A Hierarchy of Mendler style iteration/recursion combinators:

taming recursive types with negative occurrences

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Nax language design: collaborating with
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Context of the work: **Trellys** Project

- Dependently typed language aiming for both a programming language and a reasoning system
- Able to logically reason about programs and compute over data structure containing proofs

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Characteristics of programming languages and reasoning systems

Typed Functional Programming Languages (e.g., ML, Haskell)

- Can define arbitrary recursive types definition
- No guarantee for normalization

Typed Logical Reasoning Systems (e.g., Coq, HOL)

- Can NOT define arbitrary recursive types (only positive ones)
- Guaranteed to normalize

Nax: a middle ground taking good properties of the both worlds

Typed Functional Programming Languages (e.g., ML, Haskell)

- Can define arbitrary recursive types definition
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Typed Logical Reasoning Systems (e.g., Coq, HOL)

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- Guaranteed to normalize

Spoiler: Aren't there well known calculi with such properties?

Like System F and F_ω ...

Yes, Nax is going to be closely related to them.

Nax

- Named after Nax P. Mendler
- An extension of System F (or, F_ω)
- Allow formation of any recursive types (i.e. both positive and negative recursive types)
- Strongly normalizing
  - A rich family of principled iteration/recursion combinators over recursive types
  - These combinators were discovered and designed following the style of Mendler's

Why are we designing Nax?

in the context of Trellys project

- The goal of the Trellys project is to design and implement a unified system that is both
  - a full functional programming language
    Prog (programmatic sublanguage): $G \vdash_{\mathrm{Prog}} e : t$
  - and, a sound logical reasoning system
    Log (logical sublanguage): $G \vdash_{\mathrm{Log}} e : t$
- What are the minimal requirements of Log?

Nax $\left\{ \right.$

  - Normalizing (logical proof should be finite)
  - Support arbitrary Recursive Datatypes
    (to be able to refer to any program in Prog)

Outline

- Background & Motivation
- Preliminary Concepts
    - Recursive types
    - Mendler style iteration/recursion
- Current design of Nax
- Future work

Recursive Types (a.k.a. Fixpoint Types)

- Solutions for for recursive type equations
  - $X = 1 + X$ natural number
  - $X = 1 + (A \times X)$ list containing A element
  - $X = A + (X \times X)$ binary tree with A leaves
- That is, fixpoint μF such that $\mu F = F(\mu F)$
  - $N(X) = 1 + X$ natrual number type: μN
  - $L(X) = 1 + (A \times X)$ list type: μL
  - $T(X) = A + (X \times X)$ binary tree type: μT

Two styles of recursive types

- Equi-recursive (implicit conversion)

```
G |- e : μF
---------------- μ-elim
G |- e : F(μF)
```

```
G |- e : F(μF)
---------------- μ-intro
G |- e : μF
```

- Iso-recursive (explicit conversion)

```
G |-      e : μF
--------------------- μ-elim
G |- unIn e : F(μF)
```

```
G |- e    : F(μF)
------------------- μ-intro
G |- In e : μF
```

```
unIn (In e) → e
```

Two styles of recursive types

- Equi-recursive (implicit conversion)

```
type X = Either () X
data Either a b = Left a | Right b
```

This is only an analogy ... cyclic type synonym is a type error in Haskell

- Iso-recursive (explicit conversion)

```
data N r = Z | S r
type Nat = Mu N
zero     = In Z
succ n  = In (S n)
newtype Mu f = In (f (Mu f))   -- definition of μ
unIn (In x) = x          -- recall the reduction rule
```

Encoding of iso-recursive types a.k.a. two-level types

- Usual one-level recursive type definition Nat can be thought as an abstract interface (Nat, zero, succ) of the two-level implementation that hides more primitive constructs, that is, the recursion operator (Mu, In, out) and the base structure (N, Z, S)

```
data Nat = Zero | Succ Nat
```

```
data N r = Z | S r
type Nat = Mu N
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newtype Mu f = In (f (Mu f))   -- definition of μ
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```

Exercise on two-level types

- Natrual numbers

```
data Nat  =   Zero
          |   Succ Nat
```

```
data N = Z | S r
type Nat = Mu N
zero   = In N
succ n = In (S n)
```

- Lists

