad \{ \quad \text{[Bac02, exercise 67(c)]} \quad \}$$
$$\langle \mu X :: R \circ (X^{\cup} \circ R)^{*} \rangle \quad .$$

So

$$\langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle \subseteq \langle \mu X :: R \cup X \circ R^{\cup} \circ X \rangle$$
$$= \quad \{ \quad \text{above} \quad \}$$
$$\langle \mu X :: R \circ (X^{\cup} \circ R)^{*} \rangle \subseteq R \circ (R^{\cup} \circ R)^{*}$$
$$\Leftarrow \quad \{ \quad \text{fixed-point induction} \quad \}$$
$$R \circ ((R \circ (R^{\cup} \circ R)^{*})^{\cup} \circ R)^{*} \subseteq R \circ (R^{\cup} \circ R)^{*}$$
$$= \quad \{ \quad \text{properties of converse} \quad \}$$
$$R \circ ((R^{\cup} \circ R)^{*} \circ R^{\cup} \circ R)^{*} \subseteq R \circ (R^{\cup} \circ R)^{*}$$
$$\Leftarrow \quad \{ \quad \text{Leibniz and reflexivity of the subset relation} \quad \}$$
$$((R^{\cup} \circ R)^{*} \circ R^{\cup} \circ R)^{*} = (R^{\cup} \circ R)^{*}$$
$$= \quad \{ \quad \text{properties of reflexive-transitive closure} \quad \}$$
$$\mathsf{true} \quad .$$

For the converse, we have:

$$\langle \mu X :: R \cup X \circ R^{\cup} \circ X \rangle \subseteq \langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle$$
$$= \quad \{ \quad \text{for brevity, let } rhs \text{ denote } \langle \mu X :: R \cup X \circ X^{\cup} \circ X \rangle \quad \}$$
$$\langle \mu X :: R \cup X \circ R^{\cup} \circ X \rangle \subseteq rhs$$
$$\Leftarrow \quad \{ \quad \text{fixed-point induction} \quad \}$$
$$R \cup rhs \circ R^{\cup} \circ rhs \subseteq rhs$$
$$= \quad \{ \quad \text{fixed-point computation and definition of } rhs \quad \}$$
$$R \cup rhs \circ R^{\cup} \circ rhs \subseteq R \cup rhs \circ rhs^{\cup} \circ rhs$$
$$\Leftarrow \quad \{ \quad \text{monotonicity} \quad \}$$
$$R \subseteq rhs$$
$$= \quad \{ \quad \text{fixed-point computation and definition of } rhs \quad \}$$
$$\mathsf{true} \quad .$$

□

Theorem is observed by Jaoua et al [JMBD91, Proposition 4.12] but is expressed using the definition of S^* as the sum of powers of S. Their (incomplete) proof uses induction over the natural numbers. Just as the notion of the “différence” of a relation is due to Riguet [Rig51], theorem 147 is also due to Riguet [Rig50]. He calls the relation $R\circ(R^{\cup}\circ R)^{+}$ the “difunctional closure” (“fermeture difonctionelle”) of R. Note the difference. This suggests that there is a mistake in Riguet’s definition or in theorem 147. In fact, both are correct:

Lemma 148 For arbitrary relation R,

$$R \subseteq R\circ R^{\cup}\circ R \quad .$$

It follows that, for all relations R,

$$R\circ(R^{\cup}\circ R)^{+} \;=\; R\circ(R^{\cup}\circ R)^{*} \quad .$$

Proof We have:

$$R\circ R^{\cup}\circ R$$

$\supseteq$ { monotonicity }

$$R\circ(I\cap R^{\cup}\circ R)$$

$\supseteq$ { modular law }

$$R\circ I\cap R$$

$=$ { I is identity of composition, infimum is idempotent }

$$R \quad .$$

So,

$$R\circ(R^{\cup}\circ R)^{+} \;=\; R\circ(R^{\cup}\circ R)^{*}$$

$=$ { fixed-point computation and distributivity }

$$R\circ(R^{\cup}\circ R)^{+} \;=\; R\circ(R^{\cup}\circ R)^{+}\cup R$$

$=$ { supremum }

$$R \subseteq R\circ(R^{\cup}\circ R)^{+}$$

$\Leftarrow$ { fixed-point computation and distributivity }

$$R \subseteq R\circ R^{\cup}\circ R$$

$=$ { above }

$$\mathsf{true} \quad .$$

□

7 The Diagonal

This section anticipates the study of block-ordered relations in section 8. We introduce the notion of the “diagonal” of a relation in section 7.1 and formulate some basic properties. We then introduce the notion of a “non-redundant”, “polar” covering of a relation by rectangles in section 7.2. We prove that every relation has a polar covering but that not every relation has a non-redundant polar covering. Then, in section 7.3, we explore conditions under which the diagonal of the relation guarantees the non-redundancy of the covering.

7.1 Definition and Basic Properties

Straightforwardly from the definition of factors, properties of converse and set intersection,

(149) R is difunctional $\quad\equiv\quad \mathsf{R} = \mathsf{R}\cap(\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup}$.

More generally, we have:

Lemma 150 For all R, $\mathsf{R}\cap(\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup}$ is difunctional.

Proof Let S denote $\mathsf{R}\cap(\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup}$. We have to prove that S is difunctional. That is, by definition,

$$\mathsf{S}\circ\mathsf{S}^{\cup}\circ\mathsf{S} \subseteq \mathsf{S} \ .$$

Since the right side is an intersection, this is equivalent to

$$\mathsf{S}\circ\mathsf{S}^{\cup}\circ\mathsf{S} \subseteq \mathsf{R} \ \wedge\ \mathsf{S}\circ\mathsf{S}^{\cup}\circ\mathsf{S} \subseteq (\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup} \ .$$

The first is (almost) trivial:

$\mathsf{S}\circ\mathsf{S}^{\cup}\circ\mathsf{S}$

$\subseteq$ { $\mathsf{S}\subseteq\mathsf{R}$, $\mathsf{S}\subseteq(\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup}$,
converse, monotonicity }

$\mathsf{R}\circ\mathsf{R}\backslash\mathsf{R}/\mathsf{R}\circ\mathsf{R}$

$\subseteq$ { cancellation }

R .

In the above calculation, the trick was to replace the outer occurrences of S on the left side by R and the middle occurrence by $(\mathsf{R}\backslash\mathsf{R}/\mathsf{R})^{\cup}$. The replacement is done the opposite way around in the second calculation.

$$
\begin{array}{ll}
& S\circ S^{\cup}\circ S \;\subseteq\; (R\backslash R/R)^{\cup} \\
\Leftarrow & \{ \quad S\subseteq (R\backslash R/R)^{\cup}\,,\; S\subseteq R\,,\ \text{monotonicity and transitivity} \quad \} \\
& (R\backslash R/R)^{\cup}\circ R^{\cup}\circ (R\backslash R/R)^{\cup} \;\subseteq\; (R\backslash R/R)^{\cup} \\
= & \{ \quad \text{converse} \quad \} \\
& R\backslash R/R\circ R\circ R\backslash R/R \;\subseteq\; R\backslash R/R \\
= & \{ \quad \text{Galois connection} \quad \} \\
& R\circ R\backslash R/R\circ R\circ R\backslash R/R\circ R \;\subseteq\; R \\
= & \{ \quad \text{cancellation, monotonicity and transitivity} \quad \} \\
& \mathsf{true}\ \ .
\end{array}
$$

□

In order to reflect the mental picture of a difunctional relation, we call the relation $R\cap(R\backslash R/R)^{\cup}$ the *diagonal* of R; Riguet [Rig51] calls it the “différence” of the relation. (Riguet’s definition does not use factors but is equivalent.)

Definition 151 (Diagonal) The *diagonal* of relation R is the relation $R\cap(R\backslash R/R)^{\cup}$. For brevity, $R\cap(R\backslash R/R)^{\cup}$ will sometimes be denoted by ΔR.

□

Primarily for notational convenience, we note a simple property of the diagonal:

Lemma 152

$$(\Delta R)^{\cup} \;=\; \Delta(R^{\cup})\ \ .$$

Proof

$$
\begin{array}{ll}
& (\Delta R)^{\cup} \\
= & \{ \quad \text{definition and distributivity} \quad \} \\
& R^{\cup}\cap R\backslash R/R \\
= & \{ \quad \text{factors} \quad \} \\
& R^{\cup}\cap (R^{\cup}\backslash R^{\cup}/R^{\cup})^{\cup} \\
= & \{ \quad \text{definition} \quad \} \\
& \Delta(R^{\cup})\ \ .
\end{array}
$$

□

A consequence of lemma 152 is that we can write $\Delta R^{\cup}$ without ambiguity. This we do from now on.

Very straightforwardly, the relation $R \circ R^{\cup}$ is a per if R is difunctional. For a difunctional R, the relation $R \circ R^{\cup}$ is the per representation of the left domain of R. Symmetrically, $R^{\cup} \circ R$ is the per representation of the right domain of R. Thus $\Delta R \circ (\Delta R)^{\cup}$ is the per representation of the left domain of the diagonal of R. The following lemma is the basis of the construction, in certain cases, of an economic representation of the diagonal of R and, hence, of R itself. See definition 155 and theorems 161 and 167.

Lemma 153 For all relations R,

$$\Delta R \circ \Delta R^{\cup} \;=\; (\Delta R)_{<} \circ R /\!\!/ R \quad .$$

Dually,

$$\Delta R^{\cup} \circ \Delta R \;=\; (\Delta R)_{>} \circ R \backslash\!\!\backslash R \quad .$$

Consequently,

$$\Delta R^{\cup} \;=\; \Delta R^{\cup} \circ R /\!\!/ R$$

and

$$\Delta R \;=\; \Delta R \circ R \backslash\!\!\backslash R \quad .$$

Proof We prove the first equation by mutual inclusion. First,

$$\begin{array}{ll}
 & \Delta R \circ \Delta R^{\cup} \subseteq (\Delta R)_{<} \circ R /\!\!/ R \\
= & \{ \quad \text{domains} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \subseteq R /\!\!/ R \\
= & \{ \quad \text{definition of } R /\!\!/ R\text{, converse and factors} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ R \subseteq R \\
= & \{ \quad \Delta R \subseteq R\text{; } \Delta R^{\cup} \subseteq R \backslash R / R \text{ and cancellation} \quad \} \\
 & \text{true} \quad .
\end{array}$$

Second,

$$\begin{array}{ll}
 & (\Delta R)_{<} \circ R /\!\!/ R \subseteq \Delta R \circ \Delta R^{\cup} \\
\Leftarrow & \{ \quad \text{domains} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ R /\!\!/ R \subseteq \Delta R \circ \Delta R^{\cup} \\
\Leftarrow & \{ \quad \text{monotonicity and converse} \quad \} \\
 & R /\!\!/ R \circ \Delta R \subseteq \Delta R
\end{array}$$

$$
\begin{array}{ll}
= & \{ \quad \text{definition of diagonal} \quad \} \\
 & R/\!\!/R \circ \Delta R \subseteq R \;\; \wedge \;\; R/\!\!/R \circ \Delta R \subseteq (R\backslash R/R)^{\cup} \\
\Leftarrow & \{ \quad \Delta R \subseteq R \text{ ; converse} \quad \} \\
 & R/\!\!/R \circ R \subseteq R \;\; \wedge \;\; \Delta R^{\cup} \circ R/\!\!/R \subseteq R\backslash R/R \\
= & \{ \quad \text{cancellation; factors} \quad \} \\
 & \text{true} \;\; \wedge \;\; R \circ \Delta R^{\cup} \circ R/\!\!/R \circ R \subseteq R \\
\Leftarrow & \{ \quad \text{cancellation and } \Delta R^{\cup} \subseteq R\backslash R/R \quad \} \\
 & R \circ R\backslash R/R \circ R \subseteq R \\
= & \{ \quad \text{cancellation} \quad \} \\
 & \text{true} \;\; .
\end{array}
$$

Also,

$$
\begin{array}{ll}
 & \Delta R^{\cup} \\
= & \{ \quad \Delta R^{\cup} \text{ is difunctional} \quad \} \\
 & \Delta R^{\cup} \circ \Delta R \circ \Delta R^{\cup} \\
= & \{ \quad \text{above} \quad \} \\
 & \Delta R^{\cup} \circ (\Delta R)_{<} \circ R/\!\!/R \\
= & \{ \quad \text{domains} \quad \} \\
 & \Delta R^{\cup} \circ R/\!\!/R \;\; .
\end{array}
$$

The dual properties are obtained by instantiating R to $R^{\cup}$ and simplifying using properties of converse.

□

The following corollary of lemma 153 proves to be crucial later: see the discussion following lemma 193.

Lemma 154 For all relations R,

$$\Delta R \circ \Delta R^{\cup} \circ R = R \;\; \equiv \;\; (\Delta R)_{<} = R_{<} \;\; .$$

Dually,

$$R \circ \Delta R^{\cup} \circ \Delta R = R \;\; \equiv \;\; (\Delta R)_{>} = R_{>} \;\; .$$

Proof

$$\Delta R \circ \Delta R^{\cup} \circ R \ = \ R$$

$= \quad \{ \quad$ lemma 153 $\quad \}$

$$(\Delta R)_{<} \circ R /\!\!/ R \circ R \ = \ R$$

$= \quad \{ \quad$ cancellation $\quad \}$

$$(\Delta R)_{<} \circ R \ = \ R$$

$= \quad \{ \quad$ domains $\quad \}$

$$(\Delta R)_{<} \ \supseteq \ R_{<}$$

$= \quad \{ \quad \Delta R \subseteq R$; so, by monotonicity, $(\Delta R)_{<} \ \subseteq \ R_{<}$
anti-symmetry of the subset relation $\quad \}$

$$(\Delta R)_{<} = R_{<} \quad .$$

□

7.2 Completely Disjoint Subrectangles

We continue our investigation of the consequences of the case where the left and right domains of R equal the left and right domains (respectively) of ΔR. Specifically, we present a refinement of theorem 134. The refined theorem is relevant to the construction of maximal, "non-redundant" coverings of arbitrary relations.

Definition 155 (Polar covering) Suppose $\mathcal{R}$ is an indexed set of rectangles. (See definition 85.) Then $\mathcal{R}$ is said to be *polar* if, for all elements U and V of $\mathcal{R}$,

$$U_{<} \subseteq V_{<} \ \equiv \ U_{>} \supseteq V_{>} \quad .$$

Also, $\mathcal{R}$ is said to be *linear* if, for all elements U and V of $\mathcal{R}$,

$$U_{<} \subseteq V_{<} \quad \vee \quad V_{<} \subseteq U_{<} \quad .$$

(Equivalently,

$$U_{>} \subseteq V_{>} \quad \vee \quad V_{>} \subseteq U_{>} \quad .)$$

A relation R is *covered* by $\mathcal{R}$ if $R = \cup \mathcal{R}$. The covering $\mathcal{R}$ is *non-redundant* if there is a total function $\mathcal{D}$ from indices of $\mathcal{R}$ to a set of completely disjoint subrectangles of $\cup \mathcal{R}$ that "defines" the elements of $\mathcal{R}$. To be precise, the covering $\mathcal{R}$ is *non-redundant* if there is a function $\mathcal{D}$ with the same source as $\mathcal{R}$ such that

$$\begin{aligned} & \langle \forall k \ :: \ \text{rectangle}.(\mathcal{D}.k) \wedge \mathcal{D}.k \subseteq \mathcal{R}.k \rangle \\ \wedge \ & \langle \forall j,k \ :: \ \mathcal{D}.j = \mathcal{D}.k \ \not\equiv \ (\mathcal{D}.j)_{<} \cap (\mathcal{D}.k)_{<} = \perp\!\!\perp \ \wedge \ (\mathcal{D}.j)_{>} \cap (\mathcal{D}.k)_{>} = \perp\!\!\perp \rangle \\ \wedge \ & \langle \forall j,k \ :: \ \mathcal{D}.j = \mathcal{D}.k \ \equiv \ \mathcal{R}.j = \mathcal{R}.k \rangle \quad . \end{aligned}$$

In such a case, we call the indexed set $\mathcal{D}$ a *definiens* of $\mathcal{R}$.
□

The adjective "polar" alludes to the property that the left and right domains of a covering are "polar" opposites: the larger the one, the smaller the other. The notion was introduced by Riguet [Rig51] in the context of a theorem on "relations de Ferrers". More precisely, Riguet introduced the notion of a *linear* polar covering. For further details of Riguet's theorem see section 9.

As we shall see, Riguet's theorem is straightforward. The following, equally straightforward theorem, is a generalisation of the "only-if" part of the theorem.

Theorem 156 Suppose R is a relation of type $A{\sim}B$. Define the function $\mathcal{R}$ with index set $\{b: b \subseteq R{>}\}$ by

$$\mathcal{R} \;=\; \langle b : b \subseteq R{>} : R \circ b \circ R\backslash R \rangle \quad .$$

Then $\mathcal{R}$ is a polar covering of R.

Proof The elements of $\mathcal{R}$ are obviously rectangles because its index set is a set of points. (See lemma 81.) The property

$$R \;=\; \langle \cup b : b \subseteq R{>} : R \circ b \circ R\backslash R \rangle$$

is immediate from the saturation axiom (16), distributivity and cancellation.

The "polar" property is established as follows. For all b, b' such that $b \subseteq R{>}$ and $b' \subseteq R{>}$,

$$
\begin{array}{ll}
& (R \circ b' \circ R\backslash R){>} \subseteq (R \circ b \circ R\backslash R){>} \\
= & \{ \quad \text{assumption: } b \subseteq R{>} \text{ and } b' \subseteq R{>} \text{, domains} \quad \} \\
& (b' \circ R\backslash R){>} \subseteq (b \circ R\backslash R){>} \\
= & \{ \quad \text{lemma 48 with } R,a,a' := R\backslash R\,,b\,,b' \quad \} \\
& b \circ \top\!\!\top \circ b' \subseteq (R\backslash R)/(R\backslash R) \\
= & \{ \quad (30) \quad \} \\
& b \circ \top\!\!\top \circ b' \subseteq R\backslash R \\
= & \{ \quad \text{lemma 48} \quad \} \\
& (R \circ b){<} \subseteq (R \circ b'){<} \\
= & \{ \quad I \subseteq R\backslash R\text{, domains} \quad \} \\
& (R \circ b \circ R\backslash R){<} \subseteq (R \circ b' \circ R\backslash R){<} \quad .
\end{array}
$$

□

Example 157 The less-than relation has a polar covering. Specifically, suppose x is a real number. Let $lt.x$ denote $\{y : y\in\mathbb{R} : y < x\}$ and $al.x$ denote $\{y : y\in\mathbb{R} : x \leq y\}$. Theorem 156 predicts that

$$\{x : x\in\mathbb{R} : lt.x \circ \top\!\!\top \circ al.x\}$$

is a polar covering of the less-than relation. (The only non-trivial part is to check that the at-most relation $\leq$ equals $<\backslash<$.)

This covering is, of course, not unique. More significantly, it is *not* non-redundant since

$$\langle \forall u,v \quad : \quad u < x \leq v \quad : \quad x \neq \tfrac{1}{2}(u{+}x) \;\wedge\; u < \tfrac{1}{2}(u{+}x) \leq v \rangle \quad .$$

For any real number x, it is possible to remove the rectangle defined by x without affecting the supremum.
□

Given the straightforwardness of theorem 156, it is inevitable that our focus is not on the polarity of coverings but on the existence of *non-redundant* coverings. The adjective "non-redundant" is meant to express the property that removal of any element from a covering $\mathcal{R}$ will have the effect of strictly reducing $\cup\mathcal{R}$. (Removal of an element may involve removing several elements of K since there is no requirement that $\mathcal{R}$ is injective.) Example 157 demonstrates that the less-than relation on real numbers has a polar covering but, as we shall see, the less-than relation is an example of a relation for which there is no non-redundant covering.

The notation "$\mathcal{D}$" in definition 155 is chosen primarily to express the property that $\mathcal{D}.k$ uniquely "defines" (or "identifies") $\mathcal{R}.k$. Conveniently, it also expresses the property that the relation covered by a definiens (the relation $\cup\mathcal{D}$) is always difunctional: see theorem 134.

A polar covering is not *obviously redundant* in the sense that, for all elements U and V of $\mathcal{R}$,

$$U \subseteq V \;\equiv\; U = V \quad .$$

(The easy proof is left to the reader.) That is, it is not possible to identify two elements U and V such that U is a proper subset of V and, thus, U can be removed from $\mathcal{R}$ without affecting $\cup\mathcal{R}$. Example 157 shows that the less-than relation on real numbers has a polar covering that has non-obvious redundancies. Example 158 is an example of a finite relation for which the polar covering constructed by theorem 156 has a non-obvious redundancy.

Aside: This does not mean that there are no duplications in a polar covering of a given relation R: our definition of a polar covering does not exclude the possibility of there being distinct indices j and k such that $\mathcal{R}.j = \mathcal{R}.k$. In general, this will be the case for the construction given in theorem 156. This can be remedied by taking as index set the equivalence classes of the per $R\succ$. With req being a functional relation such that $R\succ = req^{\cup} \circ req$ (so, for all b such that $b \subseteq R_{>}$, $req.b$ is the equivalence class of b according to the right per domain $R\succ$), the function $\mathcal{R}$ defined by

$$\mathcal{R} \;=\; \langle c : c \subseteq req_{<} : R \circ req^{\cup} \circ c \circ req \circ R\backslash R \rangle$$

is a polar covering of R with the property that all elements are distinct. **End of Aside**

Our definition of a definiens does not include any maximality requirement. (In general, given a definiens $\mathcal{D}$ of a covering $\mathcal{R}$, a *minimal* definiens can be constructed by choosing exactly one point of each element of $\mathcal{D}$. On the other hand, maximality means that no additional points can be added without invalidating the definiens property.) It is possible that the definiens that we construct are indeed maximal but this is something we have not investigated.

If R is a finite relation, the construction of theorem 156 can be used to construct a non-redundant polar covering and its definiens. The covering is initialised to $\mathcal{R}$ as constructed by theorem 156 and the index set K of $\mathcal{R}$ is initialised to all points b in $R_{>}$. The index set K' of $\mathcal{D}$ is initialised to the empty set. Then each point b in K is examined, one by one. If $R \circ b \circ R\backslash R$ is redundant (i.e. b can be removed from K without affecting $\cup\mathcal{R}$) then b is removed from K. If not, b is retained in K and added to K'. Also $\mathcal{D}.b$ is defined by

$$\mathcal{D}.b \;=\; R \circ b \circ R\backslash R \cap \neg\langle \cup b' : b' \in K \wedge b \neq b' : R \circ b' \circ R\backslash R \rangle \;\; .$$

(So $\mathcal{D}.b$ is that part of the covering identified by b.) Assuming $R_{>}$ is finite, this process will terminate with a non-redundant polar covering of R indexed by K.

