Relators, Fans and Membership

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Allegories

Categorical formulation of (point-free) relation algebra.

Category (objects A, B, C, arrows —"relations"— R, S)

$$\mathsf{R}\circ\mathsf{S} : A \leftarrow \mathsf{B} \quad \Leftarrow \quad \mathsf{R} : A \leftarrow \mathsf{C} \wedge \mathsf{S} : \mathsf{C} \leftarrow \mathsf{B} \quad ,$$

$$\mathsf{id}_A : A \leftarrow A \quad .$$

Arrows of same type are partially ordered by $\subseteq$.

$$\mathsf{S}_1\circ\mathsf{T}_1 \subseteq \mathsf{S}_2\circ\mathsf{T}_2 \quad \Leftarrow \quad \mathsf{S}_1 \subseteq \mathsf{S}_2 \;\wedge\; \mathsf{T}_1 \subseteq \mathsf{T}_2 \quad .$$

$$\mathsf{X} \subseteq \mathsf{R} \wedge \mathsf{X} \subseteq \mathsf{S} \quad \equiv \quad \mathsf{X} \subseteq \mathsf{R}\cap\mathsf{S} \quad .$$

Converse

$$\mathsf{R}^{\cup} \subseteq \mathsf{S} \quad \equiv \quad \mathsf{R} \subseteq \mathsf{S}^{\cup} \quad ,$$

$$(\mathsf{R}\circ\mathsf{S})^{\cup} = \mathsf{S}^{\cup}\circ\mathsf{R}^{\cup} \quad ,$$

$$\mathsf{R}\circ\mathsf{S} \cap \mathsf{T} \subseteq (\mathsf{R} \cap \mathsf{T}\circ\mathsf{S}^{\cup})\circ\mathsf{S} \quad .$$

Relator

Relator: functor that is monotonic and respects converse.

Let $\mathcal{A}$ and $\mathcal{B}$ be allegories. A mapping F from objects of $\mathcal{A}$ to objects of $\mathcal{B}$ and arrows of $\mathcal{A}$ to arrows of $\mathcal{B}$ is a relator iff

$$\mathsf{F.R} : \mathsf{F}.A \leftarrow \mathsf{F.B} \quad \Leftarrow \quad \mathsf{R} : A \leftarrow \mathsf{B} \quad ,$$

$$\mathsf{F.R} \circ \mathsf{F.S} = \mathsf{F.(R \circ S)} \quad \text{for each } \mathsf{R} : A \leftarrow \mathsf{B} \text{ and } \mathsf{S} : \mathsf{B} \leftarrow \mathsf{C} \quad ,$$

$$\mathsf{F.id}_A = \mathsf{id}_{\mathsf{F}.A} \quad \text{for each object } A \quad ,$$

$$\mathsf{F.R} \subseteq \mathsf{F.S} \quad \Leftarrow \quad \mathsf{R} \subseteq \mathsf{S} \quad \text{for each } \mathsf{R} : A \leftarrow \mathsf{B} \text{ and } \mathsf{S} : A \leftarrow \mathsf{B} \quad ,$$

$$(\mathsf{F.R})^{\cup} = \mathsf{F.(R^{\cup})} \quad \text{for each } \mathsf{R} : A \leftarrow \mathsf{B} \quad .$$

Examples: List is an endorelator. $\times$ is a binary relator.

Functions

Relation $R : A \leftarrow B$ is *total* iff

$$\mathsf{id}_B \subseteq R^{\cup} \circ R \quad ,$$

and relation R is single-valued or *simple* iff

$$R \circ R^{\cup} \subseteq \mathsf{id}_A \quad .$$

A function is a relation that is total and simple.

Relators preserve totality

$$
\begin{array}{ll}
 & (F.R)^{\cup} \circ F.R \\
= & \{ \quad \text{relators respect converse} \quad \} \\
 & F.(R^{\cup}) \circ F.R \\
= & \{ \quad \text{relators distribute through composition} \quad \} \\
 & F.(R^{\cup} \circ R) \\
\supseteq & \{ \quad \text{assume } id_B \subseteq R^{\cup} \circ R\text{, relators are monotonic} \quad \} \\
 & F.id_B \\
= & \{ \quad \text{relators preserve identities} \quad \} \\
 & id_{F.B} \quad .
\end{array}
$$

Similarly, relators preserve simplicity. Hence relators preserve functions.