```
data List a =  Nil
            |  Cons a (List a)
```

```
data L a r = N | C a r
type List a = Mu (L a)
nil         = In N
cons x xs = In (C x xs)
```

- Trees

```
data Tree a
     =  Leaf a
     |  Node (Tree a) (Tree a)
```

```
data T a r = L a | N r r
type Tree a = Mu (T a)
leaf x      = In (L x)
node tl tr = In (N tl tr)
```

Recursive types and Normalization

- Unrestricted use of general recursion at term level can cause diverging computation
  - e.g. “`let f x = f x in f 0`” loops
- Unrestricted formation and elimination (i.e., pattern matching) over recursive types can also cause diverging computation, even without any use recursion at term level
  - With recursive types, it is possible to encode diverging self application $(\lambda x.xx)\ (\lambda x.xx)$ of the untyped lambda calculus in a type correct way
    - Also observed by Nax P. Mendler

Diverging computation just using Recursive types

- Mendler's example in Haskell: encoding of a classical self application $(\lambda x.xx)\ (\lambda x.xx)$

$$
\begin{array}{l|l}
\textbf{data}\ T = C\ (T \to ()) & \quad w\ (C\ w) \\
p :: T \to (T \to ()) & \leadsto (p\ (C\ w))\ (C\ w)) \\
p\ (C\ f) = f & \leadsto w\ (C\ w) \\
w :: T \to () & \leadsto (p\ (C\ w))\ (C\ w)) \\
w\ x = (p\ x)\ x & \leadsto \cdots
\end{array}
$$

- Ability to pattern match (eliminate) freely over recursive types is enough to cause divergence
  - didn't have to use term level recursion at all

Two design choices of normalization with recursive types

- Restrict Formation rule

$$\frac{G \mid\!- F : * \to *}{G \mid\!- \mu F : *}\ \mu\text{-form}$$

- Restrict Elimination rule

$$\frac{G \mid\!- e : \mu F}{G \mid\!- \mathrm{unIn}\ e : F(\mu F)}\ \mu\text{-elim}$$

$$\frac{G \mid\!- e : F(\mu F)}{G \mid\!- \mathrm{In}\ e : \mu F}\ \mu\text{-intro}$$

$$\mathrm{unIn}\ (\mathrm{In}\ e) \to e$$

Positive vs. Negative occurrences in recursive types

- Interpreting $(A \rightarrow B)$ logically as implication, which is equivalent to $(\neg A \bigvee B)$
- So, left of → is negative position and right of → is positive position
- Positive datatype: all recursive occurrences are in positive position
  data Tree = Leaf Int | InfBranch (Nat→Tree)
- Negative datatype: exist recursive occurrences in one or more negative positions
  data Exp = Lam (Exp→Exp) | App Exp Exp

Strictly Positive vs. Positive

- data $A = C\ ((A \to \mathrm{Bool}) \to \mathrm{Bool})$
  - Positive since A is in doubly negated position, but not strictly positive since A appears inside the left hand side of the top level $\to$
  - Considered to be non-set theoretic since it asserts the proposition that powerset of powerset of A being isomorphic to A, which is a set theoretic nonsense
- All strictly positive types have set theoretic interpretation
- Some positive, but not strictly positive, types CAN be considered set theoretically
  - data $SN = SN\ (\forall b.\ b \to (SN \to b) \to b)$
    Scott Numerals - an encoding of natural numbers

Diverging computation using Negative recursive types

- Mendler's example in Haskell: encoding of a classical self application $(\lambda x.xx)\ (\lambda x.xx)$

$$
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\mathbf{data}\ T = C\ (T \to ()) & \quad w\ (C\ w) \\
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- Ability to pattern match (eliminate) freely over recursive types is enough to cause divergence
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Why care about negative datatypes?
(Example 1: Reducibility)

- Definition of Reducibility for System T
  - Red{Nat}(M) iff M reduce to canonical form of Nat
  - Red{A→B}(M) iff for all N, Red{A}(N) implies Red{B}(M N)

- In proof assistants like Coq, this most natural definition will be rejected