Example 158 Fig. 5 shows a relation R of type $\{A,B,C\} \sim \{\alpha,\beta,\gamma,\delta\}$. The four relations depicted in fig. 6 are rectangles of type $\{A,B,C\} \sim \{\alpha,\beta,\gamma,\delta\}$ (as indicated by the surrounding rectangular boxes); for greater clarity only edges connecting nodes in their left and right domains have been displayed.

These four rectangles are the elements of the polar covering constructed by theorem 156. The (reflexive-transitive reduction of the) ordering on the elements of the covering is depicted by arrowed blue lines. Take care to note how the depicted edges correspond to the ordering of the left domains of the rectangles:

$$\{B\} \subseteq \{A,B\} \;\wedge\; \{B\} \subseteq \{B,C\} \;\wedge\; \{A,B\} \subseteq \{A,B,C\} \;\wedge\; \{B,C\} \subseteq \{A,B,C\} \;\; ,$$

and to the "polar" ordering of their right domains:

$$\{\alpha,\beta,\gamma,\delta\}\supseteq\{\alpha,\delta\} \;\wedge\; \{\alpha,\beta,\gamma,\delta\}\supseteq\{\beta,\delta\} \;\wedge\; \{\alpha,\delta\}\supseteq\{\delta\} \;\wedge\; \{\beta,\delta\}\supseteq\{\delta\} \;\;.$$

The bottom rectangle is redundant (but not "obviously" so). By removing this rectangle, one obtains a non-redundant polar covering. The definiens of this covering is depicted by the bold red edges in fig. 6.

We return to this example later; see example 202.

□

For reasoning about polar coverings, an important property is that equality of elements is determined by the equality of their left or right domains. Specifically:

Lemma 159 Suppose U and V are elements of a polar covering. Then

$$(U=V) \;=\; (U_{>} = V_{>}) \;=\; (U_{<} = V_{<}) \;\;.$$

Proof This is straightforward:

$$\begin{array}{cl}
 & U=V \\
= & \{ \quad U \text{ and } V \text{ are rectangles, lemma 82} \quad \} \\
 & U_{<} = V_{<} \;\wedge\; U_{>} = V_{>} \\
= & \{ \quad \text{by definition 155, } U_{<}\subseteq V_{<} \equiv U_{>}\supseteq V_{>} \\
 & \quad\;\; \text{so, by anti-symmetry, } U_{<} = V_{<} \;\equiv\; U_{>} = V_{>} \quad \} \\
 & U_{>} = V_{>} \\
= & \{ \quad \text{see last step} \quad \} \\
 & U_{<} = V_{<} \;\;.
\end{array}$$

□

Consequently:

Lemma 160 A completely disjoint set of rectangles is a polar covering.

Proof Suppose $\mathcal{R}$ is a completely disjoint set of rectangles and suppose U and V are elements of $\mathcal{R}$. Then

$$\begin{array}{cl}
 & U_{<} \subseteq V_{<} \\
= & \{ \quad \text{infimum} \quad \} \\
 & U_{<}\cap V_{<} \;=\; U_{<} \\
\Rightarrow & \{ \quad \text{(left and right domains of) elements of a completely disjoint}
\end{array}$$

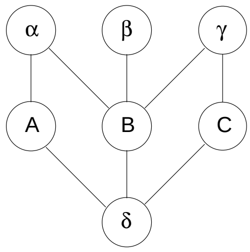


Figure 5: A Relation of Type {A,B,C}∼{α,β,γ,δ}

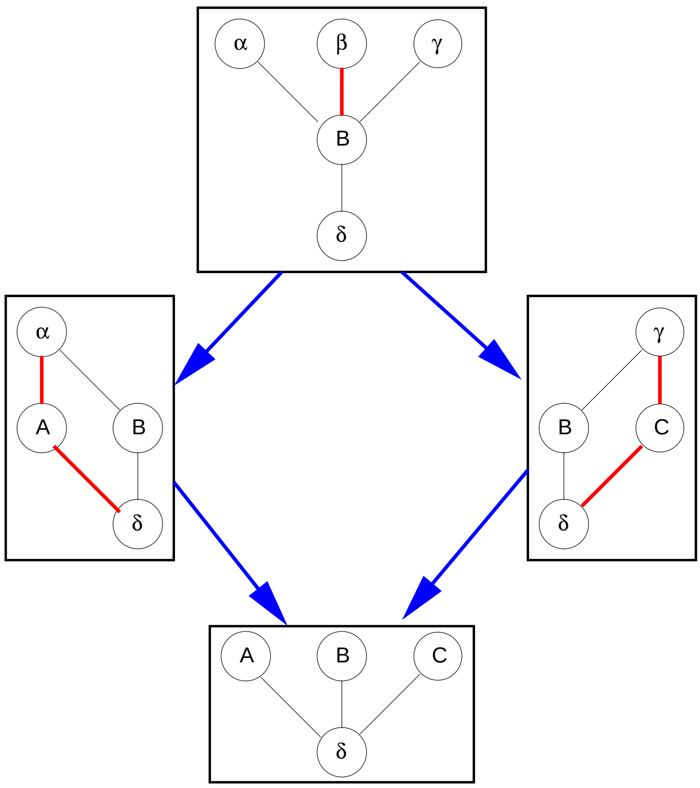


Figure 6: Polar Covering

set of rectangles are non-empty }

$U{<} \cap V{<} \neq \bot\!\!\bot$

$\Rightarrow$ { definition of completely disjoint }

$U = V$

$\Rightarrow$ { Leibniz }

$U{>} \supseteq V{>}$.

Symmetrically, $U{>} \supseteq V{>} \Rightarrow U{<} \subseteq V{<}$. The lemma follows by mutual implication.
□

Combining lemma 160 with theorem 134, it follows that every difunctional relation has a polar covering.

7.3 Non-Redundant Coverings

Now we consider circumstances in which the covering is non-redundant. In the case that R is difunctional, it is straightforward to show that the covering constructed in theorem 156 is non-redundant and is its own definiens. This suggests that, in general, a covering of the diagonal of a relation R can be used as the definiens of a covering of R. This is indeed true so long as the diagonal is sufficiently large[4]. Specifically, we prove below that, for all relations R, if $(\Delta R){>} = R{>}$, the covering $\mathcal{R}$ defined by theorem 156 is non-redundant as witnessed by the function $\mathcal{D}$ defined by

$$\mathcal{D}.b \;=\; \Delta R \circ b \circ R\backslash\!\backslash R \quad .$$

First, we show that this function is a covering of ΔR.

Theorem 161 Suppose R is a relation of type $A{\sim}B$ such that $(\Delta R){>} = R{>}$. Then the function $\mathcal{D}$ defined by

$$\mathcal{D} \;=\; \langle b : b \subseteq R{>} : \Delta R \circ b \circ R\backslash\!\backslash R \rangle$$

is a covering of ΔR. That is,

$$\Delta R \;=\; \langle \cup b : b \subseteq R{>} : \Delta R \circ b \circ R\backslash\!\backslash R \rangle \quad .$$

Moreover, for all points b and b' such that $b \subseteq R{>}$ and $b' \subseteq R{>}$, the following four properties are equivalent:

$$b \circ \top\!\!\top \circ b' \subseteq R\backslash\!\backslash R \quad ,$$

[4]But note example 169 below.

$$\top\!\top \circ \mathcal{D}.b \circ (\mathcal{D}.b')^{\cup} \circ \top\!\top = \top\!\top \quad ,$$

$$\top\!\top \circ (\mathcal{D}.b)^{\cup} \circ \mathcal{D}.b' \circ \top\!\top = \top\!\top \quad ,$$

$$\mathcal{D}.b = \mathcal{D}.b' \quad .$$

It follows that $\mathcal{D}$ is a completely disjoint covering of ΔR.

Proof That each element of $\mathcal{D}$ is a rectangle is a consequence of lemma 81. Now we show that $\mathcal{D}$ covers ΔR:

$$\begin{array}{ll}
& \langle \cup b : b \subseteq R_{>} : \Delta R \circ b \circ R \backslash\!\backslash R \rangle \\
= & \{ \quad \text{distributivity} \quad \} \\
& \Delta R \circ \langle \cup b : b \subseteq R_{>} : b \rangle \circ R \backslash\!\backslash R \\
= & \{ \quad \text{saturation axiom: (16)} \quad \} \\
& \Delta R \circ R_{>} \circ R \backslash\!\backslash R \\
= & \{ \quad \text{assumption: } (\Delta R)_{>} = R_{>} \quad \} \\
& \Delta R \circ (\Delta R)_{>} \circ R \backslash\!\backslash R \\
= & \{ \quad \text{domains and lemma 153} \quad \} \\
& \Delta R \quad .
\end{array}$$

That the four listed properties are equivalent is established by showing that all are equivalent to the first. Beginning with the equivalence of the first and second properties, we have:

$$\begin{array}{ll}
& \top\!\top \circ \mathcal{D}.b \circ (\mathcal{D}.b')^{\cup} \circ \top\!\top \\
= & \{ \quad \text{definition of } \mathcal{D} \quad \} \\
& \top\!\top \circ \Delta R \circ b \circ R \backslash\!\backslash R \circ (\Delta R \circ b' \circ R \backslash\!\backslash R)^{\cup} \circ \top\!\top \\
= & \{ \quad \text{converse and } R \backslash\!\backslash R \text{ is a per (so } R \backslash\!\backslash R = R \backslash\!\backslash R \circ (R \backslash\!\backslash R)^{\cup}) \quad \} \\
& \top\!\top \circ \Delta R \circ b \circ R \backslash\!\backslash R \circ b' \circ \Delta R^{\cup} \circ \top\!\top \\
= & \{ \quad \text{assumption: } (\Delta R)_{>} = R_{>} \text{ and } b \subseteq R_{>} \text{ and } b' \subseteq R_{>} \quad \} \\
& \top\!\top \circ b \circ R \backslash\!\backslash R \circ b' \circ \top\!\top \quad .
\end{array}$$

Applying the all-or-nothing rule, we conclude that

(162) $\top\!\top \circ \mathcal{D}.b \circ (\mathcal{D}.b')^{\cup} \circ \top\!\top = \top\!\top \quad \equiv \quad b \circ \top\!\top \circ b' \subseteq R \backslash\!\backslash R \quad .$

Also,

$$
\begin{array}{ll}
& \top\!\!\top \circ (\mathcal{D}.b)^{\cup} \circ \mathcal{D}.b' \circ \top\!\!\top \\
= & \{ \quad \text{definition of } \mathcal{D} \quad \} \\
& \top\!\!\top \circ (\Delta R \circ b \circ R\backslash\!\backslash R)^{\cup} \circ \Delta R \circ b' \circ R\backslash\!\backslash R \circ \top\!\!\top \\
= & \{ \quad \text{converse and } R\backslash\!\backslash R \text{ is an equivalence relation} \quad \} \\
& \top\!\!\top \circ b \circ \Delta R^{\cup} \circ \Delta R \circ b' \circ \top\!\!\top \\
= & \{ \quad \text{lemma 153} \quad \} \\
& \top\!\!\top \circ b \circ R\backslash\!\backslash R \circ (\Delta R)_{>} \circ b' \circ \top\!\!\top \\
= & \{ \quad \text{assumption: } (\Delta R)_{>} = R_{>} \text{ and } b' \subseteq R_{>} \quad \} \\
& \top\!\!\top \circ b \circ R\backslash\!\backslash R \circ b' \circ \top\!\!\top \quad .
\end{array}
$$

Applying the all-or-nothing rule again, we conclude that

(163) $\quad \top\!\!\top \circ (\mathcal{D}.b)^{\cup} \circ \mathcal{D}.b' \circ \top\!\!\top = \top\!\!\top \quad \equiv \quad b \circ \top\!\!\top \circ b' \subseteq R\backslash\!\backslash R \quad .$

The equivalence of the first and fourth properties is established by mutual implication. Suppose $b \circ \top\!\!\top \circ b' \subseteq R\backslash\!\backslash R$. Then

$$
\begin{array}{ll}
& \Delta R \circ b \circ R\backslash\!\backslash R \\
\subseteq & \{ \quad \text{assumption: } b \circ \top\!\!\top \circ b' \subseteq R\backslash\!\backslash R \text{; hence } b \subseteq (R\backslash\!\backslash R \circ b')_{<} \quad \} \\
& \Delta R \circ (R\backslash\!\backslash R \circ b')_{<} \circ R\backslash\!\backslash R \\
\subseteq & \{ \quad \text{domains} \quad \} \\
& \Delta R \circ R\backslash\!\backslash R \circ b' \circ b' \circ (R\backslash\!\backslash R)^{\cup} \circ R\backslash\!\backslash R \\
= & \{ \quad \text{lemma 153, } b \text{ is a point and } R\backslash\!\backslash R \text{ is a per} \quad \} \\
& \Delta R \circ b' \circ R\backslash\!\backslash R \quad .
\end{array}
$$

Now, since $R\backslash\!\backslash R = (R\backslash\!\backslash R)^{\cup}$, $b \circ \top\!\!\top \circ b' \subseteq R\backslash\!\backslash R \equiv b' \circ \top\!\!\top \circ b \subseteq R\backslash\!\backslash R$. Thus, interchanging b and b', we get, for all points b and b' such that $b \circ \top\!\!\top \circ b' \subseteq R\backslash\!\backslash R$,

$$\Delta R \circ b' \circ R\backslash\!\backslash R \ \subseteq \ \Delta R \circ b \circ R\backslash\!\backslash R \quad .$$

That is, by mutual inclusion, for all points b and b' such that $b \subseteq R_{>} \wedge b' \subseteq R_{>}$,

(164) $\quad \mathcal{D}.b = \mathcal{D}.b' \quad \Leftarrow \quad b \circ \top\!\!\top \circ b' \subseteq R\backslash\!\backslash R \quad .$

Now suppose b and b' are points such that $b \subseteq R_{>} \wedge b' \subseteq R_{>}$ and $\mathcal{D}.b = \mathcal{D}.b'$. Then

$$\begin{array}{ll}
& b{\circ}\top\!\top{\circ}b' \subseteq R\backslash\!\backslash R \\
= & \{ \quad (163) \text{ and } \mathcal{D}.b = \mathcal{D}.b' \quad \} \\
& \top\!\top \circ (\mathcal{D}.b)^{\cup} \circ \mathcal{D}.b \circ \top\!\top \;=\; \top\!\top \\
= & \{ \quad \text{definition of } \mathcal{D} \quad \} \\
& \top\!\top \circ (\Delta R \circ b \circ R\backslash\!\backslash R)^{\cup} \circ \Delta R \circ b \circ R\backslash\!\backslash R \circ \top\!\top \;=\; \top\!\top \\
= & \{ \quad \text{converse and } R\backslash\!\backslash R \text{ is reflexive} \quad \} \\
& \top\!\top \circ b \circ \Delta R^{\cup} \circ \Delta R \circ b \circ \top\!\top \;=\; \top\!\top \\
= & \{ \quad \text{lemma 153} \quad \} \\
& \top\!\top \circ b \circ (\Delta R){>} \circ R\backslash\!\backslash R \circ b \circ \top\!\top \;=\; \top\!\top \\
= & \{ \quad \text{assumption: } (\Delta R){>} = R{>} \text{ and } b \subseteq R{>} \quad \} \\
& \top\!\top \circ b \circ R\backslash\!\backslash R \circ b \circ \top\!\top \;=\; \top\!\top \\
= & \{ \quad R\backslash\!\backslash R \text{ is reflexive and } b \text{ is a point} \quad \} \\
& \text{true} \;\; .
\end{array}$$

That is, for all points b and b' such that $b \subseteq R{>} \wedge b' \subseteq R{>}$,

(165) $\quad b{\circ}\top\!\top{\circ}b' \subseteq R\backslash\!\backslash R \;\Leftarrow\; \mathcal{D}.b = \mathcal{D}.b'$.

Combining (164) and (165), we have

(166) $\quad b{\circ}\top\!\top{\circ}b' \subseteq R\backslash\!\backslash R \;\equiv\; \mathcal{D}.b = \mathcal{D}.b'$.

That $\mathcal{D}$ is a completely disjoint covering of ΔR is a straightforward consequence of propositional calculus. (See the alternative to definition 86, in particular (90).)
□

Theorem 167 Suppose R is a relation of type $A{\sim}B$ such that $(\Delta R){>} = R{>}$. Then the set $\mathcal{R}$ of rectangles defined by

$$\mathcal{R} \;=\; \langle b : b \subseteq R{>} : R \circ b \circ R\backslash R \rangle$$

is a non-redundant polar covering of R. A definiens of the covering is the function $\mathcal{D}$ defined by

$$\mathcal{D} \;=\; \langle b : b \subseteq R{>} : \Delta R \circ b \circ R\backslash\!\backslash R \rangle \;\; .$$

Proof Theorem 156 shows that $\mathcal{R}$ is a polar covering of R. It remains to show that it is non-redundant as witnessed by the function $\mathcal{D}$.

For all b such that $b \subseteq R_>$, the property $\mathcal{D}.b \subseteq \mathcal{R}.b$ is immediate from $\Delta R \subseteq R$, $R\backslash\!\backslash R \subseteq R\backslash R$ and monotonicity of composition. That the elements of $\mathcal{D}$ form a completely disjoint set of rectangles was shown in theorem 161. It remains to show that $\mathcal{D}$ "defines" $\mathcal{R}$. We have, for all b and b' such that $b \subseteq R_>$ and $b' \subseteq R_>$,

$$
\begin{array}{ll}
 & \mathcal{R}.b = \mathcal{R}.b' \\
= & \{ \quad \mathcal{R} \text{ is a polar covering, lemma 159} \quad \} \\
 & (\mathcal{R}.b)_< = (\mathcal{R}.b')_< \\
= & \{ \quad \text{definition of } \mathcal{R}\text{, domains} \quad \} \\
 & (R \circ b)_< = (R \circ b')_< \\
= & \{ \quad \text{corollary 49 and definition of } R\backslash\!\backslash R \quad \} \\
 & b \circ \top\!\!\top \circ b' \subseteq R\backslash\!\backslash R \\
= & \{ \quad \text{lemma 73} \quad \} \\
 & (b \circ R\backslash\!\backslash R)_> = (b \circ R\backslash\!\backslash R)_> \\
= & \{ \quad \text{definition of } \mathcal{D}\text{, domains} \quad \} \\
 & (\mathcal{D}.b)_> = (\mathcal{D}.b')_> \\
= & \{ \quad \mathcal{D} \text{ is a completely disjoint set of rectangles,} \\
 & \quad\quad \text{lemmas 159 and 160} \quad \} \\
 & \mathcal{D}.b = \mathcal{D}.b' \quad .
\end{array}
$$

□

Example 168

Fig. 7 pictures a small example of the theorems in this section. Fig. 7(a) depicts a relation R of type $\{\alpha,\beta,\gamma\} \sim \{A,B\}$; other parts of the figure depict the result of applying different functions to the relation R. (Heterogeneous relations are depicted as bipartite graphs whereas homogeneous relations are depicted as directed graphs.) Specifically, these are as follows.

$$
\begin{array}{ll}
\text{(a) } R \quad , & \text{(b) } \Delta R \quad , \\
\text{(c) } R\backslash R \quad , & \text{(d) } R/R \quad , \\
\text{(e) } R \circ A \circ R\backslash R \quad , & \text{(f) } R \circ B \circ R\backslash R \quad , \\
\text{(g) } \Delta R \circ A \circ R\backslash\!\backslash R \quad , & \text{(h) } \Delta R \circ B \circ R\backslash\!\backslash R \quad ,
\end{array}
$$

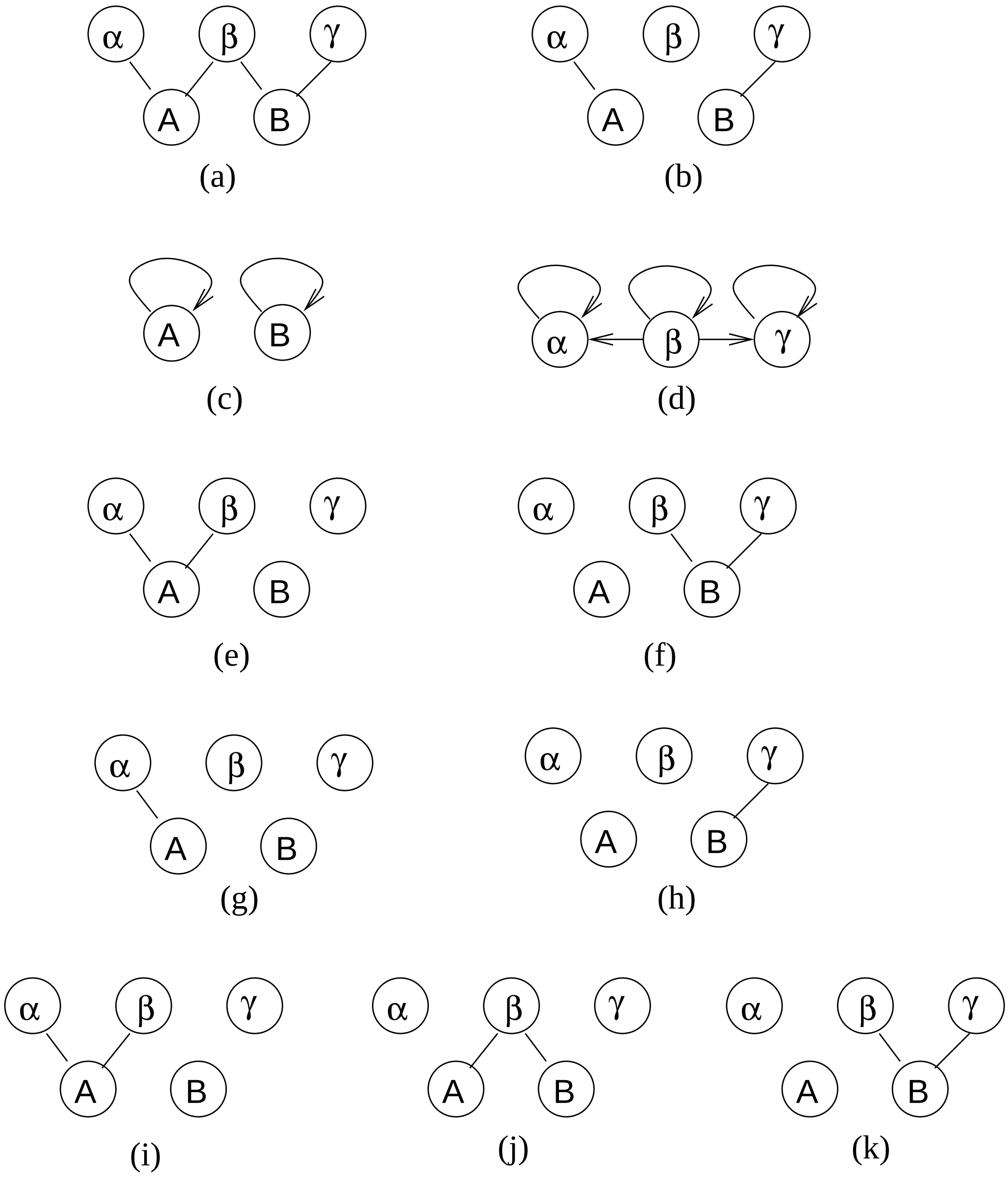


Figure 7: A Small Example

(i) $R/R\circ\alpha\circ R$, (j) $R/R\circ\beta\circ R$, (k) $R/R\circ\gamma\circ R$.