Parametricity — point-free

Recall

$$(f, g) \in R \leftarrow S \quad \equiv \quad \langle \forall c,d :: (f.c\,,\, g.d) \in R \Leftarrow (c, d) \in S \rangle \quad .$$

Point-free:

$$(f, g) \in R \leftarrow S \;\equiv\; f^{\cup} \circ R \circ g \supseteq S \quad .$$

Equivalently, using *shunting* rule:

$$(f, g) \in R \leftarrow S \;\equiv\; R \circ g \supseteq f \circ S \quad .$$

Relators are Parametric

Type:

$$F.R : F.A \leftarrow F.B \quad \Leftarrow \quad R : A \leftarrow B \quad .$$

That is,

$$F \;:\; \langle \forall \alpha,\beta :: (F.\alpha \leftarrow F.\beta) \leftarrow (\alpha \leftarrow \beta) \rangle \quad .$$

F is parametric iff, for all relations R and S, and all functions f and g,

$$(F.f\,,\,F.g) \in F.R \leftarrow F.S \quad \Leftarrow \quad (f,g) \in R \leftarrow S \quad .$$

Exercise: verify that this is the case using point-free definition of $R \leftarrow S$.

Natural Transformations

Parametricity of reverse function, rev, on lists, and of fork:

$$\mathsf{List.R} \circ \mathsf{rev}_B \ \supseteq \ \mathsf{rev}_A \circ \mathsf{List.R}$$

$$\mathsf{R}{\times}\mathsf{R} \circ \mathsf{fork}_B \ \supseteq \ \mathsf{fork}_A \circ \mathsf{R}$$

In fact,

$$\mathsf{List.R} \circ \mathsf{rev}_B \ = \ \mathsf{rev}_A \circ \mathsf{List.R} \ .$$

But, it is *not* the case that, for all R,

$$\mathsf{R}{\times}\mathsf{R} \circ \mathsf{fork}_B \ = \ \mathsf{fork}_A \circ \mathsf{R} \ .$$

For example,

$$\{(0,0)\,,(1,0)\} \times \{(0,0)\,,(1,0)\} \circ \mathsf{fork}_B \ \neq \ \mathsf{fork}_A \circ \{(0,0)\,,(1,0)\} \ .$$

fork is a (lax) *natural transformation*, rev is a *proper* natural transformation.

Natural Transformations

$$\theta : F \hookleftarrow G \;=\; F.R \circ \theta_B \supseteq \theta_A \circ G.R \quad \text{for each } R : A \leftarrow B$$

$$\theta : F \hookrightarrow G \;=\; F.R \circ \theta_B \subseteq \theta_A \circ G.R \quad \text{for each } R : A \leftarrow B \quad .$$

Facts:

$$(F.f \circ \theta_B = \theta_A \circ G.f \quad \text{for each function } f : A \leftarrow B) \;\Leftarrow\; \theta : F \hookleftarrow G \quad .$$

In a “tabular allegory”,

$$\theta : F \hookleftarrow G \;\Leftarrow\; (F.f \circ \theta_B = \theta_A \circ G.f \quad \text{for each function } f : A \leftarrow B) \quad .$$

In words, $\theta : F \hookleftarrow G$ iff θ is a (categorical) natural transformation in the underlying category of maps.

Conclusion: we take $\theta : F \hookleftarrow G$ to be the definition of a *natural transformation* in an allegory.

Division

An allegory is *locally complete* if for each set $\mathcal{S}$ of relations of type $A \leftarrow B$, the union $\cup\mathcal{S} : A \leftarrow B$ exists and, furthermore, intersection and composition distribute over arbitrary unions.

$\bot\!\!\bot_{A,B}$ is the smallest relation of type $A \leftarrow B$ and $\top\!\!\top_{A,B}$ is the largest relation of the same type.

In a *division* allegory, composition distributes through union. That is, there are two *division* operators "\" and "/", such that, for all $R : A \leftarrow B$, $S : B \leftarrow C$ and $T : A \leftarrow C$,

$$R \circ S \subseteq T \quad \equiv \quad S \subseteq R \backslash T \quad ,$$

$$R \circ S \subseteq T \quad \equiv \quad R \subseteq T / S \quad ,$$

$$S \subseteq R \backslash T \quad \equiv \quad R \subseteq T / S \quad .$$

Domain and Range

The *range* of a relation R is the set of all x such that $(x,y) \in R$ for some y.