```
Inductive Red: ty → exp → Prop
 := RedN : forall n, Const n → Red nat n
  | RedA : forall e A B, (forall A e', Red A e' → Red B (e e'))
                          → Red (A → B)
```

Why care about negative datatypes?
(Example 2: HOAS)

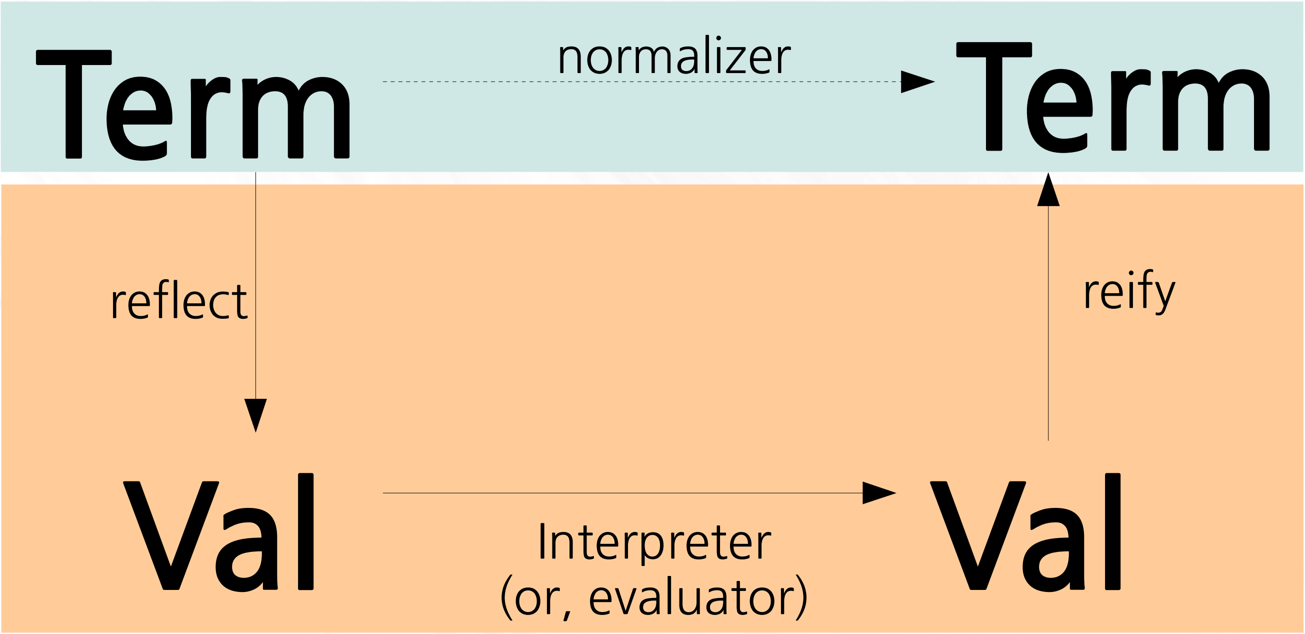
- HOAS for untyped lambda calculus (in Haskell)

  $\mathrm{data\ Exp} = \mathrm{Lam\ (Exp} \to \mathrm{Exp)} \mid \mathrm{App\ Exp\ Exp}$

  - Since Exp models the untyped lambda calculus, its eval function $\mathrm{eval} :: \mathrm{Exp} \to \mathrm{Exp}$ is partial
  - But, there can be many useful total functions over Exp, such as $\mathrm{showExp} :: \mathrm{Exp} \to \mathrm{String}$ that formats an HOAS term into a printable string
- More complex transformations using HOAS for typed languages have been studied in the context of type preserving compilers

Why care about negative datatypes?
(Example 3: Normalization by Evaluation)

- Define normalization of terms (positive datatype) using evaluation of values (negative datatype)



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Two styles of iteration/recursion Squiggol style vs. Mendler style

- Developed in the context of Functional Programming Languages with Type Inference (Hindley-Milner type system)

  No wonder why this style has been more popular in functional programming

- Iteration/Recursion well-defined for positive datatypes, but not for negative datatypes
- Defined for regular datatypes, but not easy to generalize to non-regular datatypes (e.g. nested datatypes, GADTs)

- Developed in the context of Nuprl, an interactive theorem prover for constructive math, with a very powerful type system (dependent types, higher rank polymorphism)
- Iteration/Recursion well-defined for arbitrary datatypes including negative datatypes

  Mendler didn't notice this himself, later discovered by others

- Naturally generalize to non-regular datatypes (more generally to type constructors of arbitrary higher kinds)

Exercise on two-level types

- Natrual numbers