We have chosen to depict the relation as a graph (rather than a boolean matrix) because —for very small examples such as this— it is much easier for a human being to perform the necessary calculations by manipulating the graphs. For example, computing the composition of two relations is executed by chasing edges. Also —again for such very small examples— the definition of factors in terms of nested complements is much easier to use. This said, we leave the reader to check our calculations.

The example has been chosen deliberately to illustrate a number of aspects simultaneously. Note particularly that, for the relation depicted, $(\Delta R)_> = R_>$ but $(\Delta R)_< \neq R_<$. This means that theorem 167 is applicable but its dual is not.

Considering the application of theorem 167, note that the combination of figs. 7(e) and 7(f) covers the relation R; also the relation depicted by 7(g) uniquely identifies the rectangle $R\circ A\circ R\backslash R$ shown in fig. 7(e) whilst 7(h) uniquely identifies the rectangle $R\circ A\circ R\backslash R$ shown in fig. 7(f). In contrast, figs. 7(i), (j) and (k) depict the relations $R/R\circ\alpha\circ R$, $R/R\circ\beta\circ R$ and $R/R\circ\gamma\circ R$ but none of these is identified by any subrectangle: the rectangles depicted by figs. 7(i)and (k) are disjoint but both have a non-empty intersection with the rectangle depicted by fig. 7(j).
□

Example 168 is an example of a relation R such that $(\Delta R)_> = R_>$ but $(\Delta R)_< \neq R_<$. It is thus the case that, for this example,

$$R = \langle \cup b : b \subseteq (\Delta R)_> : R\circ b\circ R\backslash R \rangle \ .$$

(Note the range restriction on the dummy b.) Curiously, in spite of the fact that $(\Delta R)_< \neq R_<$, it is also the case that

$$R = \langle \cup a : a \subseteq (\Delta R)_< : R/R\circ a\circ R \rangle \ .$$

(Again, note the range restriction on the dummy a. To check the validity of the equation, it suffices to observe that the relation R is the union of the relations depicted by figs. 7(i)and (k).) This is also a non-redundant polar covering of R. One might thus conjecture that, in general, the diagonal ΔR is the key to finding a non-redundant polar covering of a given relation R. However, this is not always the case, as evidenced by the following example.

Example 169

The top diagram of fig. 8 pictures a relation R of type $\{\alpha,\beta,\gamma\}\sim\{A,B,C,D\}$ such that ΔR is the empty relation. The example is a simplification of the example on p.161 of [KGJ00].

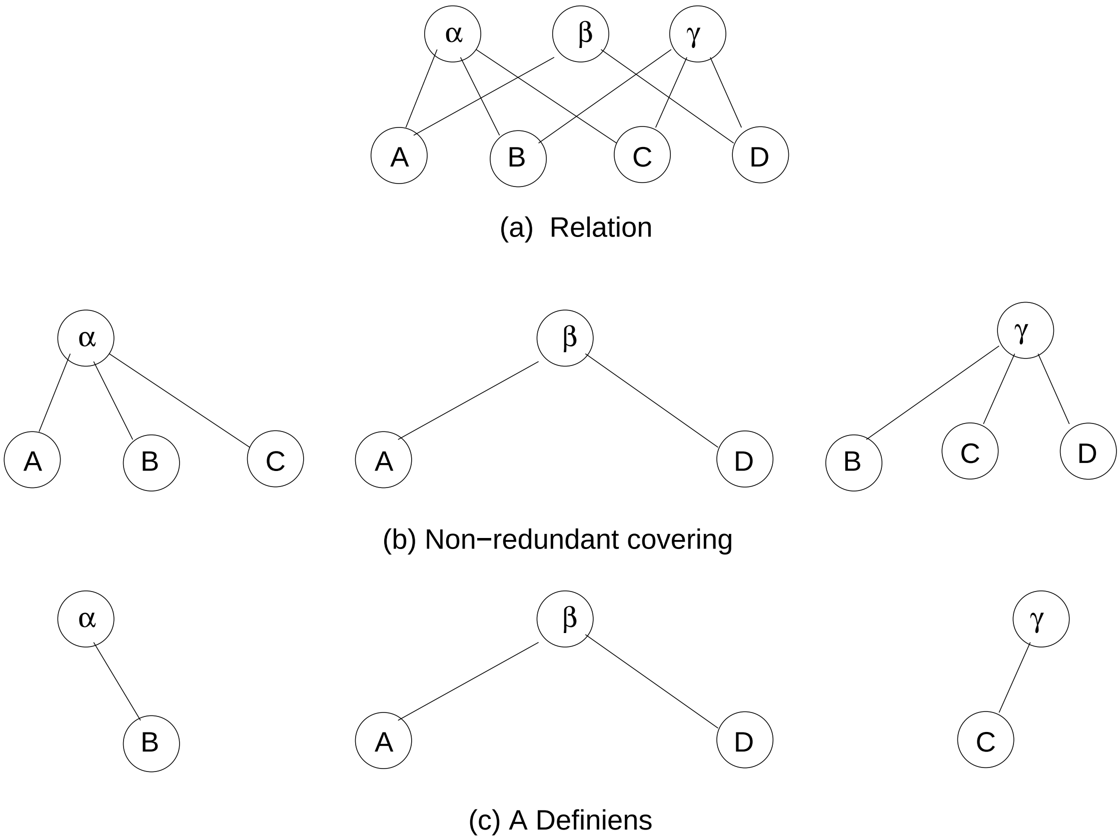


Figure 8: Empty Diagonal and Non-redundant Covering

The three components of the polar covering predicted by the dual of theorem 167 are depicted in the second row. (The index set of the covering is $\{\alpha,\beta,\gamma\}$.) Note that the covering is non-redundant: the third row pictures a function that satisfies the definition of a definiens of the covering. (Again, the index set is $\{\alpha,\beta,\gamma\}$.) This contradicts [KGJ00, theorem 1, p.159]: each of the edges in this third row is what [KGJ00] calls an "isolated point" in a "maximal rectangle" but none is a point in the diagonal.

Note that, although the definiens shown in fig. 8 is maximal, it is not unique: the edges from α to B and from γ to C may be replaced by edges from α to C and from γ to B.

□

8 Block-Ordered Relations

In general, dividing a subset of a set A into blocks is formulated by specifying a functional relation f, say, with source[5] the set A; elements $a0$ and $a1$ are in the same block equivales $f.a0$ and $f.a1$ are both defined and $f.a0 = f.a1$. In mathematical terminology, a functional relation f defines the *partial equivalence relation* $f^{\cup} \circ f$ and the "blocks" are the equivalence classes of $f^{\cup} \circ f$. (Partiality means that some elements may not be in an equivalence class.)

Given functional relations f and g with sources A and B, respectively, and equal left domains, relation R of type $A{\sim}B$ is said to be *block-structured* by f and g if there is a relation S such that $R = f^{\cup} \circ S \circ g$. Informally, whether or not a and b are related by R depends entirely on the "block" $(f.a, g.b)$ to which they belong. Note that it is not required that f and g be total functions: it suffices that $f_{>} = R_{<}$ and $g_{>} = R_{>}$. The type of S is $C{\sim}C$ where C includes $\{a\colon a \circ f_{>} = a\colon f.a\}$ (equally $\{b\colon b \circ f_{>} = b\colon g.b\}$).

Definition 170 (Block-Ordered Relation) Suppose T is a relation of type $C{\sim}C$, f is a relation of type $C{\sim}A$ and g is a relation of type $C{\sim}B$. Suppose further that T is a provisional ordering, i.e. that

$$(171) \quad T \cap T^{\cup} \subseteq I \quad \wedge \quad T = (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad \wedge \quad T \circ T \subseteq T \enspace .$$

Suppose also that f and g are functional and onto the domain of T. That is, suppose

$$(172) \quad f \circ f^{\cup} \; = \; f_{<} \; = \; T \cap T^{\cup} \; = \; g_{<} \; = \; g \circ g^{\cup} \enspace .$$

Then we say that the relation $f^{\cup} \circ T \circ g$ is a *block-ordered relation* A relation R of type $A{\sim}B$ is said to be *block-ordered* by f, g and T if $R = f^{\cup} \circ T \circ g$ and $f^{\cup} \circ T \circ g$ is a block-ordered relation.
□

Identifying a block-ordering of a relation —if it exists— is important for efficiency. Although a relation is defined to be a set of pairs, relations —even relations on finite sets— are rarely stored as such; instead some base set of pairs is stored and an algorithm used to generate, on demand, additional information about the relation. This is particularly so of ordering relations. For example, a test $m < n$ on integers m and n in a computer program is never implemented as a table lookup; instead an algorithm is used to infer from the basic relations $0 < 1$ together with the internal representation of m and n what the value of the test is. In the case of block-structured relations, functional relations f and g define partial equivalence relations $f^{\cup} \circ f$ and $g^{\cup} \circ g$ on their respective sources. (The relations $f^{\cup} \circ f$ and $g^{\cup} \circ g$ are partial because f and g are

[5] In the terminology we use, a relation of type $A{\sim}B$ has *target* A and *source* B.

not required to be total.) Combining the functional relations with an ordering relation on their (common) target is an effective way of implementing a relation (assuming the ordering relation is also implemented effectively).

It is important to note the very strict requirement (172) on the functionals f and g. Note its similarity with the requirement on functionals f and g in the definition of the characterisation of a difunctional: definition 133. Were this requirement to be omitted (retaining only that f and g are functional relations *into* —not onto— the domain of T), there would be no guarantee of non-redundancy. As we shall see, our definition of block-ordering does guarantee the existence of a non-redundant polar covering (theorem 183) but not vice-versa (corollary 192). This suggests that the requirement may be too strong. See section 8.3 and the conclusions for further discussion.

8.1 Pair Algebras and Galois Connections

In order to find lots of examples of block-ordered relations one need look no further than the theory of Galois connections (which are, of course, ubiquitous). In this section, we briefly review the notion of a "pair algebra" —due to Hartmanis and Stearns [HS64, HS66]— and its relation to Galois connections.

Hartmanis and Stearns studied a particular practical problem: the so-called "state assignment problem". This is the problem of how to encode the states and inputs of a sequential machine in such a way that state transitions can be implemented economically using logic circuits. However, as they made clear in the preface of their book [HS66], their contribution was to "information science" in general:

> It should be stressed, however, that although many structure theory results describe possible physical realizations of machines, the theory itself is independent of the particular physical components of technology used in the realization.
>
> ...
>
> The mathematical foundations of this structure theory rest on an algebraization of the concept of "information" in a machine and supply the algebraic formalism necessary to study problems about the flow of this information.

Hartmanis and Stearns limited their analysis to finite, complete posets, and their analysis was less general than is possible. This work was extended in [Bac98] to non-finite posets and the current section is a short extract.

A Galois connection involves two posets $(\mathcal{A},\sqsubseteq)$ and $(\mathcal{B},\preceq)$ and two functions, $F\in\mathcal{A}\leftarrow\mathcal{B}$ and $G\in\mathcal{B}\leftarrow\mathcal{A}$. These four components together form a *Galois connection* iff for all $b \in \mathcal{B}$ and $a \in \mathcal{A}$

(173) $F.b \sqsubseteq a \equiv b \preceq G.a$.

We refer to F as the *lower adjoint* and to G as the *upper adjoint*.

A Galois connection is thus a connection between two functions between posets. Typical accounts of the properties of Galois connections (for e.g. [GHK$^+$80]) focus on the properties of these *functions*. For example, given a function F, one may ask whether F is a lower adjoint in a Galois connection. The question posed by Hartmanis and Stearns was, however, rather different.

To motivate their question, note that the statement $F.b \sqsubseteq a$ defines a *relation* between $\mathcal{B}$ and $\mathcal{A}$. So too does $b \preceq G.a$. The existence of a Galois connection states that these two relations are equal. A natural question is therefore: under which conditions does an arbitrary (binary) relation between two posets define a Galois connection between the sets?

Exploring the question in more detail leads to two separate questions. The first is: suppose R is a relation between posets $(\mathcal{A}, \sqsubseteq)$ and $(\mathcal{B}, \preceq)$. What is a necessary and sufficient condition that there exist a function F such that

$$(a, b) \in R \;\equiv\; F.b \sqsubseteq a \quad ?$$

The second is the dual of the first: given relation R, what is a necessary and sufficient condition that there exist a function G such that

$$(a, b) \in R \;\equiv\; b \preceq G.a \quad ?$$

The conjunction of these two conditions is a necessary and sufficient condition for a relation R to define a Galois connection. Such a relation is called a *pair algebra*.

Example 174 It is easy to demonstrate that the two questions are separate. To this end, fig. 9 depicts two posets and a relation between them. The posets are $\{\alpha,\beta\}$ and $\{A,B\}$; both are ordered lexicographically: the reflexive transitive reduction of the lexicographic ordering is depicted by the directed edges. The relation of type $\{\alpha,\beta\} \sim \{A,B\}$ is depicted by the undirected edges.

Let the relation be denoted by R. Define the function F of type $\{\alpha,\beta\} \leftarrow \{A,B\}$ by $F.B = \alpha$ and $F.A = \beta$. Then it is easy to check that. for $a \in \{\alpha,\beta\}$ and $b \in \{A,B\}$,

$$(a, b) \in R \;\equiv\; F.b \sqsubseteq a \quad .$$

(There are just four cases to be considered.) On the other hand, there is no function G of type $\{A,B\} \leftarrow \{\alpha,\beta\}$ such that

$$(a, b) \in R \;\equiv\; b \preceq G.a \quad .$$

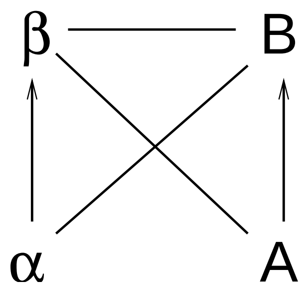


Figure 9: A Relation on Two Posets

To check that this is indeed the case, it suffices to check that the assignment $G.A = \alpha$ is invalid (because $\alpha \sqsubseteq \alpha$ but $(\alpha, A) \notin R$) and the assignment $G.A = \beta$ is also invalid (because $\alpha \sqsubseteq \beta$ but $(\alpha, A) \notin R$).
□

With respect to the current topic, the two separate questions are each of interest in their own right: a positive answer to either question may predict that a given relation has a block-ordering of a specific form: in the case of the first question, where the functional g in definition 170 is the identity function, and, in the case of the second question, where the functional f in definition 170 is the identity function. In both cases, a further step is to check the requirement on f and g: in the first case, one has to check that the function F is surjective and in the second case that the function G is surjective. (In the case of the existence of a Galois connection, the connection is said to be "perfect".)

The relevant theory predicting exactly when the first of the two questions has a positive answer is as follows. Suppose $(\mathcal{B}, \sqsubseteq)$ is a complete poset. Let $\sqcap$ denote the infimum operator for $\mathcal{B}$ and suppose p is a predicate on $\mathcal{B}$. Then we define *inf-preserving* by

(175) $\quad p \text{ is inf-preserving} \quad \equiv \quad \langle \forall g :: p.(\sqcap g) \equiv \langle \forall x :: p.(g.x) \rangle \rangle \quad .$

So, for a given a, the predicate $\langle b :: (a, b) \in R \rangle$ is inf-preserving equivales

$$\langle \forall g \;::\; (a\,, \sqcap g) \in R \;\equiv\; \langle \forall x :: (a\,,\, g.x) \in R \rangle \rangle \quad .$$

Then we have:

Theorem 176 Suppose $\mathcal{A}$ is a set and $(\mathcal{B}, \sqsubseteq)$ is a complete poset. Suppose $R \subseteq \mathcal{A} \times \mathcal{B}$ is a relation between the two sets. Define F by

(177) $\quad F.a \quad = \quad \langle \sqcap b \,:\, (a, b) \in R \,:\, b \rangle \quad .$

Then the following two statements are equivalent.

- $\langle \forall a,b \ : \ a \in \mathcal{A} \wedge b \in \mathcal{B} \ : \ (a,b) \in R \ \equiv \ F.a \sqsubseteq b \rangle$.
- For all a , the predicate $\langle b{::}\ (a,b) \in R \rangle$ is inf-preserving.

□

The answer to the second question is, of course, obtained by formulating the dual of theorem 176.

In general, for most relations occurring in practical information systems the answer to the pair-algebra questions will be negative: the required inf- and sup-preserving properties just do not hold. However, a common way to define a pair algebra is to extend a given relation to a relation between sets in such a way that the infimum and supremum preserving properties are automatically satisfied. Hartmanis and Stearns' [HS64, HS66] solution to the state assignment problem was to consider the lattice of partitions of a given set; in so-called "concept analysis", the technique is to extend a given relation to a relation between rectangles. For more detail and historical references, see [Bac98].

An important property of Galois connections is the (well-known) theorem we call the "unity of opposites": if F and G are the adjoint functions in a Galois connection of the posets $(\mathcal{A}, \sqsubseteq)$ and $(\mathcal{B}, \preceq)$, then there is an isomorphism between the posets $(F.\mathcal{B}, \sqsubseteq)$ and $(G.\mathcal{A}, \preceq)$. ($F.\mathcal{B}$ denotes the "image" of the function F, and similarly for $G.\mathcal{A}$.) Knowledge of the unity-of-opposites theorem suggests theorem 184, which expresses an isomorphism between different representations of block-ordered relations.

8.2 Analogie Frappante

In this section, we relate block-orderings to diagonals. The main results are theorems 183, 184 and 196. We have named theorem 196 the "analogie frappante" because it generalises Riguet's "analogie frappante" connecting "relation de Ferrers" to diagonals.

Lemma 178 Suppose T is a provisional ordering of type $C{\sim}C$. That is, suppose

$$T \cap T^{\cup} \subseteq I_C \quad \wedge \quad T = (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad \wedge \quad T \circ T \subseteq T \ .$$

Suppose also that f and g are functional and onto the domain of T . That is, suppose[6] that

$$f \circ f^{\cup} \ = \ f< \ = \ T \cap T^{\cup} \ = \ g< \ = \ g \circ g^{\cup} \ .$$

[6]The ordering T must be homogeneous but f and g may be heterogeneous and of different type, so long as both have target C . The restrictions on the left domains of f and g may be weakened: it is sufficient to require that $f< \ = g< \ \subseteq T \cap T^{\cup}$. The stronger restriction proves useful and entails no loss of generality.

Let R denote $f^{\cup}\circ T\circ g$. Then

(179) $R_{>} \;=\; g_{>} \;\wedge\; R_{<} = f_{>}$,

(180) $f^{\cup}\circ T^{\cup}\circ g \;=\; R_{<}\circ(R\backslash R/R)^{\cup}\circ R_{>}$, and

(181) $f^{\cup}\circ g \;=\; R \;\cap\; (R\backslash R/R)^{\cup}$.

Proof Property (179) is a straightforward application of domain calculus:

$$
\begin{array}{ll}
 & R_{>} \\
= & \{ \quad \text{definition: } R = f^{\cup}\circ T\circ g \quad \} \\
 & (f^{\cup}\circ T\circ g)_{>} \\
= & \{ \quad \text{domains (specifically, } [\,(U\circ V)_{>} = (U_{>}\circ V)_{>}\,] \text{ and } [\,(U^{\cup})_{>} = U_{<}\,]) \quad \} \\
 & (f_{<}\circ T\circ g)_{>} \\
= & \{ \quad \text{assumption: } T = f_{<}\circ T\circ g_{<} \text{ (so } T = f_{<}\circ T) \quad \} \\
 & (T\circ g)_{>} \\
= & \{ \quad \text{domains (specifically, } [\,(U\circ V)_{>} = (U_{>}\circ V)_{>}\,]) \quad \} \\
 & (T_{>}\circ g)_{>} \\
= & \{ \quad \text{lemma 79 and assumption: } T\cap T^{\cup} = g_{<} \quad \} \\
 & g_{>} \;\; .
\end{array}
$$

By a symmetric argument, $(f^{\cup}\circ T\circ g)_{<} = f_{>}$.

Now we consider (180). The raison d'être of (180) is that it expresses the left side as a function of $f^{\cup}\circ T\circ g$. In a pointwise calculation a natural step is to use indirect ordering. In a point-free calculation, this corresponds to using factors. That is, we exploit lemma 76:

$$
\begin{array}{ll}
 & f^{\cup}\circ T^{\cup}\circ g \\
= & \{ \quad \text{assumption: } T \text{ is a provisional ordering} \\
 & \quad\;\; \text{lemmas 75 and 76 (with } p := T\cap T^{\cup}) \quad \} \\
 & f^{\cup}\circ(T\cap T^{\cup})\circ T^{\cup}\backslash T^{\cup}/T^{\cup}\circ(T\cap T^{\cup})\circ g \\
= & \{ \quad \text{assumption: } f_{<} = T\cap T^{\cup} = g_{<} \quad \} \\
 & f^{\cup}\circ T^{\cup}\backslash T^{\cup}/T^{\cup}\circ g \\
= & \{ \quad \text{lemma 63 and assumption: } T = f_{<}\circ T\circ g_{<} \quad \}
\end{array}
$$

$$f{>} \circ (g^{\cup} \circ T^{\cup} \circ f) \backslash (g^{\cup} \circ T^{\cup} \circ f) / (g^{\cup} \circ T^{\cup} \circ f) \circ g{>}$$

$= \quad \{ \quad (179)$ and definition of $R \quad \}$

$$R{<} \circ R^{\cup} \backslash R^{\cup} / R^{\cup} \circ R{>}$$

$= \quad \{ \quad$ factors $\quad \}$

$$R{<} \circ (R\backslash R/R)^{\cup} \circ R{>} \quad .$$

Note the use of lemma 63. The discovery of this lemma is driven by the goal of the calculation.