Formally, the range operator "$<$" is defined by, for all $R : A \leftarrow B$ and all $X \subseteq \mathsf{id}_A$,

$$R{<} \subseteq X \quad \equiv \quad R \subseteq X \circ \top\!\!\top_{A,B} \quad .$$

The *domain* $R{>}$ is defined by

$$R{>} \quad = \quad (R^{\cup}){<} \quad .$$

Membership

The membership relation of a relator F is a family of relations mem_A, indexed by objects A, such that

$$\mathsf{mem}_A \ : \ A \leftarrow \mathsf{F}.A \quad \text{, and}$$

for all A, all $\mathsf{X} \subseteq \mathsf{id}_A$ and $\mathsf{Y} \subseteq \mathsf{id}_{\mathsf{F}.A}$,

$$\mathsf{F}.\mathsf{X} \supseteq \mathsf{Y} \ \equiv \ (\mathsf{mem}_A \circ \mathsf{Y})^{<} \subseteq \mathsf{X} \quad .$$

In words, $\mathsf{F}.\mathsf{X}$ is the largest subset Y of F-structures, each of type $\mathsf{F}.A$, such that the data stored in elements is in the set X.

Weakest Liberal Precondition

For all $X \subseteq id_A$ and $Y \subseteq id_{F.A}$,

$$(mem_A \circ Y)_< \;\subseteq\; X$$

$= \qquad \{ \qquad$ definition of range $\quad \}$

$$mem_A \circ Y \;\subseteq\; X \circ \top\!\!\top$$

$= \qquad \{ \qquad$ division $\quad \}$

$$Y \;\subseteq\; mem_A \backslash (X \circ \top\!\!\top)$$

$= \qquad \{ \qquad Y \subseteq id_{F.A} \quad \}$

$$Y \;\subseteq\; mem_A \backslash (X \circ \top\!\!\top) \cap id_{F.A} \quad .$$

For those familiar with the wp calculus: $mem_A \backslash (X \circ \top\!\!\top) \cap id_{F.A}$ is the weakest liberal precondition guaranteeing a state satisfying X after "execution" of mem.

Properties of F structures

For all A, all $\mathsf{X} \subseteq \mathsf{id}_A$ and $\mathsf{Y} \subseteq \mathsf{id}_{\mathsf{F}.A}$,

$$\mathsf{F.X} \supseteq \mathsf{Y} \quad \equiv \quad \mathsf{mem}_A \backslash (\mathsf{X} \circ \top\!\!\top) \cap \mathsf{id}_{\mathsf{F}.A} \ \supseteq \ \mathsf{Y} \quad .$$

So,

$$\mathsf{F.X} \ = \ \mathsf{mem}_A \backslash (\mathsf{X} \circ \top\!\!\top) \cap \mathsf{id}_{\mathsf{F}.A} \quad .$$

Interpreting $\mathsf{X} \subseteq \mathsf{id}_A$ as a property of values of type A, $\mathsf{F.X}$ is a property of values of type $\mathsf{F}.A$. The identity says that a property of an F-structure is characterised by properties of the values stored in the structure (its "members").

Largest Natural Transformations

Recall: for each object A,

$$\mathsf{mem}_A \ : \ A \leftarrow \mathsf{F}.A \quad .$$

Membership is parametric: for all R,

$$\mathsf{R} \circ \mathsf{mem} \ \supseteq \ \mathsf{mem} \circ \mathsf{F}.\mathsf{R} \quad .$$

Equivalently,

$$\mathsf{mem} : \mathrm{Id} \hookleftarrow \mathsf{F} \quad .$$

Also,

$$\mathsf{mem}\backslash\mathsf{id} : \mathsf{F} \hookleftarrow \mathrm{Id} \quad .$$

Theorem: The fan of relator F, $\mathsf{mem}\backslash\mathsf{id}$, is the largest natural transformation of type $\mathsf{F} \hookleftarrow \mathrm{Id}$. The membership of relator F is the largest natural transformation of type $\mathrm{Id} \hookleftarrow \mathsf{F}$.

Understanding Natural Transformations

Theorem: Suppose F and G are relators with memberships $\mathsf{mem.F}$ and $\mathsf{mem.G}$ respectively. Then the largest natural transformation of type $\mathsf{F} \hookleftarrow \mathsf{G}$ is $\mathsf{mem.F} \backslash \mathsf{mem.G}$.

Interpretation: A natural transformation of type $\mathsf{F} \hookleftarrow \mathsf{G}$ changes structure only. Stored values may be lost or duplicated, but no computation is performed on them.

A *proper* natural transformation to F from G changes the structure without loss or duplication of stored values.