```
data Nat  =   Zero
          |   Succ Nat
```

```
data N = Z | S r
type Nat = Mu N
zero    = In N
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data T a r = L a | N r r
type Tree a = Mu (T a)
leaf x       = In (L x)
node tl tr = In (N tl tr)
```

Iteration (a.k.a. catamorhpism) in Squiggol style

```
iter :: Functor f ⇒
        (f a → a) → Mu f → a
iter φ (In x) =
  φ (fmap (iter φ) x)
```

```
instance Functor (L x) where
  -- fmap :: (a→ b) → L x a → L x b
  fmap h N       = N
  fmap h (C x a) = C x (h a)
```

```
lenList = Mu (L x) → Int
lenList = iter phi
  where
    phi :: L x Int → Int
    phi N            = 0
    phi (C x xslen) = 1 + xslen
```

- Generalization of folds, expressed using 2-level types
- Recursion is captured by `iter` at term level, and `Mu` at type level (non-recursive elsewhere)
- `fmap` guides where to invoke recursive calls
- User supplies `phi :: f a → a`, which defines how to process the base structure containing answers of the already processed subcomponents

Iteration (a.k.a. catamorhpism) in Mendler style

```
miter :: (∀r.(r → a) → f r → a) →
          Mu f → a
miter φ (In x) =
  φ (miter φ) x
```

```
lenList = Mu (L x) → Int
lenList = miter phi
  where
   phi :: ∀r.(r→Int) → L x r → Int
   phi len N        = 0
   phi len (C x xs) = 1 + len xs
```

- Key idea: φ expects another argument, which enables user to control recursive calls instead of relying on predefined fmap
- No requirement on the base (f) being a positive functor
- Higher rank polymorphism (∀r. ...) enforce recursive subcomponents in the base structure (f r) be abstract inside φ, which is Mendler's ingenious idea to guarantee normalization
- phi looks almost the same as what you'd write in a functional language with general recursion

Does miter really normalize?

Isn't it dangerous to allow users to control recursive calls?

```
miter :: (     (Mu f → a) → f (Mu f) → a) → Mu f → a   -- Naïve type
miter :: (∀r. ( r      → a) → f  r       → a) → Mu f → a   -- Mendler's type
```

```
cons :: x → Mu (L x) → Mu (L x)

lenListBad = mcata phi
  where
   phi len N          = 0
   phi len (C x xs) = 1 + len (cons x xs)
```

The function lenListBad diverges when applied to a non-empty list, since it recursively calls on the same list (cons x xs) rather than its tail (xs).

- lenListBad will type check with the naïve type of miter
- len (cons x xs) is a type error with Mendler's type
  - cons expects its 2nd arg of type Mu (L x) but xs :: r, where r is parametric (or, abstract)
  - len :: (r→a) expects an arg of abstract type r but the result type of cons is Mu (L x)

can't cons with xs

won't work even if you could cons

Does miter really normalize?

- Yes, we've seen the intuition
- The formal proof can be done by a reduction preserving embedding into a well known normalizing language Fω (i.e., one reduction step involving miter is simulated by one or more constantly bound reduction steps in Fω)

```
newtype Mu f = In (f (Mu f))

mcata :: (∀r.(r → a) → f r → a) → Mu f
→ a
mcata φ (In x) = φ (mcata φ) x

{- lambda abstraction, application, and
   non recursive base structures have
   trivial embeddings into Fw -}
```

Fω

Primitive Recursion vs. Iteration

- Primitive Recursion gives you access to both the values of the subcomponents and the results of processing the subcomponents
- Iteration only gives you access to the results of processing the subcomponents
- Although miter looks like giving you access to the the subcomponent values, it really isn't. (try to define factorial if you are in doubt)