The pointwise interpretation of $f^{\cup} \circ g$ is a relation expressing equality between values of f and g. This suggests that, in order to prove (181), we begin by exploiting the anti-symmetry of T:

$$f^{\cup} \circ g$$

$= \quad \{ \quad f{<} = T \cap T^{\cup} = g{<}$ and domains $\quad \}$

$$f^{\cup} \circ (T \cap T^{\cup}) \circ g$$

$= \quad \{ \quad$ distributivity (valid because f and g are functional) $\quad \}$

$$f^{\cup} \circ T \circ g \;\cap\; f^{\cup} \circ T^{\cup} \circ g$$

$= \quad \{ \quad$ definition of R and (180) $\quad \}$

$$f^{\cup} \circ T \circ g \;\cap\; f{>} \circ ((f^{\cup} \circ T \circ g) \backslash (f^{\cup} \circ T \circ g) / (f^{\cup} \circ T \circ g))^{\cup} \circ g{>}$$

$= \quad \{ \quad$ (182) (see below) $\quad \}$

$$f{>} \circ f^{\cup} \circ T \circ g \circ g{>} \;\cap\; ((f^{\cup} \circ T \circ g) \backslash (f^{\cup} \circ T \circ g) / (f^{\cup} \circ T \circ g))^{\cup}$$

$= \quad \{ \quad$ domains (specifically, $f{>} \circ f^{\cup} = f^{\cup}$ and $g \circ g{>} = g$) $\quad \}$

$$f^{\cup} \circ T \circ g \;\cap\; ((f^{\cup} \circ T \circ g) \backslash (f^{\cup} \circ T \circ g) / (f^{\cup} \circ T \circ g))^{\cup}$$

$= \quad \{ \quad$ definition of $R \quad \}$

$$R \;\cap\; (R\backslash R/R)^{\cup} \quad .$$

A crucial step in the above calculation is the use of the property

(182) $\quad U \cap p{\circ}V{\circ}q \quad = \quad p{\circ}(U{\cap}V){\circ}q \quad = \quad p{\circ}U{\circ}q \cap V$

for all relations U and V and coreflexive relations p and q. This is a frequently used property of domain restriction.

□

Theorem 183 Suppose R is a block-ordered relation. Then R has a non-redundant polar covering such that the definiens of the covering is a covering of the diagonal of R.

Proof The theorem is a consequence of theorems 167 and 161, and lemma 178. Specifically, suppose $R = f^{\cup} \circ T \circ g$ where f, g and T have the properties stated in definition 170. Then lemma 178 (in particular (181)) states that the diagonal ΔR of R equals $f^{\cup} \circ g$. In order to apply theorem 167, it suffices to check that $(\Delta R)_> = R_>$. This is straightforward:

$$
\begin{array}{ll}
 & (\Delta R)_> \\
= & \{ \quad \text{definition of } \Delta R \text{ and (181)} \quad \} \\
 & (f^{\cup} \circ g)_> \\
= & \{ \quad \text{domains and assumption: } f_< = g_< \quad \} \\
 & g_> \\
= & \{ \quad (179) \quad \} \\
 & R_> \ .
\end{array}
$$

Applying theorem 167, R has a non-redundant polar covering of which the definiens is the function $\mathcal{D}$ defined by

$$\mathcal{D} \;=\; \langle b : b \subseteq R_> : \Delta R \circ b \circ R \backslash\!\backslash R \rangle \ .$$

Moreover, by theorem 161, $\mathcal{D}$ is a covering of ΔR.
□

Theorem 184 makes precise the statement that block orderings —where they exist— are unique "up to isomorphism".

Theorem 184 Suppose T is a provisional ordering. That is, suppose

$$T \cap T^{\cup} \subseteq I \quad \wedge \quad T = (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad \wedge \quad T \circ T \subseteq T \ .$$

Suppose also that f and g are functional and onto the domain of T. That is, suppose

$$f \circ f^{\cup} \;=\; f_< \;=\; T \cap T^{\cup} \;=\; g_< \;=\; g \circ g^{\cup} \ .$$

Suppose further[7] that S, h and k satisfy the same properties as T, f and g (respectively) and that

(185) $f^{\cup} \circ T \circ g \;=\; h^{\cup} \circ S \circ k$.

[7]The types of T and S may be different. The types of f and h, and of g and k will then also be different. As in lemma 178, the requirement is that the types are compatible with the type restrictions on the operators in all assumed properties. The symbol " I " in (190) is overloaded: if the type of T is $A \sim A$ and the type of S is $B \sim B$, $\phi \circ \phi^{\cup}$ has type $A \sim A$ and $\phi^{\cup} \circ \phi$ has type $B \sim B$.

Then

(186) $f{>} = h{>} \;\wedge\; g{>} = k{>}$,

(187) $f^{\cup} \circ g = h^{\cup} \circ k$,

(188) $f^{\cup} \circ T^{\cup} \circ g \;=\; h^{\cup} \circ S^{\cup} \circ k$, and

(189) $f \circ h^{\cup} \;=\; g \circ k^{\cup}$.

Also, letting ϕ denote $f \circ h^{\cup}$ (equally, by (189), $g \circ k^{\cup}$),

(190) $\phi \circ \phi^{\cup} = T \cap T^{\cup} \;\wedge\; \phi^{\cup} \circ \phi = S \cap S^{\cup} \;\wedge\; \phi \circ T = S \circ \phi$.

In words, ϕ is an order isomorphism of the domains of T and S .

Proof In combination with the assumption (185), properties (186), (188) and (187) are immediate from (179), (180) and (181), respectively.

Proof of (189) is a step on the way to proving (190). From symmetry considerations, it is an obvious first step.

$$
\begin{array}{ll}
 & f \circ h^{\cup} \\
= & \{ \quad \text{assumption:} \quad k \circ k^{\cup} = h{<} \quad \} \\
 & f \circ h^{\cup} \circ k \circ k^{\cup} \\
= & \{ \quad (187) \quad \} \\
 & f \circ f^{\cup} \circ g \circ k^{\cup} \\
= & \{ \quad \text{assumption:} \quad f \circ f^{\cup} = g{<} \quad \} \\
 & g \circ k^{\cup} \quad .
\end{array}
$$

Now,

$$
\begin{array}{ll}
 & \phi \circ \phi^{\cup} \\
= & \{ \quad \text{definition of } \phi \text{, converse} \quad \} \\
 & f \circ h^{\cup} \circ h \circ f^{\cup} \\
= & \{ \quad (189) \quad \} \\
 & g \circ k^{\cup} \circ h \circ f^{\cup} \\
= & \{ \quad (187) \text{ and converse} \quad \} \\
 & g \circ g^{\cup} \circ f \circ f^{\cup} \\
= & \{ \quad \text{assumption:} \quad f \circ f^{\cup} = T \cap T^{\cup} = g \circ g^{\cup} \quad \} \\
 & T \cap T^{\cup} \quad .
\end{array}
$$

Symmetrically, $\phi^\cup \circ \phi = T \cap T^\cup$. Finally,

$$
\begin{array}{ll}
& T \circ \phi \\
= & \{ \quad \text{definition of } \phi \quad \} \\
& T \circ f \circ h^\cup \\
= & \{ \quad \text{assumptions: } \; f \circ f^\cup = T \cap T^\cup = g \circ g^\cup \\
& \qquad T \;=\; (T \cap T^\cup) \circ T \circ (T \cap T^\cup) \quad \} \\
& f \circ f^\cup \circ T \circ g \circ g^\cup \circ f \circ h^\cup \\
= & \{ \quad \text{assumption: } \; f^\cup \circ T \circ g \;=\; h^\cup \circ S \circ k \;\; \text{, (187) and converse} \quad \} \\
& f \circ h^\cup \circ S \circ k \circ k^\cup \circ h \circ h^\cup \\
= & \{ \quad \text{assumption: } \; h \circ h^\cup = S \cap S^\cup = k \circ k^\cup \quad \} \\
& f \circ h^\cup \circ S \\
= & \{ \quad \text{definition of } \phi \quad \} \\
& \phi \circ S \;\; .
\end{array}
$$

□

Lemma 191 If R is a block-ordered relation then $R< = (\Delta R)<$ and $R> = (\Delta R)>$.

Proof Suppose f , g and T are as specified in definition 170. Then

$$
\begin{array}{ll}
& (\Delta R)< \\
= & \{ \quad \text{lemma 178 (in particular (181)) and definition of } \Delta \quad \} \\
& (f^\cup \circ g)< \\
= & \{ \quad \text{domains and assumption: } \; f< = g< \quad \} \\
& f> \\
= & \{ \quad \text{assumption: } \; f< \;=\; T \cap T^\cup \quad \} \\
& ((T \cap T^\cup) \circ f)> \\
= & \{ \quad \text{domains and converse} \quad \} \\
& (f^\cup \circ (T \cap T^\cup))< \\
= & \{ \quad \text{lemma 79 and domains} \quad \} \\
& (f^\cup \circ T)< \\
= & \{ \quad \text{domains and assumption: } \; g< = T \cap T^\cup
\end{array}
$$

and lemma 79 }

$(f^{\cup} \circ T \circ g)_{<}$.

That is $(\Delta R)_{<} = (f^{\cup} \circ T \circ g)_{<}$. A similar calculation shows that $(\Delta R)_{>} = (f^{\cup} \circ T \circ g)_{>}$:

$(\Delta R)_{>}$

= { lemma 178 (in particular (181)) and definition of Δ }

$(f^{\cup} \circ g)_{>}$

= { domains and assumption: $f_{<} = g_{<}$ }

$g_{>}$

= { assumption: $g_{<} = T \cap T^{\cup}$ }

$((T \cap T^{\cup}) \circ g)_{>}$

= { lemma 79 and domains }

$(T \circ g)_{>}$

= { domains and assumption: $f_{<} = T \cap T^{\cup}$
and lemma 79 }

$(f^{\cup} \circ T \circ g)_{>}$.

□

Lemma 191 has as immediate corollary that the converse of theorem 183 is invalid.

Corollary 192 There are relations that have a non-redundant polar covering but are not block-ordered.

Proof Examples 168 and 169 are both examples of finite relations that have non-redundant polar coverings. Example 168 has the property that $(\Delta R)_{<} \neq R_{<}$; however, $(\Delta R)_{>} = R_{>}$. Example 169 has an empty diagonal; that is, $(\Delta R)_{<} \neq R_{<}$ (and $(\Delta R)_{>} \neq R_{>}$). So by (the converse of) lemma 191, neither relation is block-ordered.
□

We now prove the converse of lemma 191.

Lemma 193 A relation R is block-ordered if $R_{<} = (\Delta R)_{<}$ and $R_{>} = (\Delta R)_{>}$.

Proof Suppose $R_{<} = (\Delta R)_{<}$ and $R_{>} = (\Delta R)_{>}$. Our task is to construct relations f, g and T such that

$$R = f^{\cup} \circ T \circ g \quad ,$$

$$T \cap T^{\cup} \subseteq I \quad \wedge \quad T = (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup}) \quad \wedge \quad T \circ T \subseteq T \quad \text{and}$$

$$f \circ f^{\cup} \;=\; f< \;=\; T \cap T^{\cup} \;=\; g< \;=\; g \circ g^{\cup} \quad .$$

Since ΔR is difunctional, theorem 132 is the obvious place to start. Applying the theorem, we can construct f and g such that $\Delta R = f^{\cup} \circ g$ and

$$\Delta R = f^{\cup} \circ g \;\wedge\; f \circ f^{\cup} \;=\; f< \;=\; g \circ g^{\cup} \;=\; g< \quad .$$

(The proof of theorem 132 gives several ways of doing this.) Using standard properties of the domain operators together with the assumption that $R< = (\Delta R)<$ and $R> = (\Delta R)>$, it follows that

$$R< = f> \;\wedge\; R> = g> \quad .$$

It remains to construct the provisional ordering T. The appropriate construction is suggested by lemma 178, in particular (180). Specifically, we define T by the equation

(194) $\quad T \;=\; g \circ R\backslash R/R \circ f^{\cup} \quad .$

The proof that $R = f^{\cup} \circ T \circ g$ is by mutual inclusion. First note that

(195) $\quad f^{\cup} \circ T \circ g \;=\; \Delta R \circ R\backslash R/R \circ \Delta R$

since

$$\begin{array}{ll} & f^{\cup} \circ T \circ g \\ = & \{ \quad (194) \quad \} \\ & f^{\cup} \circ g \circ R\backslash R/R \circ f^{\cup} \circ g \\ = & \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\ & \Delta R \circ R\backslash R/R \circ \Delta R \quad . \end{array}$$

So

$$\begin{array}{ll} & f^{\cup} \circ T \circ g \\ \subseteq & \{ \quad (195) \text{ and } \Delta R \subseteq R \quad \} \\ & R \circ R\backslash R/R \circ R \\ \subseteq & \{ \quad \text{cancellation} \quad \} \\ & R \quad . \end{array}$$

Also,

$$
\begin{array}{ll}
 & R \subseteq f^{\cup} \circ T \circ g \\
= & \{ \quad (195) \quad \} \\
 & R \subseteq \Delta R \circ R\backslash R/R \circ \Delta R \\
= & \{ \quad \text{assumption: } R_{<} = (\Delta R)_{<} \text{ and } R_{>} = (\Delta R)_{>} \text{, lemma 154} \quad \} \\
 & \Delta R \circ \Delta R^{\cup} \circ R \circ \Delta R^{\cup} \circ \Delta R \subseteq \Delta R \circ R\backslash R/R \circ \Delta R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & \Delta R^{\cup} \circ R \circ \Delta R^{\cup} \subseteq R\backslash R/R \\
\Leftarrow & \{ \quad \Delta R^{\cup} \subseteq R\backslash R/R \text{, monotonicity} \quad \} \\
 & R\backslash R/R \circ R \circ R\backslash R/R \subseteq R\backslash R/R \\
= & \{ \quad \text{factors} \quad \} \\
 & R \circ R\backslash R/R \circ R \circ R\backslash R/R \circ R \subseteq R \\
= & \{ \quad \text{cancellation} \quad \} \\
 & \text{true} \ .
\end{array}
$$

Combining the two inclusions we conclude that indeed $R = f^{\cup} \circ T \circ g$.

We now establish the requirements on T. First,

$$
\begin{array}{ll}
 & T \cap T^{\cup} \\
= & \{ \quad \text{definition and converse} \quad \} \\
 & g \circ R\backslash R/R \circ f^{\cup} \ \cap \ f \circ (R\backslash R/R)_{\cup} \circ g^{\cup} \\
\subseteq & \{ \quad \text{modular law} \quad \} \\
 & f \circ (f^{\cup} \circ g \circ R\backslash R/R \circ f^{\cup} \circ g \ \cap \ (R\backslash R/R)_{\cup}) \circ g^{\cup} \\
= & \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\
 & f \circ (\Delta R \circ R\backslash R/R \circ \Delta R \ \cap \ (R\backslash R/R)_{\cup}) \circ g^{\cup} \\
\subseteq & \{ \quad \Delta R \subseteq R \text{, monotonicity and cancellation} \quad \} \\
 & f \circ (R \ \cap \ (R\backslash R/R)_{\cup}) \circ g^{\cup} \\
= & \{ \quad \Delta R = R \ \cap \ (R\backslash R/R)_{\cup} \quad \} \\
 & f \circ \Delta R \circ g^{\cup} \\
= & \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\
 & f \circ f^{\cup} \circ g \circ g^{\cup} \\
= & \{ \quad f \circ f^{\cup} = f_{<} = g \circ g^{\cup} = g_{<} \quad \}
\end{array}
$$

$$f<\ .$$

Thus $T \cap T^{\cup} \subseteq f<$. So $T \cap T^{\cup} \subseteq I$. Now

$$
\begin{array}{ll}
 & f< \subseteq T \cap T^{\cup} \\
= & \{ \quad \text{infima and } f< \text{ is coreflexive} \quad \} \\
 & f< \subseteq T \\
\Leftarrow & \{ \quad \text{domains} \quad \} \\
 & f \circ f^{\cup} \subseteq T \\
\Leftarrow & \{ \quad \text{definition of } T \text{ and monotonicity} \quad \} \\
 & f \subseteq g \circ R\backslash R/R \\
\Leftarrow & \{ \quad f< \,=\, g \circ g^{\cup}\text{, domains and monotonicity} \quad \} \\
 & g^{\cup} \circ f \subseteq R\backslash R/R \\
= & \{ \quad f^{\cup} \circ g = \Delta R \quad \} \\
 & \Delta R^{\cup} \subseteq R\backslash R/R \\
= & \{ \quad \Delta R \,=\, R \cap (R\backslash R/R)_{\cup}\text{, converse} \quad \} \\
 & \mathsf{true}\ .
\end{array}
$$

So, by anti-symmetry we have established that $T \cap T^{\cup} = f<$. Since also $f< = g<$, we conclude from the definition of T and properties of domains that

$$T \,=\, (T \cap T^{\cup}) \circ T \circ (T \cap T^{\cup})\ .$$

The final task is to show that T is transitive:

$$
\begin{array}{ll}
 & T \circ T \\
= & \{ \quad \text{definition} \quad \} \\
 & g \circ R\backslash R/R \circ f^{\cup} \circ g \circ R\backslash R/R \circ f^{\cup} \\
= & \{ \quad \Delta R = f^{\cup} \circ g \quad \} \\
 & g \circ R\backslash R/R \circ \Delta R \circ R\backslash R/R \circ f^{\cup} \\
\subseteq & \{ \quad \Delta R \subseteq R \quad \} \\
 & g \circ R\backslash R/R \circ R \circ R\backslash R/R \circ f^{\cup} \\
\subseteq & \{ \quad \text{factors} \quad \} \\
 & g \circ R\backslash R/R \circ f^{\cup}
\end{array}
$$

$= \quad \{ \quad \text{definition} \quad \}$

$\quad \top$.

□

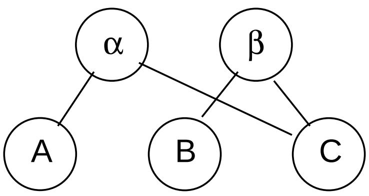
It is interesting to reflect on the proof of lemma 193. The hardest part is to find appropriate definitions of f, g and T such that $R = f^{\cup} \circ T \circ g$. The key to constructing f and g is Riguet's "analogie frappante" [Rig51] whereby he introduced the "différence" —the diagonal ΔR— of the relation R. Expressing the diagonal in terms of factors as we have done makes many parts of the calculations very straightforward. One much less straightforward step is the use of lemma 154 in the proof that $R \subseteq f^{\cup} \circ T \circ g$. Note how calculational needs drive the search for the lemma: in order to simplify the inclusion it is necessary to expose the term $R\backslash R/R$ on the right side, and that is precisely what the lemma enables.

We conclude with the theorem that we call the "analogie frappante". It is not the theorem that Riguet presented but we have chosen to give it this name in order to recognise Riguet's contribution.

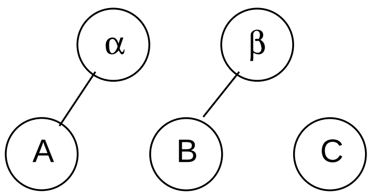
Theorem 196 (Analogie Frappante) A relation R is block-ordered if and only if $R_< = (\Delta R)_<$ and $R_> = (\Delta R)_>$.

Proof Lemma 191 establishes "only-if" and lemma 193 establishes "if".
□

Example 197 Fig. 10 is a second example of a relation that is not block-ordered. The relation is shown on the left and its diagonal on the right. Note that, for this example, $R_< = (\Delta R)_<$ but $R_> \neq (\Delta R)_>$.



(a) Relation



(b) Diagonal

Figure 10: A Relation That Is Not Block-Ordered

Because of the simplicity of the example, it is possible to check, by exhausting all possible assignments to f and g, that the relation is not block-ordered. For suppose, on the contrary, that $R = f^{\cup} \circ T \circ g$, where f, T and g satisfy the conditions for a block-ordering. Then it must be the case that $f.\alpha \neq f.\beta$ (since $(\alpha \circ R)_> \neq (\beta \circ R)_>$). But also it

must be the case that $g.A$, $g.B$ and $g.C$ are distinct (because, eg., $(R{\circ}A)_< \neq (R{\circ}B)_<$). This has the consequence that $f_< \neq g_<$. But, by defining $f.\alpha = x$, $f.\beta = y$, $g.A = x$, $g.B = y$, $g.C = z$ and $x \sqsubseteq z$ and $y \sqsubseteq z$, it is the case that $R = f^{\cup} \circ \sqsubseteq \circ g$. We say that the relation has an "imperfect" block-ordering. See section 8.3.
□

8.3 Imperfect Block-Orderings

Following definition 170 we remarked that the condition on the functional relations f and g in a block-ordering is very strict. Later we remarked that a Galois connection satisfies the condition only if it is so-called "perfect". In this section we present a theorem on the existence of what might be called "(possibly) imperfect" block-orderings. The theorem is exploited later to show that finite "staircase relations" are indeed block-ordered.

Theorem 198 Suppose R is an arbitrary relation. Define the set C of rectangles by

$$C \;=\; \{b : b \subseteq R_> : R \circ b \circ R\backslash R\} \;\;.$$

Define the provisional ordering $\sqsubseteq$ on the set C by, for all b and b' such that $b \subseteq R_>$ and $b' \subseteq R_>$,

(199) $\quad R \circ b \circ R\backslash R \sqsubseteq R \circ b' \circ R\backslash R \quad \equiv \quad (R{\circ}b)_< \subseteq (R{\circ}b')_< \;\;.$

Suppose the provisional ordering is complete. Then $R = f^{\cup} \circ \sqsubseteq \circ g$ where the functional relations f and g are defined by, for all points b such that $b \subseteq R_>$,

(200) $\quad g.b \;=\; R \circ b \circ R\backslash R$

and, for all points a such that $a \subseteq R_<$,

(201) $\quad f.a \;=\; \langle \sqcap b : a{\circ}\top\!\top{\circ}b \subseteq R : R \circ b \circ R\backslash R \rangle \;\;.$

Proof Recall from theorem 156 that $\langle b : b \subseteq R_> : R \circ b \circ R\backslash R \rangle$ is a polar covering of R. That $\sqsubseteq$ satisfies (171) follows from its being a polar ordering (see theorem 156). Also, $g_> = g \circ g^{\cup} = R_>$ and $g_< = C$, by definition.

Expressed pointwise, the requirement that $R = f^{\cup} \circ \sqsubseteq \circ g$ is the property that, for all a and b,

$$a{\circ}\top\!\top{\circ}b \subseteq R \;\equiv\; f.a \sqsubseteq R \circ b \circ R\backslash R$$

This suggests the definition (201) of f. We must first show that f is well-defined. We have

$$
\begin{array}{ll}
 & a \subseteq R< \\
= & \{ \quad \text{lemma 47} \quad \} \\
 & \langle \exists b : b \subseteq R> : a \circ \top\!\top \circ b \subseteq R \rangle \\
\Rightarrow & \{ \quad \text{assumption: the provisional ordering } \sqsubseteq \text{ is complete} \quad \} \\
 & \langle \sqcap b : a \circ \top\!\top \circ b \subseteq R : R \circ b \circ R \backslash R \rangle \text{ is well-defined} \quad .
\end{array}
$$

Now, for all points a and b,

$$
\begin{array}{ll}
 & a \circ \top\!\top \circ b \subseteq R \\
= & \{ \quad \text{domains; definition of infimum} \quad \} \\
 & a \subseteq R< \ \wedge \ b \subseteq R> \ \wedge \ \langle \sqcap b : a \circ \top\!\top \circ b \subseteq R : R \circ b \circ R \backslash R \rangle \ \sqsubseteq \ R \circ b \circ R \backslash R \\
= & \{ \quad \text{definitions of } f \text{ and } g \quad \} \\
 & a \subseteq R< \ \wedge \ b \subseteq R> \ \wedge \ a \circ \top\!\top \circ b \ \subseteq \ f^{\cup} \circ \sqsubseteq \circ g \\
\Rightarrow & \{ \quad \text{weakening} \quad \} \\
 & a \circ \top\!\top \circ b \ \subseteq \ f^{\cup} \circ \sqsubseteq \circ g \\
= & \{ \quad \text{definitions of } f \text{ and } g \quad \} \\
 & \langle \sqcap b : a \circ \top\!\top \circ b \subseteq R : R \circ b \circ R \backslash R \rangle \ \sqsubseteq \ R \circ b \circ R \backslash R \\
= & \{ \quad \text{definition of } \sqsubseteq \quad \} \\
 & \langle \cap b : a \circ \top\!\top \circ b \subseteq R : (R \circ b)< \rangle \ \subseteq \ (R \circ b)< \\
= & \{ \quad \begin{array}{ll} & a \subseteq \langle \cap b : a \circ \top\!\top \circ b \subseteq R : (R \circ b)< \rangle \\ = & \{ \quad \text{definition of infimum} \quad \} \\ & \langle \forall b : a \circ \top\!\top \circ b \subseteq R : a \subseteq (R \circ b)< \rangle \\ = & \{ \quad \text{lemma 46} \quad \} \\ & \text{true} \quad \} \end{array} \\
 & a \subseteq \langle \cap b : a \circ \top\!\top \circ b \subseteq R : (R \circ b)< \rangle \subseteq (R \circ b)< \\
\Rightarrow & \{ \quad \text{transitivity of the subset relation} \quad \} \\
 & a \subseteq (R \circ b)< \\
= & \{ \quad \text{lemma 46} \quad \} \\
 & a \circ \top\!\top \circ b \subseteq R \quad .
\end{array}
$$

We have thus proved by mutual implication that, for all points a and b,

$$
a \circ \top\!\top \circ b \subseteq R \quad \equiv \quad a \circ \top\!\top \circ b \ \subseteq \ f^{\cup} \circ \sqsubseteq \circ g \quad .
$$

The theorem follows by the saturation axiom: (16).
□

Note that theorem 198 falls short of constructing a block-ordering because the requirement (172) on f and g has not been established. The following example illustrates the shortfall.

Example 202 Recall that fig. 5 shows a relation R of type $\{A,B,C\}\sim\{\alpha,\beta,\gamma,\delta\}$ and fig. 6 shows the provisional ordering defined by theorem 198.

Recall also that the four relations depicted in fig. 6 are rectangles of the same type as R. These four rectangles are the values of the functional relation g. Specifically, the topmost rectangle depicts the relation $g.\beta$, the middle-left rectangle depicts $g.\alpha$, the middle-right rectangle depicts $g.\gamma$ and the bottom rectangle depicts $g.\delta$.

The top three rectanges are also the values of the functional relation f. Specifically, the topmost rectangle depicts the relation $f.B$, the middle-left rectangle depicts $f.A$, the middle-right rectangle depicts $f.C$.

The (reflexive-transitive reduction of the) provisional ordering $\sqsubseteq$ on the rectangles is depicted by the blue arrowed edges. We leave the reader to check that $R = f^{\cup}\circ\sqsubseteq\circ g$.

Note that we have *not* constructed a block-ordering of the relation R because $f_< \neq g_<$. (That is, f is not surjective.)

The reader might also wish to construct the diagonal ΔR. For this example, it is the case that $R_< \neq (\Delta R)_<$ but $R_> = (\Delta R)_>$.

The theorem does not apply to the dual construction (whereby the rectangles are of the form $R/R\circ a\circ R$ where a ranges over points in $R_<$) because the poset of rectangles is not complete. In this case, there are three, not four, rectangles: the top three rectangles depicted in fig. 6. The dual construction gives a method of defining functional relation g: the topmost rectangle depicts the relation $g.B$, the middle-left rectangle depicts $g.A$, the middle-right rectangle depicts $g.C$. However, the construction fails to define the functional relation f because the value of $f.\delta$ is undefined.
□

9 Staircase Relations

As mentioned immediately after its definition, the notion of a polar covering was introduced by Riguet in connection with what he called "relations de Ferrers". Riguet [Rig51] states the following theorem:

> Pour que R soit une relation de Ferrers, il faut et il suffit que R soit réunion de rectangles dont les projections de même nom sont totalement ordonnées par inclusion et tels que si la première projection de l'un des rectangles est

> contenue dans la première projection d'un autre rectangle, la seconde projection du second est contenue dans la seconde projection du premier.

(For those unable to read French, the theorem states a necessary and sufficient condition for a relation to be "de Ferrers". The formal statement and proof of the theorem is given below: see theorem 234. The theorem clearly begs the question what is the definition of a "relation de Ferrers". We postpone answering this question until later. The reason for doing so is that Riguet gives both a formal definition and a mental picture —the picture shown in fig. 1 of what we call a "staircase relation"— but it is far from obvious how Riguet's definition and the mental picture are related.)

Riguet does not give a proof of the theorem. He also states that there is a striking analogy ("analogie frappante") between the definitions and properties of "relations de Ferrers" and difunctional relations but leaves the analogy unclear. In this section, we formalise the mental picture of a "staircase relation" (fig. 1) in several different but equivalent ways, one of which is Riguet's orginal definition. We then prove Riguet's theorem. This is quite straightforward. However, clarifying the "analogie frappante" is more difficult. To this end, we formulate the notion of a "polar covering" of a staircase relation and a "non-redundant" polar covering. We show how Riguet's theorem predicts that the less-than relation on real numbers has a polar covering but not a non-redundant polar covering. The non-redundancy property is the vital link between difunctional relations and (a proper subclass of) staircase relations. It is also the link between (a proper subclass of) staircase relations and block-ordered relations.

9.1 Formal Definition

Let us now turn to the formalisation of the mental picture of a "staircase" relation.

Suppose the relation R of type $A{\sim}B$ can be depicted as a "staircase". Then, for any element b of B, the set of elements a of A such that a and b are related by R is depicted by the region where a vertical line drawn at the point that depicts b intersects with the shaded area in the staircase depiction of R. See fig. 11. (Conversely, the set of elements b of B that are related to a given element a of A is depicted by drawing a horizontal line at the point depicted by a.)

The characteristic property of a "staircase" is that such lines increase in length —of course, not strictly— as one proceeds from the left to the right of the picture. But "length" and "left" and "right" are features of pictures and not properties of relations. A better characterisation that is not specific to drawing pictures is suggested by focusing on the subset of A comprising elements related by R to a given element b of B. In relation algebra, this is denoted by $(R{\circ}b)_{<}$ and the characteristic property of a "staircase" is that, for any two elements $b0$ and $b1$ of B, either $(R{\circ}b0)_{<}$ is a subset of $(R{\circ}b1)_{<}$ or,

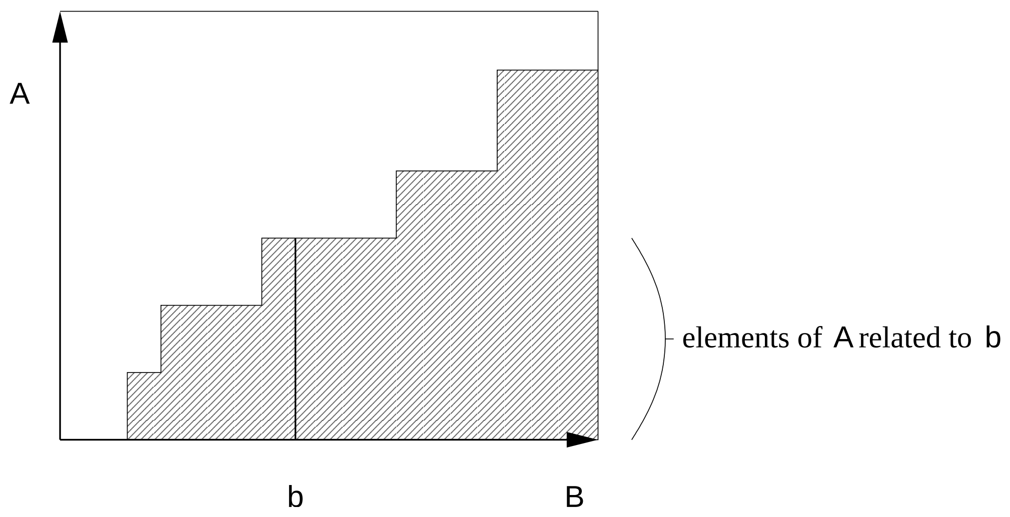


Figure 11: Preordering Defined By a Staircase Relation

vice-versa, $(R{\circ}b1)_{<}$ is a subset of $(R{\circ}b0)_{<}$. In terms of the mental picture, $b0$ is to the left or to the right of $b1$.

At this point, certain concepts central to relation algebra spring to mind. First, the subset relation is an ordering relation. This immediately leads to the observation that the relation S defined by

$$b0\llbracket S\rrbracket b1 \quad\equiv\quad (R{\circ}b0)_{<} \subseteq (R{\circ}b1)_{<}$$

is a preorder. Then the "vice-versa" statement also looks familiar: it is the statement that $S\cup S^{\cup}$ is total (i.e. equal to the universal relation).

Those familiar with factors will immediately spot a much better characterisation. For any binary relation R, the relations $R\backslash R$ and R/R are preorders. That is, both are transitive and reflexive. (If R has type $A{\sim}B$ then $R\backslash R$ has type $B{\sim}B$ and R/R has type $A{\sim}A$.) If R is itself a preorder, then $R = R\backslash R = R/R = R\backslash R/R$. (Transitivity of R is equivalent to $R\subseteq R\backslash R$ and reflexivity of R implies $R\backslash R\subseteq R$; similarly for R/R.) This fact underlies the use of the rule called indirect ordering.

The pointwise formulation of the relation $R\backslash R$ is

$$b0\llbracket R\backslash R\rrbracket b1 \quad\equiv\quad \langle\forall a : a\llbracket R\rrbracket b0 : a\llbracket R\rrbracket b1\rangle \quad .$$

That is $R\backslash R$ is the relation S defined above. This is the eureka moment in this preliminary investigation: that relation R is a "staircase" relation means formally that the preorder $R\backslash R$ is linear[8]. (Later we show that this is equivalent to R/R being linear.) For brevity, we denote this property by SC. That is:

[8]An ordering S —of any sort— is said to be *linear* if $S\cup S^{\cup} = \top\!\top$. Sometimes the word "total" is used instead of linear. For example, Riguet [Rig51] uses the term "totalement ordonnées".

Definition 203 The predicate SC on (binary) relations is defined by, for all R,

$$\mathsf{SC.R} \quad\equiv\quad \mathsf{R}\backslash\mathsf{R}\cup(\mathsf{R}\backslash\mathsf{R})^{\cup} \;=\; \top\!\top \quad .$$

□

The boolean SC.R should be read as " R is a staircase relation". This section is thus about the properties of R\R, for arbitrary relation R, when R\R is linear. The properties we investigate are driven by the need to provide further justification for the "correctness" of the formal definition with respect to the informal mental picture of such a relation.

Inevitably, we sometimes need to exploit pointwise definitions of "staircase" relations. Such a definition is formulated in lemma 204. Informally, the lemma states that there is a linear ordering on the depths of the "stairs" of a "staircase" relation. (Later we see that this is equivalent to there being a linear ordering on the heights of the "stairs".)

Lemma 204 The property SC.R is equivalent to:

$$\langle\forall\, b,b' : b\subseteq R_{>} \wedge b'\subseteq R_{>} : (R\circ b)_{<} \subseteq (R\circ b')_{<} \;\vee\; (R\circ b')_{<} \subseteq (R\circ b)_{<}\rangle \quad .$$

(Dummies b and b′ range over points of the appropriate type.)

Proof

$$\begin{array}{ll}
 & \mathsf{SC.R} \\
= & \{\quad \text{definition 203} \quad\} \\
 & R\backslash R\cup(R\backslash R)^{\cup} \;=\; \top\!\top \\
= & \{\quad \text{saturation axiom: (16)} \quad\} \\
 & \langle\forall\, b,b' :: b\circ\top\!\top\circ b' \;\subseteq\; R\backslash R\cup(R\backslash R)^{\cup}\rangle \\
= & \{\quad b\circ\top\!\top\circ b' \text{ is an (irreducible) atom, and converse} \quad\} \\
 & \langle\forall\, b,b' :: b\circ\top\!\top\circ b' \subseteq R\backslash R \;\vee\; b'\circ\top\!\top\circ b \subseteq R\backslash R\rangle \\
= & \{\quad \text{lemma 48} \quad\} \\
 & \langle\forall\, b,b' :: (R\circ b)_{<} \subseteq (R\circ b')_{<} \;\vee\; (R\circ b')_{<} \subseteq (R\circ b)_{<}\rangle \\
= & \{\quad b \text{ and } b' \text{ are points;} \\
 & \quad\;\; \text{hence, } (b\subseteq R_{>} \wedge b'\subseteq R_{>}) \;\vee\; (R\circ b)_{<} = \bot\!\bot \;\vee\; (R\circ b')_{<} = \bot\!\bot \\
 & \quad\;\; \text{case analysis (further details omitted)} \quad\} \\
 & \langle\forall\, b,b' : b\subseteq R_{>} \wedge b'\subseteq R_{>} : (R\circ b)_{<} \subseteq (R\circ b')_{<} \;\vee\; (R\circ b')_{<} \subseteq (R\circ b)_{<}\rangle \quad .
\end{array}$$

□

The final step in the proof of lemma 204 restricts the range of the dummies b and b′. This is an indication that our definition of SC demands refinement: the relation R\R typically includes irrelevant information. We return to this topic in section 9.6.

9.2 Equivalent Formulations

Lemma 34 enables a simple proof that linearity of $R\backslash R$ is equivalent to linearity of R/R. Specifically:

Lemma 205 The following are all equivalent formulations of $SC.R$:

(206) $R\backslash R \cup (R\backslash R)^{\cup} \;=\; \top\!\top$,

(207) $R/R \cup (R/R)^{\cup} \;=\; \top\!\top$,

(208) $R \cup (R\backslash R/R)^{\cup} \;=\; \top\!\top$,

(209) $R\circ\neg R^{\cup}\circ R \;\subseteq\; R$.

Proof We prove first that (207) and (209) are equivalent:

$$
\begin{array}{ll}
 & R\circ\neg R^{\cup}\circ R \;\subseteq\; R \\
= & \{ \quad \text{factors} \quad \} \\
 & R\circ\neg R^{\cup} \;\subseteq\; R/R \\
= & \{ \quad \text{complements} \quad \} \\
 & \top\!\top \;\subseteq\; R/R \cup \neg(R\circ\neg R^{\cup}) \\
= & \{ \quad (38) \text{ with } R,S := R^{\cup}, R^{\cup} \text{ (and } R = (R^{\cup})^{\cup}) \quad \} \\
 & \top\!\top \;\subseteq\; R/R \,\cup\, R^{\cup}\backslash R^{\cup} \\
= & \{ \quad (35) \text{ with } R,S := R,R \quad \} \\
 & \top\!\top \;\subseteq\; R/R \cup (R/R)^{\cup} \\
= & \{ \quad [S \subseteq \top\!\top] \text{ with } S := R/R \cup (R/R)^{\cup} \text{ and anti-symmetry} \quad \} \\
 & \top\!\top \;=\; R/R \cup (R/R)^{\cup} \;\; .
\end{array}
$$

A symmetric argument establishes the equivalence of (206) and (209):

$$
\begin{array}{ll}
 & R\circ\neg R^{\cup}\circ R \;\subseteq\; R \\
= & \{ \quad \text{factors} \quad \} \\
 & \neg R^{\cup}\circ R \;\subseteq\; R\backslash R \\
= & \{ \quad \text{complements} \quad \} \\
 & \top\!\top \;\subseteq\; R\backslash R \cup \neg(\neg R^{\cup}\circ R) \\
= & \{ \quad (38) \text{ with } S,T := R^{\cup}, R^{\cup} \quad \}
\end{array}
$$

$$\top\!\top \;\subseteq\; R\backslash R \,\cup\, R^{\cup}/R^{\cup}$$

$= \quad \{ \quad (36)$ with $R,S := R,R$ (and $R = (R^{\cup})^{\cup}$) $\quad \}$

$$\top\!\top \;\subseteq\; R\backslash R \cup (R\backslash R)^{\cup}$$

$= \quad \{ \quad [S \subseteq \top\!\top]$ with $S := R\backslash R \cup (R\backslash R)^{\cup}$ and anti-symmetry $\quad \}$

$$\top\!\top \;=\; R\backslash R \cup (R\backslash R)^{\cup} \quad .$$

Finally,

$$R \circ \neg R^{\cup} \circ R \;\subseteq\; R$$

$= \quad \{ \quad$ factors $\quad \}$

$$\neg R^{\cup} \;\subseteq\; R\backslash R/R$$

$= \quad \{ \quad$ converse and complements $\quad \}$

$$\top\!\top \;\subseteq\; R \cup (R\backslash R/R)^{\cup}$$

$= \quad \{ \quad [S \subseteq \top\!\top]$ with $S := R \cup (R\backslash R/R)^{\cup}$ and anti-symmetry $\quad \}$

$$\top\!\top \;=\; R \cup (R\backslash R/R)^{\cup} \quad .$$

□

Note that, in lemma 205, the symbol " $\top\!\top$ " denoting the universal relation is overloaded: if R has type $A{\sim}B$, its occurrence in (206) has type $B{\sim}B$, its occurrence in (207) has type $A{\sim}A$ and its occurrence in (208) has type $A{\sim}B$. This means that any attempt to prove, for example, that

$$R \cup (R\backslash R/R)^{\cup} = R/R \cup (R/R)^{\cup}$$

is doomed to fail. One might conjecture that it is possible to establish the equivalence of (206) and (207) without introducing complements by showing that both are equivalent to (208). However, the use of (209) is inevitable because of the algebraic properties of set union: when a set union is on the greater side of a set inclusion, there is no other choice but to introduce set negation.

9.3 General Constructions

Two general methods for identifying examples of staircase relations are given in lemmas 210 and 211.

Lemma 210 A linear preorder is a staircase relation. That is, for all (homogeneous) R,

$$SC.R \;\Leftarrow\; R \circ R \subseteq R \;\wedge\; I \subseteq R \;\wedge\; R \cup R^{\cup} = \top\!\top \quad .$$

Proof We have

$$R = R\backslash R/R \quad \Leftarrow \quad R \circ R \subseteq R \ \wedge\ I \subseteq R$$

since

$$
\begin{array}{ll}
 & R \subseteq R\backslash R/R \\
= & \{ \quad \text{factors} \quad \} \\
 & R \circ R \circ R \subseteq R \\
\Leftarrow & \{ \quad \text{monontonicity and transitivity} \quad \} \\
 & R \circ R \subseteq R
\end{array}
$$

and

$$
\begin{array}{ll}
 & R\backslash R/R \subseteq R \\
= & \{ \quad [\, R = I\backslash R/I \,] \quad \} \\
 & R\backslash R/R \subseteq I\backslash R/I \\
\Leftarrow & \{ \quad \text{(anti)monotonicity} \quad \} \\
 & I \subseteq R \ .
\end{array}
$$

Also,

$$R^{\cup} \circ R^{\cup} \subseteq R^{\cup} \ \wedge\ I \subseteq R^{\cup} \quad \equiv \quad R \circ R \subseteq R \ \wedge\ I \subseteq R \ .$$

(The converse of a preorder is a preorder.) So

$$
\begin{array}{ll}
 & SC.R \\
= & \{ \quad \text{lemma 205, in particular (208)} \quad \} \\
 & R \cup (R\backslash R/R)^{\cup} \ = \ \top\!\!\top \\
= & \{ \quad \text{assumption: } R \text{ is a preorder} \\
 & \quad\ \ \text{(hence, } R^{\cup} \text{ is a preorder and } R^{\cup} = R^{\cup} \backslash R^{\cup} / R^{\cup} \text{)} \\
 & \quad\ \ \text{lemma 34, in particular (37)} \quad \} \\
 & R \cup R^{\cup} \ = \ \top\!\!\top \\
= & \{ \quad \text{assumption: } R \text{ is linear (i.e. } R \cup R^{\cup} = \top\!\!\top \text{)} \quad \} \\
 & \text{true} \ .
\end{array}
$$

□

An example of a staircase relation predicted by lemma 210 is the at-most relation — on natural numbers, integers, rational numbers or reals.