```
lenList = Mu (L x) → Int
lenList = miter phi
  where
   phi :: ∀r.(r→Int) → L x Int →
Int
   phi len N          = 0
   phi len (C x xs) = 1 + len xs
```

xs :: r is an abstract handle to the tail whose only sensible use is to be applied to the abstract interface to the recursive call

len :: r → Int

Mendler style Primitive Recursion

$$\text{miter} :: (\forall r. (r \to a) \to f\ r \to a) \to \text{Mu}\ f \to a$$
$$\text{miter}\ \varphi\ (\text{In}\ x) = \varphi\ (\text{miter}\ \varphi)\ x$$

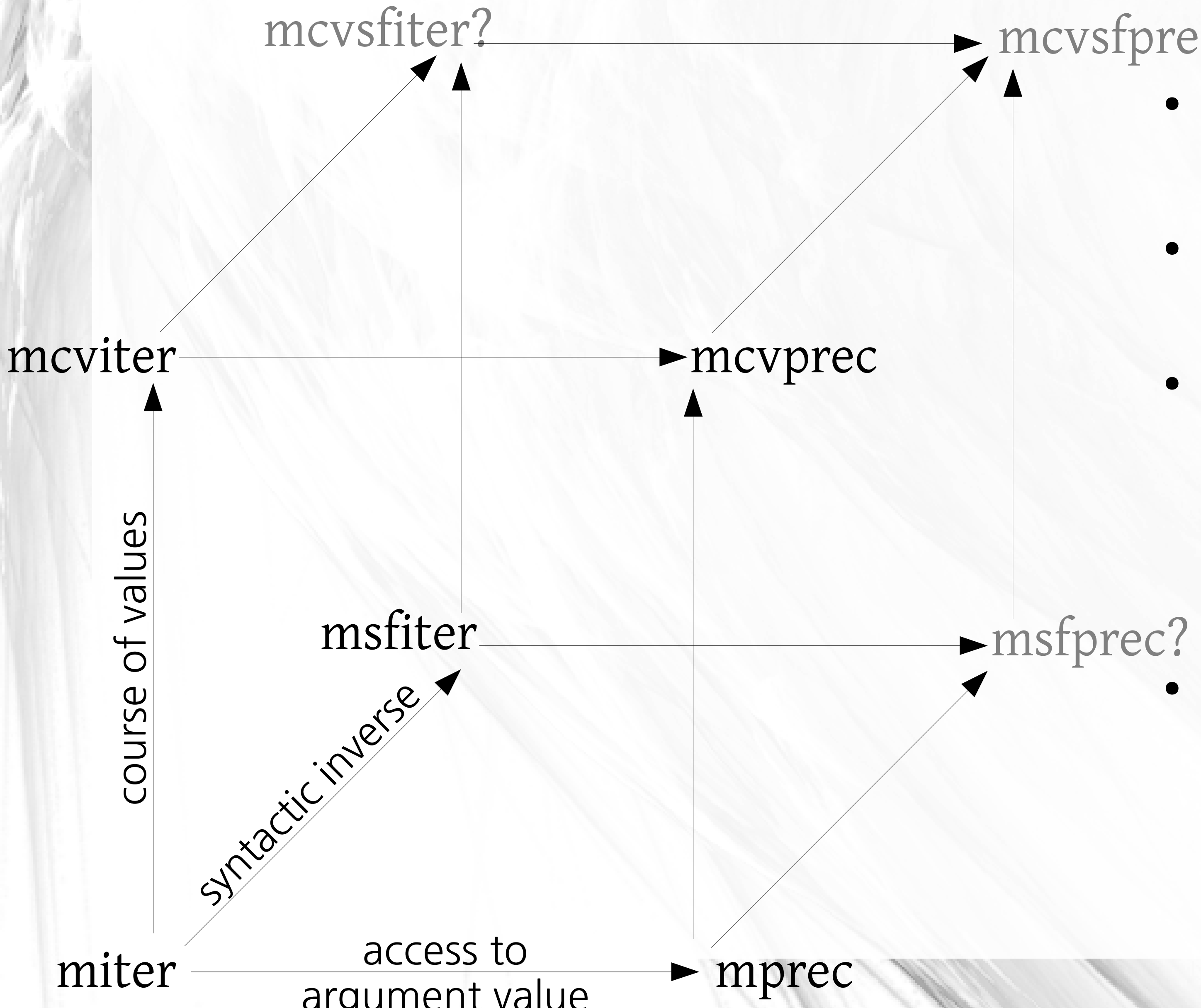
$$\text{mprec} :: (\forall r. (r \to \text{Mu}\ f) \to (r \to a) \to f\ r \to a) \to \text{Mu}\ f \to a$$
$$\text{mprec}\ \varphi\ (\text{In}\ x) = \varphi\ \text{id}\ (\text{mprec}\ \varphi)\ x$$

$$(\times) :: \text{Mu}\ N \to \text{Mu}\ N \to \text{Nat}$$

$$\text{fact} = \text{Mu}\ N \to \text{Nat}$$
$$\begin{aligned}\text{fact} = \text{mprec phi} \quad \text{where}\ & \text{phi} :: \forall r. (r \to \text{Mu}\ N) \to (r \to \text{Nat}) \to N\ r \to \text{Nat} \\ & \text{phi cast fac}\ Z = 0 \\ & \text{phi cast fac}\ (S\ n) = \text{succ}(\text{cast}\ n) \times \text{fac}\ n\end{aligned}$$

- φ for mprec expects yet another argument, which is a type casting function from an abstract type (r) to the concrete recursive type ($\text{Mu}\ f$)
- Mendler's original work (LICS '87) is about mprec

A Hierarchy of Mendler style Iteration/Recursion Combinators



- miter, msfiter can be embeded into F_ω
- mprec embeds into Fix_ω (Abel & Matthes CSL '04)
- mcv- combinators only normalize for positive datatypes (other non-cv combinators normalize for arbitrary datatypes)
- Naturally extends to higher kinds where Mu and related combinators are index by kinds
  Mu_{*} , In_{*} , miter_{*} , ...
  $\mathrm{Mu}_{*\to*}$, $\mathrm{In}_{*\to*}$, $\mathrm{miter}_{*\to*}$, ..

Mendler style course of values Iteration