The second way of constructing a staircase relation is to reduce a linear preorder by eliminating its reflexive part (making it so-called "strict"). For example, the less-than relation (on natural numbers, integers, rational numbers or reals) is a staircase relation. (Lemma 212 is an alternative way of establishing that the less-than relation is a staircase relation. See example 216.) Formally, we have:

Lemma 211 For all (homogeneous) R,

$$\mathsf{SC.R} \;\Leftarrow\; R\circ R\subseteq R \;\wedge\; R\cup I\cup R^{\cup} = \top\!\top \;.$$

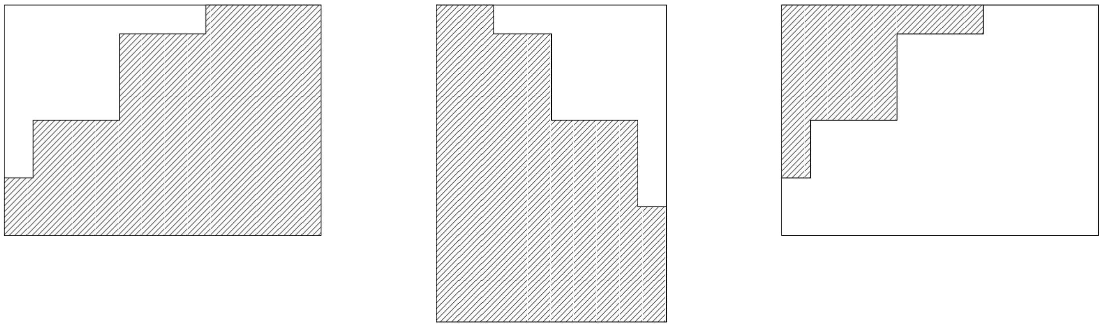
Proof

$$\begin{array}{ll}
 & \mathsf{SC.R} \\
= & \{ \quad (208) \quad \} \\
 & R \cup (R\backslash R/R)^{\cup} \;=\; \top\!\top \\
= & \{ \quad [X\subseteq \top\!\top] \text{ and antisymmetry} \quad \} \\
 & \top\!\top \subseteq R \cup (R\backslash R/R)^{\cup} \\
\Leftarrow & \{ \quad \text{assumption: } R\cup I\cup R^{\cup} = \top\!\top\text{, so } \top\!\top \subseteq R\cup I\cup R^{\cup} \\
 & \quad\;\; \text{monotonicity and transitivity} \quad \} \\
 & I\cup R^{\cup} \subseteq (R\backslash R/R)^{\cup} \\
= & \{ \quad \text{converse, factors and distributivity} \quad \} \\
 & R\circ I\circ R\cup R\circ R\circ R \;\subseteq\; R \\
= & \{ \quad \text{supremum and monotonicity} \quad \} \\
 & R\circ R \subseteq R \\
= & \{ \quad \text{assumption} \quad \} \\
 & \mathsf{true} \;\;.
\end{array}$$

□

9.4 Invariant Properties

In this section, we prove that the class of linear preorders characterised by the predicate SC is invariant under a variety of operators. Lemma 212 is supported by the mental picture shown in fig. 12.



(a) Staircase (b) Converse (c) Complement

Figure 12: Staircase Invariants

Lemma 212 For all R,

$$SC.R \;=\; SC.\neg R \;=\; SC.R^{\cup} \;.$$

(As always, equality is used conjunctionally.)

Proof

$$
\begin{array}{ll}
 & SC.R \\
= & \{ \quad \text{definition 203} \quad \} \\
 & R\backslash R \cup (R\backslash R)^{\cup} \;=\; \top\!\!\top \\
= & \{ \quad \text{corollary 39} \quad \} \\
 & \neg R\backslash \neg R \cup (\neg R\backslash \neg R)^{\cup} \;=\; \top\!\!\top \\
= & \{ \quad \text{definition 203} \quad \} \\
 & SC.\neg R \;.
\end{array}
$$

Also,

$$
\begin{array}{ll}
 & SC.R \\
= & \{ \quad \text{definition 203} \quad \} \\
 & R\backslash R \cup (R\backslash R)^{\cup} \;=\; \top\!\!\top \\
= & \{ \quad \text{lemma 205 (in particular (209))} \quad \} \\
 & R \circ \neg R^{\cup} \circ R \;\subseteq\; R \\
= & \{ \quad \text{properties of converse} \quad \}
\end{array}
$$

$$R^{\cup} \circ \neg R \circ R^{\cup} \subseteq R^{\cup}$$

$=$ { lemma 205 (in particular (209)) with $R := R^{\cup}$ }

$$SC.R^{\cup} \quad .$$

□

Lemma 213 The functions $\langle R :: R\backslash R \rangle$ and $\langle R :: R/R \rangle$ are closure operators. That is

$$(R\backslash R)\backslash(R\backslash R) = R\backslash R \ \wedge \ (R/R)/(R/R) = R/R \quad .$$

Proof This is a straightforward application of standard properties of factors:

$$(R\backslash R)\backslash(R\backslash R)$$

$=$ { $[\, R\backslash(S\backslash T) = (S{\circ}R)\backslash T \,]$ with $R,S,T := R,R,R$ }

$$(R \circ R\backslash R)\backslash R$$

$=$ { (28): $[\, R \circ R\backslash R = R \,]$ }

$$R\backslash R \quad .$$

The second equation is proved in the same way.
□

Lemma 214 For all R,

$$SC.R \ = \ SC.(R\backslash R) \ = \ SC.(R/R) \quad .$$

Proof Straightforward application of definition 203 and lemma 213.
□

Lemma 215 For all S, R and T (of appropriate type),

$$SC.(S{\circ}R{\circ}T) \ \Leftarrow \ SC.R \quad .$$

Proof

$$SC.(S{\circ}R{\circ}T)$$

$=$ { lemma 205, in particular (209) with $R := S{\circ}R{\circ}T$ }

$$S \circ R \circ T \circ \neg(S{\circ}R{\circ}T)^{\cup} \circ S \circ R \circ T \ \subseteq \ S \circ R \circ T$$

$\Leftarrow$ { monotonicity of composition }

$$R \circ T \circ \neg(S{\circ}R{\circ}T)^{\cup} \circ S \circ R \ \subseteq \ R$$

$=$ { middle-exchange rule (and double negation) }

$$(R{\circ}T)^{\cup} \circ \neg R \circ (S{\circ}R)^{\cup} \;\subseteq\; (S{\circ}R{\circ}T)^{\cup}$$

$=\quad\{\quad$ converse $\quad\}$

$$T^{\cup} \circ R^{\cup} \circ \neg R \circ R^{\cup} \circ S^{\cup} \;\subseteq\; T^{\cup} \circ R^{\cup} \circ S^{\cup}$$

$\Leftarrow\quad\{\quad$ monotonicity of composition $\quad\}$

$$R^{\cup} \circ \neg R \circ R^{\cup} \subseteq R^{\cup}$$

$=\quad\{\quad R = (R^{\cup})^{\cup}$ and lemma 205 with $R := R^{\cup} \quad\}$

$$SC.R^{\cup}$$

$=\quad\{\quad$ lemma 212 $\quad\}$

$$SC.R\ .$$

□

Example 216 The above properties allow us to identify a number of examples of staircase relations that prove to be significant later.

The at-most relation (commonly denoted by the symbol "$\leq$") is a linear ordering relation — on the integers, on the rationals and on the real numbers. By lemma 210 all three relations are staircase relations. By applying lemma 212 it is thus the case that the greater-than relation (commonly denoted by "$>$"), the less-than relation (commonly denoted by the symbol "$<$") and the at-least relation (commonly denoted by the symbol "$\geq$") are all staircase relations — again, on the integers, on the rationals and on the real numbers. This is because the greater-than relation is the complement of the at-most relation, the less-than relation is the converse of the greater-than relation, and, in turn, the at-least relation is the complement of the less-than relation.

Note that the less-than relation is not a preorder. (It is transitive but not reflexive.) Thus it is an example of a relation R such that $R \neq R\backslash R$ (and $R \neq R/R$) but is nevertheless a staircase relation according to definition 203.

The reader is invited to picture the less-than relation on the integers as a "staircase". Picturing the less-than relation on the rational numbers (or on the real numbers) as a "staircase" is, however, more difficult — in fact impossible in a formal sense to be made precise later. This raises doubts as to whether definition 203 is the appropriate abstraction from the mental picture of a "staircase".

□

We conclude this section with a property due to Riguet [Rig51]. (See the discussion following the lemma.)

Lemma 217 For all R, the relation $R\backslash R/R$ is a staircase relation if R is a staircase relation.

Proof For brevity, let S denote $R\backslash R/R$. Then

$$
\begin{array}{ll}
& SC.S \\
= & \{ \quad \text{lemma 205} \quad \} \\
& S\circ\neg S^{\cup}\circ S \ \subseteq\ S \\
= & \{ \quad \text{lemma 32 and definition of } S \quad \} \\
& R\backslash R/R\circ R\circ\neg R^{\cup}\circ R\circ R\backslash R/R \ \subseteq\ R\backslash R/R \\
= & \{ \quad \text{definition of factors} \quad \} \\
& R\circ R\backslash R/R\circ R\circ\neg R^{\cup}\circ R\circ R\backslash R/R\circ R \ \subseteq\ R \\
\Leftarrow & \{ \quad \text{cancellation} \quad \} \\
& R\circ\neg R^{\cup}\circ R \\
= & \{ \quad \text{lemma 205} \quad \} \\
& SC.R \quad .
\end{array}
$$

□

The combination of lemmas 150 and 217 is the second of two theorems stated by Riguet [Rig51]. More precisely, he states that $R\circ\neg R^{\cup}\circ R$ is a "relation de Ferrers" if R is a "relation de Ferrers" (cf. lemma 217) and their "différence" $R\cap\neg(R\circ\neg R^{\cup}\circ R)$ (i.e. ΔR) is a difunctional relation (cf. lemma 150). This explains his use of the term "différence" for what we call the "diagonal" of a relation.

9.5 Linear Orderings

In this section and the next we return to the mental picture of "staircases" as illustrated by fig. 1. An alternative perspective on a staircase relation of type $A{\sim}B$ is that it divides the elements of A into "blocks"; similarly the elements of B are also divided into "blocks". Fig. 13 is an example where A and B are each divided into five blocks. The effect is to divide the "staircase" into fifteen ($1{+}2{+}3{+}4{+}5$) blocks. A pair (a,b) is related by the staircase relation if the number assigned to a is at most the number assigned to b. Note that the at-most relation on numbers is a linear ordering.

In section 9.6, we show that every *linearly* block-ordered relation is a staircase relation. However, as we show in this section, a staircase relation does not necessarily have a block-ordering. See theorem 220. Thus, contrary to claims made in the literature —see section 10— it is not the case that these two concepts are equivalent.

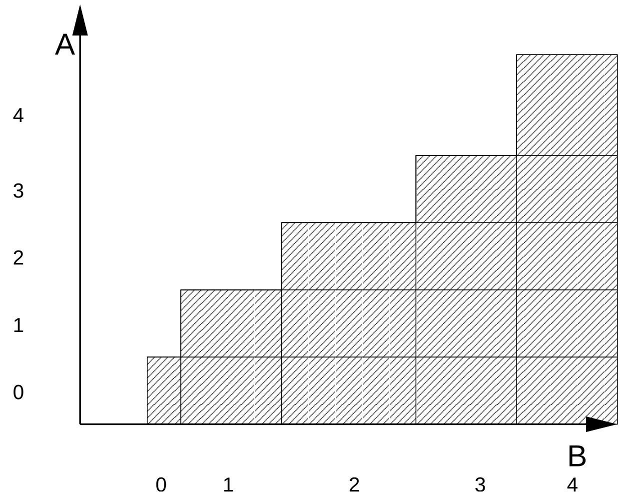


Figure 13: Block Structure of a Staircase Relation

Lemma 218 Suppose R has type $A{\sim}B$ and f and g are relations with targets A and B, respectively, such that $f\circ f^{\cup} = R{<}$ and $g\circ g^{\cup} = R{>}$. Then

$$SC.(f^{\cup}\circ R\circ g) \;\equiv\; SC.R \quad .$$

Proof The equivalence is proved by mutual implication.

$$
\begin{array}{ll}
 & SC.R \\
= & \{\quad \text{assumption: } f\circ f^{\cup} = R{<} \text{ and } g\circ g^{\cup} = R{>} \;\; ; \text{domains} \quad\} \\
 & SC.(f\circ f^{\cup}\circ R\circ g\circ g^{\cup}) \\
\Leftarrow & \{\quad \text{lemma 215 with } S,T := f\,,g^{\cup} \quad\} \\
 & SC.(f^{\cup}\circ R\circ g) \\
\Leftarrow & \{\quad \text{lemma 215 with } S,T := f^{\cup}\,,g \quad\} \\
 & SC.R \quad .
\end{array}
$$

□

Corollary 219 Suppose T of type $C{\sim}C$ is a linear ordering and suppose f and g are functional and surjective relations of types $C{\sim}A$ and $C{\sim}B$, respectively. Then $f^{\cup}\circ T\circ g$ is a staircase relation.

Proof An ordering is also a preorder (and a linear ordering is a linear preorder). So the corollary follows immediately from the combination of lemmas 210 and 218.
□

Theorem 220 Not every staircase relation is block-ordered. Specifically, the less-than relation on the real numbers (or the rational numbers) is a staircase relation but is not block-ordered.

Proof We remarked in example 216 that the less-than relation on the real numbers is a staircase relation. To show that it is not block-ordered, we exploit lemma 178.

Suppose that the less-than relation on the real numbers is block-ordered by the functions f and g and the provisional ordering T. That is, suppose

$$R \;=\; f^{\cup}\circ T\circ g \quad ,$$

where f and g are functionals of type $A{\leftarrow}\mathbb{R}$, for some A, and

$$f\circ f^{\cup} = T\cap T^{\cup} = g\circ g^{\cup} \quad\wedge\quad I_{\mathbb{R}} \subseteq f^{\cup}\circ f \cap g^{\cup}\circ g \quad ,$$

T is a provisional ordering of type $A{\sim}A$ and for all x and y of type $\mathbb{R}$, and

$$x\circ\top\!\top\circ y \subseteq R \quad\equiv\quad x<y \quad .$$

We begin by showing that $f^{\cup}\circ g$ is the empty relation. Inevitably, we need to exploit the pointwise definition of the diagonal, as formulated in lemma 40.

$$
\begin{array}{ll}
 & x\circ\top\!\top\circ y \subseteq f^{\cup}\circ g \\
= & \{ \quad \text{lemma 178, in particular (181)} \quad \} \\
 & x\circ\top\!\top\circ y \subseteq R\cap R^{\cup}\backslash R^{\cup}/R^{\cup} \\
= & \{ \quad \text{definition of intersection and lemma 40} \quad \} \\
 & x\circ\top\!\top\circ y \subseteq R \;\wedge\; \langle\forall u,v : u\circ\top\!\top\circ y \subseteq R \wedge x\circ\top\!\top\circ v \subseteq R : u\circ\top\!\top\circ v \subseteq R\rangle \\
= & \{ \quad \text{definition of } R \quad \} \\
 & x<y \;\wedge\; \langle\forall u,v : u<y \wedge x<v : u<v\rangle \\
\Rightarrow & \{ \quad u,v := y-(y-x)\times\frac{2}{3}\,,\, x+(y-x)\times\frac{1}{3} \\
 & \quad\;\; (\text{Note that } x<y \;\Rightarrow\; y-(y-x)\times\frac{2}{3} < y \,\wedge\, x < x+(y-x)\times\frac{1}{3}) \quad \} \\
 & y-(y-x)\times\frac{2}{3} < x+(y-x)\times\frac{1}{3} \\
= & \{ \quad \text{arithmetic} \quad \} \\
 & y-x < y-x \\
= & \{ \quad \text{the less-than relation is irreflexive} \quad \} \\
 & \mathsf{false} \quad .
\end{array}
$$

That is, by the saturation axiom (16), $f^{\cup}\circ g = \perp\!\!\perp_{\mathbb{R}}$. This contradicts theorem 196 since the left (and right) domain of the empty relation is the empty relation and the left and right domains of the less-than relation are both non-empty.
□

A brief, informal summary of the proof of theorem 220 is that the less-than relation on real numbers is indeed a staircase relation but has no "diagonal" (more formally its "diagonal" is the empty relation) and no such staircase relation can be block-ordered. The informal contrapositive is that a necessary step in the process of block-ordering a staircase relation is to begin by identifying its diagonal; this is a difunctional relation and can be represented by $f^{\cup}\circ g$ where f and g are functional. If the right domain of g equals the right domain, and the right domain of f equals the left domain of the given relation, the process is completed by identifying the ordering relation T.

For example, the less-than relation on the integers is block-ordered. Indeed, for all integers m and n

$$m<n \;\equiv\; m+1\leq n \quad .$$

The relation f is thus the successor function, the relation T is the at-most relation and the relation g is the identity function (on the integers). The "diagonal" is the set of pairs $(m\,,\,m+1)$.

The less-than relation on the natural numbers is also block-ordered but more care needs to be taken in the definition of the block-ordering. The relation f is the successor function; its source is the natural numbers and its target is the strictly positive natural numbers. The provisional ordering T is a subset of the at-most relation on natural numbers (specifically, the at-most relation restricted to the strictly positive natural numbers) and g is the *partial* identity relation on the natural numbers with left (and right) domain the strictly positive natural numbers. (Thus no number is related by g to the number 0.)

That the less-than relation on the real numbers is not block-ordered is a consequence of the fact that if $x<y$ the interval between x and y can always be subdivided at will. (That is, it is always possible to find a real number z such that $x<z$ and $z<y$.) The same is also true of the rationals and the proof of theorem 220 is equally valid in this case. Abstracting from the details of the less-than relation, we get the following theorem.

Theorem 221 Suppose R is a homogeneous relation such that

$$R\neq\perp\!\!\perp \;\;\wedge\;\; I\cap R=\perp\!\!\perp \;\;\wedge\;\; R=R\circ R \;\;\wedge\;\; R\cup I\cup R^{\cup}=\top\!\!\top \quad .$$

Then R is a staircase relation and $\Delta R=\perp\!\!\perp$.

It follows that any such relation is not block-ordered.

Proof Lemma 211 proves that R is a staircase relation.

Comparing the above conditions on R with those in lemma 211, the additions are the non-emptiness property $R \neq \perp\!\!\perp$, the "strictness" property $I \cap R = \perp\!\!\perp$ and the "subdivision" property $R \subseteq R \circ R$. (The less-than relation on real numbers has the subdivision property whereas the less-than relation on the integers does not.) Applying lemma 222 (below), the subdivision and strictness properties imply that $\Delta R = \perp\!\!\perp$. That R is not block-ordered follows from theorem 196 and the assumption that $R \neq \perp\!\!\perp$.
□

The lemma used to prove theorem 221 is the following:

Lemma 222

$$R \subseteq R \circ R \Rightarrow (\Delta R = \perp\!\!\perp \equiv I \cap R \subseteq \perp\!\!\perp) \quad .$$

Proof

$$
\begin{array}{ll}
 & R \subseteq R \circ \neg R^{\cup} \circ R \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & I \cap R \subseteq I \cap R \circ \neg R^{\cup} \circ R \\
\Rightarrow & \{ \quad \text{modular law} \quad \} \\
 & I \cap R \subseteq R \circ (R^{\cup} \circ R^{\cup} \cap \neg R^{\cup}) \circ R \\
= & \{ \quad \text{assumption: } R \subseteq R \circ R \quad \} \\
 & I \cap R \subseteq R \circ (R^{\cup} \cap \neg R^{\cup}) \circ R \\
= & \{ \quad \text{complements} \quad \} \\
 & I \cap R \subseteq \perp\!\!\perp \\
= & \{ \quad I = I^{\cup} \text{, converse and shunting} \quad \} \\
 & I \subseteq \neg R^{\cup} \\
\Rightarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R \circ R \subseteq R \circ \neg R^{\cup} \circ R \\
\Rightarrow & \{ \quad \text{assumption: } R \subseteq R \circ R \text{ and transitivity} \quad \} \\
 & R \subseteq R \circ \neg R^{\cup} \circ R \quad .
\end{array}
$$

That is,

$$(223) \quad R \subseteq R \circ R \ \Rightarrow \ (R \subseteq R \circ \neg R^{\cup} \circ R \ \equiv \ I \cap R \subseteq \perp\!\!\perp) \quad .$$

So

$\Delta R = \perp\!\!\perp$

$=$ { $[\perp\!\!\perp \subseteq X]$ and antisymmetry, definition of ΔR }

$R \cap (R\backslash R/R)^{\cup} \subseteq \perp\!\!\perp$

$=$ { shunting }

$R \subseteq \neg(R\backslash R/R)^{\cup}$

$=$ { (32) }

$R \subseteq R \circ \neg R^{\cup} \circ R$

$=$ { assumption: $R \subseteq R \circ R$, (223) }

$I \cap R \subseteq \perp\!\!\perp$.

□

The assumption that $R \neq \perp\!\!\perp$ in theorem 221 is necessary. The relation $\neg I_{1\!\!1}$ (see (33)) is the empty relation; it is also a block-ordered staircase relation on a finite type that satisfies all the assumptions of theorem 221 except for the assumption that it is non-empty.

Note that, if R is a homogeneous relation such that

$$R \neq \perp\!\!\perp \;\wedge\; R = R \circ R \;\wedge\; I \cap R = \perp\!\!\perp \quad ,$$

the left and right domains of R cannot be finite. (The easy proof involves constructing an infinite sequence of points $\langle i : i \in \mathbb{N} : a_i \rangle$ such that,

$$\langle \forall i :: a_i \circ \top\!\!\top \circ a_{i+1} \subseteq R \rangle \;\wedge\; \langle \forall i,j :: a_i = a_j \;\equiv\; i = j \rangle \quad .$$

This raises the question whether all finite staircase relations are linearly block-ordered.

9.6 Linear Block Ordering

Recall that, immediately following lemma 204, we remarked that the definition of SC demands refinement. This is more evident from the limitations of corollary 219: the corollary assumes a linear ordering —and not a provisional linear ordering— and, more importantly, that f and g are surjective. In practice, one might be tempted to fudge the application of the corollary by restricting a given ordering to a subset of the elements on which it is defined (for example, restricting the at-most ordering on integers to the at-most ordering on even integers). Rather than resort to such measures, we prefer to make the process precise within our axiom system. Indeed, it is necessary for us to do so in order to establish a sufficient condition for a staircase relation to be linearly block-ordered. See theorem 233 below.

In the following lemmas $R_{>\bullet}$ denotes the complement of $R_{>}$ in the lattice of coreflexives. That is, for arbitrary relation R, we have

(224) $\quad R_{>} \cup R_{>\bullet} = I \quad \wedge \quad R_{>} \cap R_{>\bullet} = \perp\!\!\!\perp$

(where I and $\perp\!\!\!\perp$ denote the identity and empty relations of appropriate type). Similarly $R_{\bullet<}$ denotes the complement of $R_{<}$ That is

(225) $\quad R_{<} \cup R_{\bullet<} = I \quad \wedge \quad R_{<} \cap R_{\bullet<} = \perp\!\!\!\perp \quad .$

Domain calculus enables the proof of the following:

(226) $\quad R \circ R_{>\bullet} = \perp\!\!\!\perp \quad \wedge \quad R_{\bullet<} \circ R = \perp\!\!\!\perp \quad .$

Given a relation R, the points in $R_{\bullet<}$ (or, dually $R_{>\bullet}$) are arguably irrelevant since they are precisely the points that are not related to any other point by R. Similar statements can be made about factors. In general, for arbitrary relations R and S, the factor $R\backslash S$ is arguably too big because its left domain includes $R_{>\bullet}$. Similarly, the factor R/S is too big because its right domain includes $S_{\bullet<}$, as is shown in the following lemma.

Lemma 227 For all R and S,

$$R_{>\bullet} \circ R\backslash S \;=\; R_{>\bullet} \circ \top\!\!\!\top \quad \wedge \quad R/S \circ S_{\bullet<} \;=\; \top\!\!\!\top \circ S_{\bullet<} \quad .$$

Proof We prove the first equation:

$$R_{>\bullet} \circ \top\!\!\!\top$$

$=\quad \{\quad$ complements $\quad\}$

$$R_{>\bullet} \circ (R\backslash S \cup \neg(R\backslash S))$$

$=\quad \{\quad$ distributivity $\quad\}$

$$R_{>\bullet} \circ R\backslash S \;\cup\; R_{>\bullet} \circ \neg(R\backslash S)$$

$=\quad \{\quad$ (38) $\quad\}$

$$R_{>\bullet} \circ R\backslash S \;\cup\; R_{>\bullet} \circ R^{\cup} \circ \neg S$$

$=\quad \{\quad$ (226) and converse $\quad\}$

$$R_{>\bullet} \circ R\backslash S \;\cup\; \perp\!\!\!\perp \circ \neg S$$

$=\quad \{\quad$ $\perp\!\!\!\perp$ is zero of composition and unit of union $\quad\}$

$$R_{>\bullet} \circ R\backslash S \quad .$$

□

The argument that factors typically include irrelevant information extends to the preorders $R\backslash R$ and R/R. In particular, note the terms involving $R_{>\bullet}$ in the following lemma.

Lemma 228 For all R,

$$R\backslash R \cup (R\backslash R)^{\cup} \;=\; R_{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R_{>} \;\cup\; R_{>\bullet} \circ \top\!\!\top \;\cup\; \top\!\!\top \circ R_{>\bullet} \;\; .$$

Proof

$$
\begin{array}{ll}
 & R\backslash R \cup (R\backslash R)^{\cup} \\
= & \{ \quad (224) \quad \} \\
 & (R_{>} \cup R_{>\bullet}) \circ R\backslash R \;\cup\; (R\backslash R)^{\cup} \circ (R_{>} \cup R_{>\bullet}) \\
= & \{ \quad \text{distributivity} \quad \} \\
 & \quad R_{>} \circ R\backslash R \;\cup\; R_{>\bullet} \circ R\backslash R \\
 & \cup \quad (R\backslash R)^{\cup} \circ R_{>} \;\cup\; (R\backslash R)^{\cup} \circ R_{>\bullet} \\
= & \{ \quad \text{lemma 227 and rearranging} \quad \} \\
 & \quad R_{>} \circ R\backslash R \;\cup\; \top\!\!\top \circ R_{>\bullet} \\
 & \cup \quad (R\backslash R)^{\cup} \circ R_{>} \;\cup\; R_{>\bullet} \circ \top\!\!\top \\
= & \{ \quad (224) \text{ and distributivity (as in first two steps)} \quad \} \\
 & \quad R_{>} \circ R\backslash R \circ R_{>} \;\cup\; R_{>} \circ R\backslash R \circ R_{>\bullet} \;\cup\; \top\!\!\top \circ R_{>\bullet} \\
 & \cup \quad R_{>} \circ (R\backslash R)^{\cup} \circ R_{>} \;\cup\; R_{>\bullet} \circ (R\backslash R)^{\cup} \circ R_{>} \;\cup\; R_{>\bullet} \circ \top\!\!\top \\
= & \{ \quad R_{>} \circ R\backslash R \subseteq \top\!\!\top \text{ and } (R\backslash R)^{\cup} \circ R_{>} \subseteq \top\!\!\top \\
 & \quad\quad \text{and definition of subset relation} \quad \} \\
 & \quad R_{>} \circ R\backslash R \circ R_{>} \;\cup\; \top\!\!\top \circ R_{>\bullet} \\
 & \cup \quad R_{>} \circ (R\backslash R)^{\cup} \circ R_{>} \;\cup\; R_{>\bullet} \circ \top\!\!\top \\
= & \{ \quad \text{rearranging and distributivity} \quad \} \\
 & R_{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R_{>} \;\cup\; \top\!\!\top \circ R_{>\bullet} \;\cup\; R_{>\bullet} \circ \top\!\!\top \;\; .
\end{array}
$$

□

(Lemma is essentially the case analysis that was omitted in the proof of lemma 204.) Avoiding the useless information introduced by the factor operators was the motivation for our introducing the notion of “provisional” (pre)orders. The following lemma enables the conventional notion of a linear ordering to be extended to provisional orderings.

Lemma 229 For all R,

$$R_{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R_{>} = R_{>} \circ \top\!\!\top \circ R_{>} \;\equiv\; R\backslash R \cup (R\backslash R)^{\cup} = \top\!\!\top \;\; .$$

Proof By mutual implication. First,

$$R\backslash R \cup (R\backslash R)^{\cup} = \top\!\top$$

$\Rightarrow$ { Leibniz }

$$R_{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R_{>} \ = \ R_{>} \circ \top\!\top \circ R_{>} \quad .$$

Second,

$$R\backslash R \cup (R\backslash R)^{\cup}$$

$=$ { lemma 228 }

$$R_{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R_{>} \ \cup \ R_{>\bullet} \circ \top\!\top \ \cup \ \top\!\top \circ R_{>\bullet}$$

$=$ { assume: $R_{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R_{>} = R_{>} \circ \top\!\top \circ R_{>}$ }

$$R_{>} \circ \top\!\top \circ R_{>} \ \cup \ R_{>\bullet} \circ \top\!\top \ \cup \ \top\!\top \circ R_{>\bullet}$$

$=$ { (224), distributivity and rearranging
(as in proof of lemma 228) }

$$R_{>} \circ \top\!\top \circ R_{>} \ \cup \ R_{>\bullet} \circ \top\!\top \circ R_{>} \ \cup \ R_{>\bullet} \circ \top\!\top$$
$$\cup \quad R_{>} \circ \top\!\top \circ R_{>} \ \cup \ R_{>} \circ \top\!\top \circ R_{>\bullet} \ \cup \ \top\!\top \circ R_{>\bullet}$$

$=$ { (224), distributivity and rearranging }

$$\top\!\top \circ R_{>} \ \cup \ R_{>\bullet} \circ \top\!\top$$
$$\cup \quad R_{>} \circ \top\!\top \ \cup \ \top\!\top \circ R_{>\bullet}$$

$=$ { (224), distributivity and rearranging }

$$\top\!\top \quad .$$

(Note the assumption in the second step.) That is,

$$R_{>} \circ (R\backslash R \cup (R\backslash R)^{\cup}) \circ R_{>} = R_{>} \circ \top\!\top \circ R_{>} \ \Rightarrow \ R\backslash R \cup (R\backslash R)^{\cup} = \top\!\top \quad .$$

□

Lemma 230 A linear provisional ordering is a staircase relation.

Proof Suppose T is a linear provisional ordering. Then

$$SC.T$$

$=$ { definition 203 }

$$T\backslash T \cup (T\backslash T)^{\cup} = \top\!\top$$

$=$ { lemma 229 }

$$T_{>}\circ(T\backslash T\cup(T\backslash T)^{\cup})\circ T_{>} = T_{>}\circ\top\!\!\top\circ T_{>}$$

$= \quad \{ \quad$ lemma 79 $\quad \}$

$$(T\cap T^{\cup})\circ(T\backslash T\cup(T\backslash T)^{\cup})\circ(T\cap T^{\cup}) \;=\; (T\cap T^{\cup})\circ\top\!\!\top\circ(T\cap T^{\cup})$$

$= \quad \{ \quad$ assumption: T is a provisional ordering

lemma 76 and definition 78 $\quad \}$

$$(T\cap T^{\cup})\circ(T\cup T^{\cup})\circ(T\cap T^{\cup}) \;=\; (T\cap T^{\cup})\circ\top\!\!\top\circ(T\cap T^{\cup})$$

$= \quad \{ \quad$ assumption: T is linear, definition 78 $\quad \}$

true .

□

Lemma 231 Suppose R is a linearly block-ordered relation. Then R is a staircase relation.

Proof This is an immediate consequence of lemmas 218 and 230. Specifically, by definition 170, R is a block-ordered relation if $R = f^{\cup}\circ T\circ g$ where f and g satisfy (172) and T is a provisional ordering (i.e. satisfies (171)). It is a linearly block-ordered relation if, in addition, T is a linear provisional ordering. Applying lemma 218 (with $R{:=}T$), R is a staircase relation if T is a staircase relation. But this is indeed the case by lemma 230.
□

Lemma 232 Suppose R is a staircase relation. Then

$$R \text{ is linearly block-ordered } \;\Leftarrow\; (\Delta R)_{<} \,=\, R_{<} \,\wedge\, (\Delta R)_{>} \,=\, R_{>} \quad .$$

Proof By lemma 193, R is block-ordered. Specifically, lemma 193 shows how to construct functionals f and g and a provisional ordering T satisfying the properties (172) and (171) such that $R = f^{\cup}\circ T\circ g$. The task is thus to prove that T is linear if R is a staircase relation.

We have:

$$R_{<}\circ(R\backslash R/R)^{\cup}\circ R_{>}$$

$= \quad \{ \quad R = f^{\cup}\circ T\circ g$ and (172) $\quad \}$

$$f_{>}\circ((f^{\cup}\circ T\circ g)\backslash(f^{\cup}\circ T\circ g)/(f^{\cup}\circ T\circ g))^{\cup}\circ g_{>}$$

$= \quad \{ \quad$ converse and factors: (37) $\quad \}$

$$f_{>}\circ(g^{\cup}\circ T^{\cup}\circ f)\backslash(g^{\cup}\circ T^{\cup}\circ f)/(g^{\cup}\circ T^{\cup}\circ f)\circ g_{>}$$

$$
\begin{array}{ll}
= & \{ \quad \text{lemma 63 with } U,V,W := T^{\cup}, T^{\cup}, T^{\cup} \quad \} \\
 & f^{\cup} \circ (T\backslash T/T)^{\cup} \circ g \\
= & \{ \quad \text{domains and (172)} \quad \} \\
 & f^{\cup} \circ (T \cap T^{\cup}) \circ (T\backslash T/T)^{\cup} \circ (T \cap T^{\cup}) \circ g \\
= & \{ \quad T \text{ is a provisional ordering, lemmas 75 and 76} \quad \} \\
 & f^{\cup} \circ T^{\cup} \circ g \quad .
\end{array}
$$

So

$$
\begin{array}{ll}
 & SC.R \\
= & \{ \quad (208) \quad \} \\
 & R \cup (R\backslash R/R)^{\cup} \;=\; \top\!\!\top \\
\Rightarrow & \{ \quad [S \subseteq \top\!\!\top], \text{ domains and monotonicity} \quad \} \\
 & R \;\cup\; R_{<} \circ (R\backslash R/R)^{\cup} \circ R_{>} \;=\; R_{<} \circ \top\!\!\top \circ R_{>} \\
= & \{ \quad R = f^{\cup} \circ T \circ g \text{ and above calculation} \quad \} \\
 & f^{\cup} \circ T \circ g \;\cup\; f^{\cup} \circ T^{\cup} \circ g \;=\; f_{>} \circ \top\!\!\top \circ g_{>} \\
= & \{ \quad \text{distributivity} \quad \} \\
 & f^{\cup} \circ (T \cup T^{\cup}) \circ g \;=\; f_{>} \circ \top\!\!\top \circ g_{>} \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & f \circ f^{\cup} \circ (T \cup T^{\cup}) \circ g \circ g^{\cup} \;\supseteq\; f \circ f_{>} \circ \top\!\!\top \circ g_{>} \circ g^{\cup} \\
= & \{ \quad \text{definition 170 of block-ordered} \\
 & \qquad \text{in particular (172); domains} \quad \} \\
 & (T \cap T^{\cup}) \circ (T \cup T^{\cup}) \circ (T \cap T^{\cup}) \;=\; (T \cap T^{\cup}) \circ \top\!\!\top \circ (T \cap T^{\cup}) \\
= & \{ \quad \text{lemma 79 and definition 78} \quad \} \\
 & T \text{ is linear} \quad .
\end{array}
$$

□

Theorem 233 Suppose R is a staircase relation. Then

$$
R \text{ is linearly block-ordered} \;\equiv\; (\Delta R)_{<} \;=\; R_{<} \;\wedge\; (\Delta R)_{>} \;=\; R_{>} \quad .
$$

Proof By mutual implication. “Only-if” is an instance of theorem 196. “If” is lemma 232.

□

9.7 Riguet's Rectangle Theorem

As mentioned earlier, the purpose of undertaking this exercise was to demonstrate how reasoning with factors is so much more straightforward than reasoning with nested negations. It was a surprise to discover an error in the extant literature. This section is about our attempt to trace the source of the material on difunctional and staircase relations and, in particular, the source of the error.

Riguet introduces the notion of a difunctional relation in [Rig48] and the notion of a staircase relation in [Rig51] — but uses the name "relation de Ferrers". His definition corresponds to property (209). He lists a number of properties related to the ones stated above. Direct comparison is slightly complicated by the fact that he does not make use of factors. For example, he states that R is a "relation de Ferrers" if and only if $R \circ \neg R^{\cup}$ is a "relation de Ferrers". This is a combination of lemma 205 (in particular (206)) and lemma 212.

Riguet does not give a proof of the theorem. Riguet [Rig51] states that there is a striking analogy ("une analogie frappante") between the definitions and properties of "relations de Ferrers" and difunctional relations. He states that the analogy is clarified by[9] a theorem similar to our lemma 150 but does not go into further details. As mentioned earlier, his theorem is that, if R is a staircase relation (a "relation de Ferrers"), then so too is $R \circ \neg R^{\cup} \circ R$ and their "différence" $R \cap \neg(R \circ \neg R^{\cup} \circ R)$ is difunctional. Lemma 150 is stronger than Riguet's difunctionality property because it does not require R to be a staircase relation.

Note that in the case that R is the less-than relation on real numbers, $R \circ \neg R^{\cup} \circ R$ is also the less-than relation and $R \cap \neg(R \circ \neg R^{\cup} \circ R)$ is trivially difunctional (since it is the empty relation). This observation leads one to wonder precisely how the "analogie frappante" is clarified by Riguet's theorem. (We invite the reader to verify the claims we have just made and then work out the difference when "real number" is replaced by "integer".)

As announced earlier, the proof of Riguet's theorem is straightforward[10]:

Theorem 234 (Riguet's theorem) A relation is a staircase relation if and only if it has a linear polar covering.

Proof By mutual implication.

For the "only-if" part, theorem 156 establishes that every relation has a polar covering. So it suffices to show that if R is a staircase relation the covering is linear. Recall the construction of $\mathcal{R}$ in theorem 156. If R is a staircase relation, that the set $\mathcal{R}_{<}$ is linearly ordered by inclusion is immediate from lemma 204.

[9] "Cette analogie s'éclaire par"

[10] This may explain why he didn't provide a proof.

For the "if" part, suppose R of type $A{\sim}B$ has a linear polar covering $\mathcal{R}$. Our task is to show that R is a staircase relation. Aiming to apply lemma 204, we consider points b and b' such that $b \subseteq R{>}$ and $b' \subseteq R{>}$. Our task becomes to show that

$$(R{\circ}b){<} \subseteq (R{\circ}b'){<} \;\vee\; (R{\circ}b'){<} \subseteq (R{\circ}b){<} \quad .$$

This is achieved as follows:

$$
\begin{array}{ll}
 & (R{\circ}b){<} \subseteq (R{\circ}b'){<} \;\vee\; (R{\circ}b'){<} \subseteq (R{\circ}b){<} \\
= & \{ \quad R = \cup\mathcal{R} \quad \} \\
 & (\cup\mathcal{R} \circ b){<} \subseteq (R{\circ}b'){<} \;\vee\; (\cup\mathcal{R} \circ b'){<} \subseteq (R{\circ}b){<} \\
= & \{ \quad \text{distributivity properties} \quad \} \\
 & \langle \forall U : U{\in}\mathcal{R} : (U{\circ}b){<} \subseteq (R{\circ}b'){<} \rangle \;\vee\; \langle \forall U : U{\in}\mathcal{R} : (U{\circ}b'){<} \subseteq (R{\circ}b){<} \rangle \\
= & \{ \quad \text{lemma 84,} \\
 & \qquad \text{case analyses on } (b' \subseteq U{>} \wedge (U{\circ}b'){<} = U{<}) \vee (U{\circ}b'){<} = \perp\!\!\perp \\
 & \qquad \text{and } (b \subseteq U{>} \wedge (U{\circ}b){<} = U{<}) \vee (U{\circ}b){<} = \perp\!\!\perp \quad \} \\
 & \quad \langle \forall U : U{\in}\mathcal{R} \wedge b \subseteq U{>} : U{<} \subseteq (R{\circ}b'){<} \rangle \\
 & \vee \;\; \langle \forall U : U{\in}\mathcal{R} \wedge b' \subseteq U{>} : U{<} \subseteq (R{\circ}b){<} \rangle \\
\Leftarrow & \{ \quad R = \cup\mathcal{R}\text{, monotonicity and lemma 84} \quad \} \\
 & \quad \langle \forall U : U{\in}\mathcal{R} \wedge b \subseteq U{>} : \langle \exists V : V{\in}\mathcal{R} \wedge b' \subseteq V{>} : U{<} \subseteq V{<} \rangle \rangle \\
 & \vee \;\; \langle \forall U : U{\in}\mathcal{R} \wedge b' \subseteq U{>} : \langle \exists V : V{\in}\mathcal{R} \wedge b \subseteq V{>} : U{<} \subseteq V{<} \rangle \rangle \\
= & \{ \quad \text{assumption: } \mathcal{R} \text{ is a polar covering} \\
 & \qquad \text{so } U{<} \subseteq V{<} \;\equiv\; U{>} \supseteq V{>} \quad \} \\
 & \quad \langle \forall U : U{\in}\mathcal{R} \wedge b \subseteq U{>} : \langle \exists V : V{\in}\mathcal{R} \wedge b' \subseteq V{>} : U{>} \supseteq V{>} \rangle \rangle \\
 & \vee \;\; \langle \forall U : U{\in}\mathcal{R} \wedge b' \subseteq U{>} : \langle \exists V : V{\in}\mathcal{R} \wedge b \subseteq V{>} : U{>} \supseteq V{>} \rangle \rangle \\
= & \{ \quad [\, p \vee q \;\equiv\; (\neg q \Rightarrow p) \,] \text{ together with the calculation below} \quad \} \\
 & \text{true} \quad .
\end{array}
$$

The justification of the final step is as follows.

$$
\begin{array}{ll}
 & \neg\langle \forall U : U{\in}\mathcal{R} \wedge b' \subseteq U{>} : \langle \exists V : V{\in}\mathcal{R} \wedge b \subseteq V{>} : U{>} \supseteq V{>} \rangle \rangle \\
= & \{ \quad \text{predicate calculus (and dummy change: } U,V := V,U \text{)} \quad \} \\
 & \langle \exists V : V{\in}\mathcal{R} \wedge b' \subseteq V{>} : \langle \forall U : U{\in}\mathcal{R} \wedge b \subseteq U{>} : \neg(V{>} \supseteq U{>}) \rangle \rangle \\
= & \{ \quad \text{assumption: } \mathcal{R} \text{ is a linear polar covering}
\end{array}
$$

$$\text{in particular, the inclusion ordering on left domains is linear } \}$$

$$\langle \exists V : V{\in}\mathcal{R} \wedge b' \subseteq V{>} : \langle \forall U : U{\in}\mathcal{R} \wedge b \subseteq U{>} : V{>} \subset U{>} \rangle\rangle$$

$$\Rightarrow \quad \{ \quad \text{predicate calculus and } V{>} \subset U{>} \Rightarrow U{>} \supseteq V{>} \quad \}$$

$$\langle \forall U : U{\in}\mathcal{R} \wedge b \subseteq U{>} : \langle \exists V : V{\in}\mathcal{R} \wedge b' \subseteq V{>} : U{>} \supseteq V{>} \rangle\rangle \ .$$

□

In the proof of theorem 234 we have chosen a covering that is indexed by points in the *source* of the given relation R. We could, of course, have chosen a covering that is indexed by points in the relation's *target*. Fig. 14 is a mental picture of the different choices.

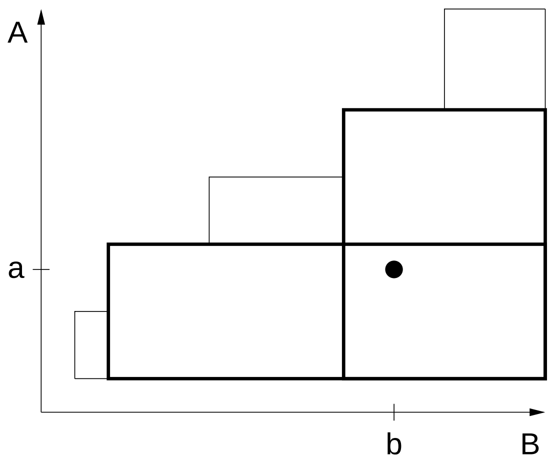


Figure 14: Choices of Polar Covering

Highlighted in fig. 14 are a point —the point $a{\circ}\top\!\top{\circ}b$ in our formalism— and two rectangles. The (highlighted) long, low rectangle depicts the relation

$$R/R \circ a \circ R \ ,$$

whilst the (highlighted) short, tall rectangle depicts the relation

$$R \circ b \circ R \backslash R \ .$$

Rather than choosing the latter as the elements of the polar covering —as we did— , we could have chosen the former. The (highlighted) corner rectangle depicts the relation

$$R/R \circ a \circ \top\!\top \circ b \circ R \backslash R \ .$$

Indeed, for all relations R,

$$R/R \circ a \circ R \ \cap \ R \circ b \circ R \backslash R \ = \ R/R \circ a \circ \top\!\top \circ b \circ R \backslash R$$

$$\Leftarrow \quad a{\circ}\top\!\top{\circ}b \subseteq R \ .$$

We leave the proof of this property to the reader. (Hint: use lemma 82.)