```
miter   :: (∀r.           (r → a) → f r → a) → Mu f → a
miter   φ (In x) = φ      (miter φ)   x
```

```
mcviter :: (∀r. (r → f r) → (r → a) → f r → a) → Mu f → a
mcviter φ (In x) = φ unIn (mcviter φ) x
```

```
(+) :: Mu N → Mu N → Nat

fibo = Mu N → Nat
fibo = mcviter phi where phi :: ∀r.(r→N r) → (r→Nat) → N r → Nat
                         phi  out  fib Z     = succ zero
                         phi  out  fib (S n) = case out n of  Z    → succ zero
                                                              S n' → fib n + fib n'
```

- φ for mcviter expects yet another argument, which is an abstract eliminator out :: r → f r passing around unIn :: Mu f → f (Mu f), giving the ability to abstractly eliminate (i.e., pattern match away) In constructor of Mu

only normalizes for positive datatype.
A counter example for negative datatype in our ICFP'11 paper.

Mendler style Sheard-Fegaras Iteration

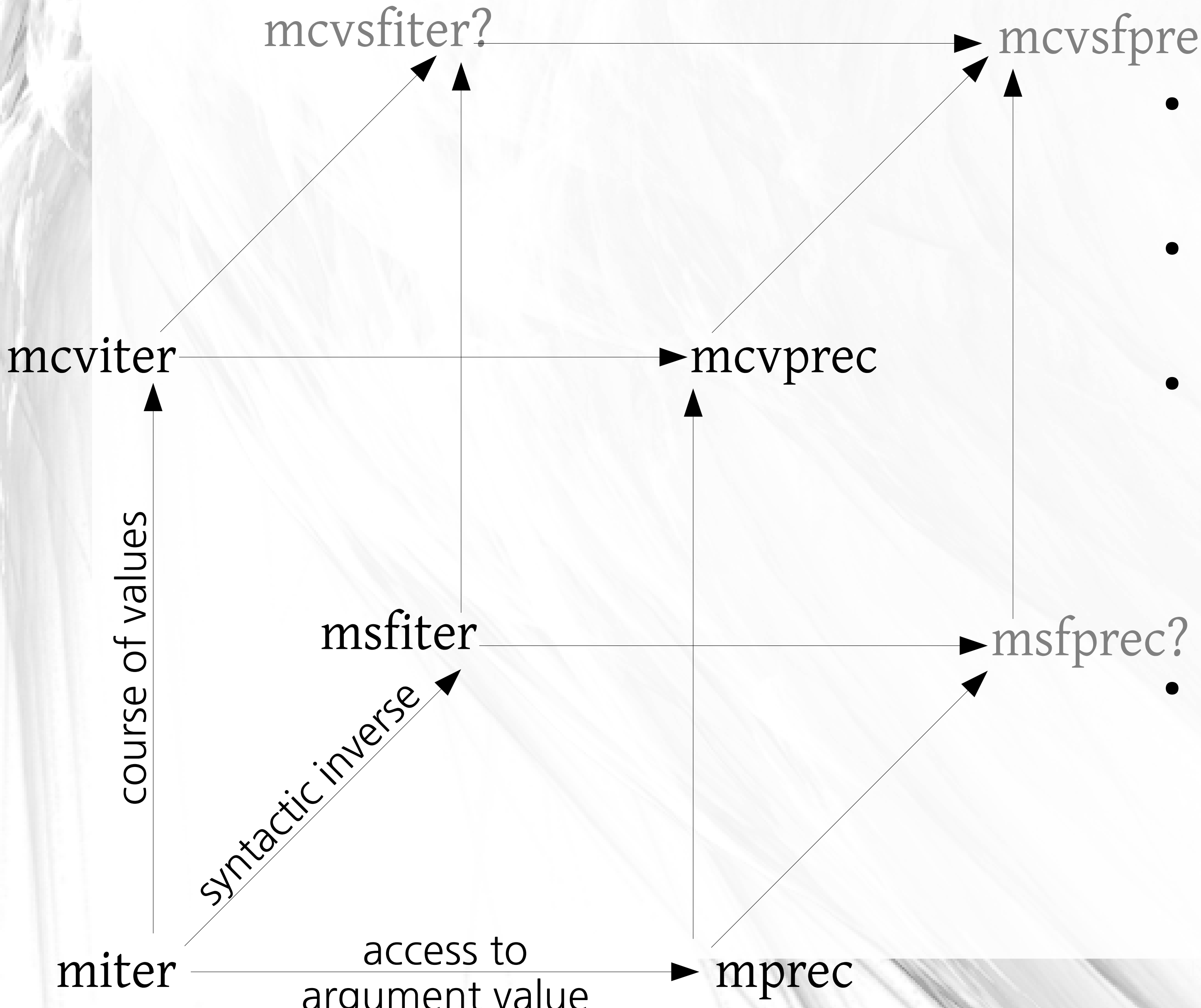
```
data Mu' f a = In' (f (Mu' f a)) | Inverse a  -- Mu with syntactic inverse

msfiter :: (∀r. (a → r a) → (r a → a) → f (r a) → a) → (∀a. Mu' f a) → a
msfiter φ (In' x)       = φ Inverse (msfiter φ) x
msfiter φ (Inverse x) = x
```

```
data E r = A r r | L (r → r)      -- base structure for HOAS
type Exp = ∀a. Mu' E a
countVar :: Exp → Int  -- count the no. of variable use. (λx.xx) is 2, (λx.λy.x) is 1
countVar = msfiter phi  where  phi :: ∀r.(Int→r Int)→(r Int→Int)→E (r Int)→Int
                               phi  inv  count (L g)        = count (g (inv 1))
                               phi  inv  count (A e1 e2) = count e1 + count e2
```

- φ for msfiter expects yet another argument, which is a syntactic Inverse allowing you to instantly create an abstract recursive value (inv 1 :: r Int) from an expected result value (1 :: Int) so that you can supply it to a function (g:: r Int → r Int) expecting an abstract recursive value

See our ICFP'11 paper for further details.

A Hierarchy of Mendler style Iteration/Recursion Combinators



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  - Recursive types (equi/iso, positive/negative)
  - Mendler style iteration/recursion
    - Well-defined for negative datatypes
    - Naturally generalize to non-regular datatypes
    - Variations of iteration/recursion schemes (course of values, syntactic inverse) have been discovered and studied
- Current design of Nax
- Future work

Some trivia

Why design a new language when you can embed that new language into F_ω or Fix_ω? Why not just use F_ω or Fix_ω?

Same reason you don't want to use Turing machine or plain lambda calculus instead of programming in high level languages

- Embedding into F_ω or Fix_ω is only a tool for showing normalization
- Encoding datatypes in F_ω or Fix_ω is tedious
- Some language design decisions can make type inference/checking more convenient
- Some recursion combinators can be simplified when we define them as language constructs

Mendler style Sheard-Fegaras Iteration