9.8 Finite Staircase Relations

As we have seen in theorems 220 and 221, not every staircase relation is block-ordered. However, for a relation to satisfy the assumptions made in theorem 221 it must be infinite. In this section we show that every finite staircase relation is indeed block-ordered.

The key is a combination of theorems 234 and 198. As exploited by theorem 198, theorem 156 suggests how to define a functional relation g with the same right domain as the given relation and with left domain a set of rectangles. As shown in theorem 234, these rectangles are linearly ordered (by the subset relation) if the given relation is a staircase relation.

Since it is the set of rectangles defined by theorem 156 that needs to be finite (rather than the given relation) we generalise to relations that are "block-finite":

Definition 235 A relation R is *block-finite* if either of the sets of rectangles

$$\{a : a \subseteq R_< : R/R \circ a \circ R\}$$

or

$$\{b : b \subseteq R_> : R \circ b \circ R\backslash R\}$$

has finite cardinality.
□

Theorem 236 Suppose R is a block-finite staircase relation. Then R has a linear block-ordering.

Proof Suppose R is a block-finite relation and R is a staircase relation. Our goal is to define relations f and g and T such that R is block-ordered by f, g and T (see definition 170) and T is linear.

Because it fits with the statement and proof of theorem 234, we assume that the set of rectangles $\{b : b \subseteq R_> : R \circ b \circ R\backslash R\}$ is finite. Now a linear provisional ordering with domain a finite set is trivially complete. Thus we may apply theorem 198. Equations (199), (201) and (200) define a provisional ordering $\sqsubseteq$ and functional relations f and g such that $R = f^{\cup} \circ \sqsubseteq \circ g$. Moreover, $\sqsubseteq$ is linear — since R is a staircase relation (see the details of the proof of theorem 234)— . Also, by theorem 198, $f_< \subseteq g_<$, $f_> = R_<$, $g_> = R_>$ and $g_< = (\sqsubseteq)_<$. It remains to show that $f_< = g_<$.

The characteristic property of a linear ordering of a finite set is that the notion of infimum coincides with the notion of minimum[11]. That is, the definition (201) of f can be rewritten as, for all a such that $a \subseteq R_<$,

(237) $f.a \;=\; \langle \mathsf{MIN}_{\sqsubseteq}\, b : a \circ \top\!\top \circ b \subseteq R : R \circ b \circ R\backslash R \rangle$.

[11]Formally, the proof is by induction on the cardinality of the set. This is such a well-known and easily proved property that we omit the proof.

Now, for all b,

$$
\begin{array}{ll}
& \langle \exists a : a \circ \top\!\!\top \circ b \subseteq R : R \circ b \circ R \backslash R = f.a \rangle \\
= & \{ \quad (237) \text{ and definition of } \mathsf{MIN}_{\sqsubseteq} \quad \} \\
& \langle \exists a : a \circ \top\!\!\top \circ b \subseteq R : \langle \forall b' : a \circ \top\!\!\top \circ b' \subseteq R : R \circ b \circ R \backslash R \sqsubseteq R \circ b' \circ R \backslash R \rangle \rangle \\
= & \{ \quad \text{definition of } \sqsubseteq \text{: (199)} \quad \} \\
& \langle \exists a : a \circ \top\!\!\top \circ b \subseteq R : \langle \forall b' : a \circ \top\!\!\top \circ b' \subseteq R : (R \circ b)_< \subseteq (R \circ b')_< \rangle \rangle \\
= & \{ \quad \text{lemma 46} \quad \} \\
& \langle \exists a : a \subseteq (R \circ b)_< : \langle \forall b' : a \subseteq (R \circ b')_< : (R \circ b)_< \subseteq (R \circ b')_< \rangle \rangle \\
= & \{ \quad \text{definition of infimum} \quad \} \\
& \langle \exists a : a \subseteq (R \circ b)_< : (R \circ b)_< \subseteq \langle \cap b' : a \subseteq (R \circ b')_< : (R \circ b')_< \rangle \rangle \\
= & \{ \quad \langle \forall b' :: a \subseteq (R \circ b')_< \Rightarrow b' \subseteq R_> \rangle \ ; \\
& \quad\ \subseteq \text{ is a linear provisional ordering on the finite set} \\
& \quad\ \{b' : b' \subseteq R_> : (R \circ b')_< \}\text{, so infimum is } \mathsf{MIN} \quad \} \\
& \langle \exists a : a \subseteq (R \circ b)_< : (R \circ b)_< \subseteq \langle \mathsf{MIN}_{\subseteq}\, b' : a \subseteq (R \circ b')_< : (R \circ b')_< \rangle \rangle \\
\Leftarrow & \{ \quad \text{definition of } \mathsf{MIN}_{\subseteq} \quad \} \\
& \langle \exists a : a \subseteq (R \circ b)_< : a \subseteq (R \circ b)_< \rangle \\
= & \{ \quad \text{lemma 47} \quad \} \\
& b \subseteq R_> \ .
\end{array}
$$

That is

$$\langle \forall b : b \subseteq R_> : \langle \exists a : a \circ \top\!\!\top \circ b \subseteq R : g.b = f.a \rangle \rangle \ .$$

It follows that $g_< \subseteq f_<$. But, by theorem 198, $f_< \subseteq g_<$. We conclude that $f_< = g_<$ as required.
□

10 Discussion

The writing of this paper began after reading a paper by Wolfram Kahl which included a section on “Ferrers-type relations” citing not Riguet [Rig51] (where the notion is introduced) but the textbook by Schmidt and Ströhlein [SS93]. Although Schmidt and Ströhlein also do not cite [Rig51], they do use Riguet’s definitions. It was immediately clear that substantial improvements could be made to Schmidt and Ströhlein's calculations by exploiting the properties of the factors of a relation. Further study also revealed an obvious error in their "definition" [SS93, Definition 4.4.11]. (Schmidt and Ströhlein's "definitions" often include what they call "definition variants" which, in most cases, they deem to be obviously equivalent. This is not the case here — see below.) This led to an investigation of the origin of the error which, in turn, led to the discovery of the original paper by [Rig51]. Several more recent publications were also discovered where the opportunity to correct Schmidt and Ströhlein's error is not taken. Intrigued, it was decided to embark on a thorough investigation of the notions introduced in [Rig51]: the notion of the "différence" of a relation and the notion of a "relation de Ferrers" as well as Riguet's "analogie frappante" connecting the two. In the process, it became clear that a more general notion of "block-ordering" was relevant than the total ordering demanded by Riguet. This led to the four goals enumerated in the introduction.

The need for the first two goals is clear from a study of Riguet's paper. Although his work is comprehensive (in particular [Rig48]), the typography of publications written 70 years ago makes them difficult to read; the notation chosen by Riguet is also often rather quaint (and in some cases impossible to reproduce!). Ironically in a paper about "correspondances de Galois", Riguet does not introduce the Galois connection defining the factors of a relation and, instead, makes copious use of (nested) complements. Also, Riguet states many properties without proof: for example, [Rig51] lists ten definitions of a "relation de Ferrers" with justification that it is easy to see ("il est facile de voir") that they are all equivalent. Moreover, subsequent literature leaves many gaps. For example, we have been unable to find any proof of theorem 236, even though we have seen several publications that assume the theorem (correctly in the case of finite relations).

Experience shows that the most important concepts —the ones with wide applicability— tend to be discovered and rediscovered, often quite independently, in several different and apparently unrelated contexts. Different formulations, that turn out to be equivalent, and different terminology, reflecting particular application areas, is introduced, making the task of proper attribution almost impossible. All that an author can be expected to do is to cite the publications that have had a significant influence on their own work — which is what we have done here.

For the reasons given above, the initial steps in the writing of this document were influenced by section 4.4 of the textbook by Schmidt and Ströhlein [SS93]. Like Riguet, Schmidt and Ströhlein do not introduce the factor operators and, implicitly, use the equivalent definition in terms of nested complements. (See lemma 32.) The longest and arguably most opaque calculation in this section of Schmidt and Ströhlein's book is their proof of proposition 4.4.13(ii). Aside from its extensive use of nested complements, it fails to make clear what is being proved, why it is being proved and where and when assumptions are invoked (at least in our view). The proposition is formulated in theorem

184. Various properties are used in their proof which we have formulated and proven in lemma 178 in terms of factors. Properties (179) and —more significantly— (180) are not observed by Schmidt and Ströhlein. Their derivation of (181) is asymmetric in f and g and involves several unexplained steps.

We have not been able to avoid the use of complements altogether. As pointed out at the time, the equivalence of several different formulations of the notion of a staircase relation formulated in lemma 205 uses the definition of factors in terms of nested complements. Also, for concrete examples of (small) finite relations, such as examples 168, 169 and 202, the use of complements often makes calculations easier. Nevertheless our use of complements has been minimal.

We have attributed the two principal concepts of a "relation difonctionelle" and a "relation de Ferrers" to Riguet ([Rig48] and [Rig51], respectively) but we have not explored any publications prior to Riguet's. Riguet himself cites two papers by Norbert Wiener, dated 1912–1914 and 1914-1916, as giving an equivalent definition of a "relation de Ferrers" but no other indication of their content is provided (not even their titles). We have also been unable to find publications on either topic in the fourty or so years following their publication. (Riguet [Rig51] announces a "prochaine Note" that will make precise a correspondence between "relations equivalence conjuguées" and "relations de Ferrers" but we have not been able to find the publication.) So the current work should not be regarded as a history of the concepts.

The notion of a difunctional relation is now generally attributed to Riguet [Rig48]; Jaoua et al [JMBD91] use the name "regular relation" but later publications [KGJ00] use the name "difunctional relation". Voermans [Voe99] emphasises their importance in developing a theory of datatypes with laws; Oliveira [Oli18] argues that difunctional relations are "metaphors" for program specification. Much of our presentation on difunctionals and non-redundant polar coverings is influenced by the goal of gaining a complete understanding of Riguet's "analogie frappante" [Rig51].

The notions of a rectangle and completely disjoint rectangles, and elementary facts about difunctional relations, in particular theorems 100 and 132, are discussed by Riguet [Rig48]. The corresponding properties of pers (theorems 114 and 105) are well-known. The construction given in section 6.3.3 is not made explicit in [Rig50] but was most probably the basis of Riguet's statement that the characterisation of difunctional relations as a pair of functional relations (theorem 132) is a generalisation of the theorem that a partial equivalence relation is characterised by a single functional relation (theorem 105). (Evidence for this is that Riguet effectively states lemma 142.) Our contribution has been to compare different algebraic proofs of the theorems: point-free and pointwise proofs. Perhaps surprisingly, our conclusion is that the pointwise proofs are preferable to the proofs that exploit a point-free characterisation of power transpose. This is because of the simplicity of the step from elementary characterisations of pers and difunctionals (theorem 105(ii) and (131)) to a set of rectangles ("réunions de rectangles"): see, for example, the step from theorem 115 to lemma 120.

Theorem 132 is also stated in [JMBD91, Proposition 4.12] and a proof given. Their proof assumes the relation is homogeneous; the proof of theorem 141 is inspired by their proof whilst avoiding the assumption. Winter [Win04] assumes theorem 105 and then uses it to prove theorem 132 (thus making precise Riguet's generalisation). His (very short and elegant) proof, which we have reproduced here, gives different —albeit isomorphic— characterisations of a difunctional relation.

Theorem 136 is Schmidt and Ströhlein's proposition 4.4.10(ii) . Their statement of the theorem is unclear: it appears to state that a difunctional relation has exactly one representation as a pair of functional, surjective relations but they only prove that there is at most one such representation. (Both here and in the statement of proposition proposition 4.4.13(ii) they use the phrase "may be achieved in essentially one fashion". The English is ambiguous: "may be achieved" suggests "at least one" and "in essentially one fashion" suggests "at most one", the combination being exactly one. But they only prove at most one.) Lemma 135 is novel and permits a subtle difference in presentation, in particular of theorem 136.

There is much in common between our section 7.2 and Khchérif, Gammoudi and Jaoua [KGJ00]. Khchérif, Gammoudi and Jaoua [KGJ00] correctly attribute the concept of the diagonal to Riguet but do not cite [Rig51]; like Riguet, they define the diagonal in terms of nested complements and do not exploit factors. Their notion of a covering specifies the rectangles to be "maximal". This is the property of not being "obviously redundant" as discussed immediately following definition 155. Slightly confusingly[12], Khchérif, Gammoudi and Jaoua [KGJ00] define two rectangles to be "disjoint" when they are what we call "completely disjoint". With this caveat, they list theorem 134 as a property of difunctionals. They do not seem to be aware of theorem 156. Their focus is on what they call "minimal" coverings and "isolated points"; "minimal" coverings are what we call "non-redundant" coverings whilst "isolated points" are the points of a definiens of a relation. It is not clear whether or not they are aware that the "isolated points" can be grouped into the rectangles of a difunctional. They seem to suggest a dichotomy: for each relation R, either $(\Delta R)_{<} = R_{<}$ and $(\Delta R)_{>} = R_{>}$, or $\Delta R = \perp\!\!\perp$. (See [KGJ00, p.161, **Problem**].) Example 168 shows that this is not the case: it is indeed possible to construct a non-redundant covering of a relation R where $(\Delta R)_{<} \neq R_{<}$ so long as $(\Delta R)_{>} = R_{>}$ (and, of course, dually when $(\Delta R)_{>} \neq R_{>}$ so long as $(\Delta R)_{<} = R_{<}$). Example 169 is a counterexample to their [KGJ00, Theorem 1]. The statement of [KGJ00, Theorem 2] is unclear, making it difficult to verify or refute.

[12] The term "disjoint" is commonly used to describe sets with an empty intersection; the confusion arises because relations are sets of pairs.

Schmidt and Ströhlein [SS93, p.80] cite the paper by Jacques Riguet [Rig50] with the word "difonctionelle" in the title; they also use the same definition of a "Ferrers type relation" as Riguet but do not cite [Rig51]. (They do cite [Rig48] earlier in the text but not in connection with difunctionals.) Schmidt and Ströhlein appear to claim that "staircase" and "linearly block-ordered" are equivalent properties of a relation: Their definition of "Ferrers type" [SS93, Definition 4.4.11] comprises five properties connected by the symbol "$\Leftrightarrow$". Presumably the symbol denotes logical equivalence (an implicit universal quantification over all free variables combined with boolean equality) but it is nowhere defined[13]. From definition 2.1.3, and experience with common mathematical practice, one infers that Schmidt and Ströhlein use the keyword "**Definition**" to simultaneously introduce a definition and to state properties of the defined entity that are deemed to be obvious. The problem is that the equality of the predicates "staircase" and "linearly block-ordered" is far from obvious and, as we have shown in theorem 220, it is just not true! Other papers that cite Riguet assume that the relations under consideration are finite —in which case the equivalence is valid (see lemma 231 and theorem 236)— ; consequently, it would appear that the erroneous claim was introduced by Schmidt and Ströhlein.

Winter restates the erroneous claim made by Schmidt and Ströhlein [SS93, Definition 4.4.11]:

> A concrete relation of Ferrers type may be written as a Boolean matrix in staircase block form by suitably rearranging rows and columns.

I have been unable to find a definition of the word "concrete" in the paper; the use of the word "matrix" suggests that "concrete" means "finite". In this case, the claim is a special case of theorem 236. However, I have been unable to find any proof of the theorem in the published literature: Riguet [Rig51] states the theorem but does not provide a proof; he does make very clear that his definition of a "relation de Ferrers" extends to infinite relations, specifically by giving a concrete example. (In addition to finiteness, Riguet [Rig51] adds a second condition that I do not understand.)

Winter is clearly aware that the claim is invalid in general because immediately afterwards [Win04, lemma 5] states that the claim is invalid for "dense" relations. (Winter formulates a property of " dense linear strict-orderings" that is essentially theorem 221.) Winter does not, however, give the most obvious example of a "dense" relation — the less-than relation on real numbers. Schmidt [Sch08] does observe that the less-than rela-

[13] Page 1 introduces set notation and properties of sets. It uses the symbol "$\Rightarrow$" —presumably meaning "only if"— but the symbol is also nowhere defined. The symbol "$\Leftrightarrow$" first appears on p.7 and continued equivalences first appear on p.8 in definition 2.1.3 (reflexive and irreflexive relations). No explanation is given of how a continued equivalence is to be read. (Boolean equality is associative and transitive. So a continued equivalence could be read associatively or conjunctionally.)

tion is "dense" but does not take the opportunity to correct the error in [SS93, Definition 4.4.11].

As previously stated, the notion of the diagonal of a relation is due to Riguet [Rig51]; Riguet called it the "différence". (See the discussion immediately following lemma 217.) The notion of a "polar" covering was also introduced by Riguet in [Rig51], albeit with a slightly stricter definition to fit the topic of his paper ("relations de Ferrers"): he requires the subset ordering on domains to be total ("linear" in the terminology used here).

Winter [Win04] does not give the diagonal function a name but denotes the "différence" of relation R by R^d (as do Khchérif, Gammoudi and Jaoua [KGJ00]); Winter cites [Rig51] but does not ascribe the concept to Riguet. Schmidt [Sch08] calls it the "fringe" of the relation; Schmidt [Sch08] does cite Winter [Win04] but does not cite Riguet [Rig51]. Berghammer and Winter [BW12, p.8] state that Riguet's notion of the "différence" of a relation was "introduced" by Winter [Win04] and Schmidt [Sch08]; like Schmidt [Sch08], Berghammer and Winter [BW12] do not cite Riguet [Rig51]. Although Winter [Win04] and Berghammer and Winter [BW12] define the "différence" using residuals, they frequently use Riguet's definition in terms of nested complements.

Theorem 218 introduces two constraints slightly weaker than those imposed by Schmidt and Ströhlein in their proposition 4.4.13(i); it is also stronger because it states an equality rather than an implication. Lemma 210, in combination with lemma 215 also yields a stronger theorem than their proposition 4.4.13(i). (No constraints are imposed on the parameters f and g.)

Section 8.1 has been included partly to make Hartmanis and Stearn's [HS66] pioneering contribution to information science better known. The theory of "pair algebras" anticipates results in what has since become known as "concept analysis" [DP90]. For further details, see [Bac98].

Finally, a few words on notation. The very rich algebraic properties of the converse of a relation mean that many notions and properties come in pairs, each element of the pair being the dual mirror-image of the other. For example, we have defined both the left domain and right domain of a relation; lemma 43 is an example of mirror-image properties of the relations. Some authors emphasise such mirroring by their choice of notation. Freyd and Ščedrov [Fv90], for example, denote the source and target of a relation R by $\Box R$ and $R\Box$, respectively.

A consequence of this is that it is possible to get away with defining just one of a pair of operators, leaving its mirror image to have an "obvious" definition in terms of relational converse. For example, in section 3.5 we gave only the definition of the "left" power transpose of a relation, leaving the definition of the "right" power transpose to the reader. Doing this systematically would mean introducing the notation $R{<}$ for the left domain of relation R and then using the notation $(R^{\cup}){<}$ to denote the right domain of R. Similarly, one might introduce just the left factor R/S and then write $(S^{\cup}/R^{\cup})^{\cup}$ for the left factor $R\backslash S$. This is, of course, very undesirable because then the associativity of the operators (the rule that $R\backslash(S/T)$ and $(R\backslash S)/T$ are equal, which we exploit by using the notation $R\backslash S/T$) becomes the very cumbersome

$$((S/T)^{\cup}/R^{\cup})^{\cup} \;=\; (S^{\cup}/R^{\cup})^{\cup}/T \quad .$$

Even worse is when a symmetric notation is used for an operator that has both left and right variants — as is done by both Freyd and Ščedrov [Fv90] and Schmidt and Ströhlein [SS93, p.80] in the case of the so-called "symmetric division/quotient" of a relation. By writing $\frac{R}{S}$ (or $R \div S$), the reader may be misled into supposing that either the operator has no mirror image or that the mirror image is $\frac{S}{R}$ (or $S \div R$).

11 Further Work

In retrospect, I believe that a substantial improvement can be made by making effective use of Voerman's [Voe99] left per domain $\prec$ and right per domain $\succ$ operators. Recall that

$$R\prec \;=\; R/\!\!/R \circ R< \quad \wedge \quad R\succ \;=\; R> \circ R\backslash\!\!\backslash R \quad \wedge \quad R = R\prec \circ R \circ R\succ \quad .$$

Both are pers so can be characterised by their equivalence classes. Specifically, for a given R, suppose

$$R\prec \;=\; \lambda^{\cup} \circ \lambda \quad \wedge \quad R\succ \;=\; \rho^{\cup} \circ \rho$$

where λ and ρ are functional relations[14]. Then

$$R = \lambda^{\cup} \circ \lambda \circ R \circ \rho^{\cup} \circ \rho \quad .$$

The relation $\lambda \circ R \circ \rho^{\cup}$, which we denote by $|[R]|$, is a relation on the equivalence classes. For a mental picture of such a relation, refer to fig. 13: the individual blocks of the relation R become points of the relation $|[R]|$. I expect that the decomposition

$$R = \lambda^{\cup} \circ |[R]| \circ \rho$$

is "canonical" in the sense that there are theorems like: R is block-ordered equivales $|[R]|$ is a provisional ordering, and R is a staircase relation equivales $|[R]|$ is a linear provisional ordering. Also, I conjecture that the diagonal $\Delta|[R]|$ is coreflexive. In old-fashioned mathematical vernacular, I expect that the assumption $R = |[R]|$ can be made "without loss of generality", leading to many changes in the above presentation, but have yet to pursue this line of enquiry.

[14] I believe that λ is the Greek symbol for the letter l and ρ is the Greek symbol for the letter r.

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