```
data Mu' f a = In' (f (Mu' f a)) | Inverse a  -- Mu with syntactic inverse

msfiter :: (∀r. (a → r a) → (r a → a) → f (r a) → a) → (∀a. Mu' f a) → a
msfiter φ (In' x)        = φ Inverse (msfiter φ) x
msfiter φ (Inverse x) = x
```

Instead of above implementation in Haskell, we can define msfiter as a Nax language primitive of the following type using one same Mu, and reduction rules defined as follows:

```
msfiter :: (∀r. (a → r) → (r → a) → f r → a) → Mu f → a

msfiter φ (In' x)        → φ Inverse (msfiter φ) x
msfiter φ (Inverse x) → x
```

Inverse is a transient term, which only appear during computation but cannot appear in the source code

Syntax:

Curry style System F with some extensions

$$Dec ::= F\,\overline{X}\,X.\{\,\overline{C\,\overline{T}}\,\}$$

$$Decs ::= \cdot \mid Dec; Decs$$

$$T ::= F\,\overline{T}\,T \mid T \to T \mid X \mid \forall X.T \mid \mu(F\,\overline{T})$$

$$\begin{aligned} M ::=\ & x \mid C \mid \mathsf{case}\ M\ \{\,\overline{C\ \overline{x}.M}\,\} \mid \lambda x.M \mid M\,M \\ & \mid \mathsf{in}\ M \mid \mathsf{mit}\ M \mid \mathsf{mrec}\ M \mid \mathsf{mcvit}\ M \mid \mathsf{mcvrec}\ M \mid \mathsf{msfit}\ M \\ & \mid \mathsf{out} \mid \mathsf{inv} \quad \text{-- these are transient objects cannot apppear in source code} \end{aligned}$$

$$Program ::= Decs; M$$

- This description is still at a level of a calculus
- More concrete syntax is being designed by trying out a prototype implementation

Type System

$$\text{in}\ \frac{\Gamma \vdash M : F\,\overline{T}\,(\mu(F\,\overline{T}))}{\Gamma \vdash \mathsf{in}\,M : \mu(F\,\overline{T})} \qquad \text{mit}\ \frac{\Gamma, X : \star \vdash M : (X \to T') \to F\,\overline{T}\,X \to T'}{\Gamma \vdash \mathsf{mit}\,M : \mu(F\,\overline{T}) \to T'}$$

$$\text{mrec}\ \frac{\Gamma, X : \star \vdash M : (X \to \mu(F\,\overline{T})) \to (X \to T') \to F\,\overline{T}\,X \to T'}{\Gamma \vdash \mathsf{mrec}\,M : \mu(F\,\overline{T}) \to T'}$$

$$\text{mcvit}\ \frac{\begin{array}{c}\Gamma, X : \star, \vdash M : (X \to F\,\overline{T}\,X) \to (X \to T') \to F\,\overline{T}\,X \to T' \\ F \text{ is positive}\end{array}}{\Gamma \vdash \mathsf{mcvit}\,M : \mu(F\,\overline{T}) \to T'}$$

$$\text{mcvrec}\ \frac{\begin{array}{c}\Gamma, X : \star, \vdash M : (X \to F\,\overline{T}\,X) \to (X \to F\,\overline{T}\,X) \to (X \to T') \to F\,\overline{T}\,X \to T' \\ F \text{ is positive}\end{array}}{\Gamma \vdash \mathsf{mcvrec}\,M : \mu(F\,\overline{T}) \to T'}$$

$$\text{msfit}\ \frac{\Gamma, X : \star \vdash M : (T' \to X) \to (X \to T') \to F\,\overline{T}\,X \to T'}{\Gamma \vdash \mathsf{msfit}\,M : \mu(F\,\overline{T}) \to T'}$$

Reduction

$$
\begin{aligned}
(\lambda x.M)M' &\longrightarrow M[M'/x] \\
\mathsf{mit}\,M\,(\mathsf{in}\,M') &\longrightarrow M\,(\mathsf{mit}\,M)\,M' \\
\mathsf{mrec}\,M\,(\mathsf{in}\,M') &\longrightarrow M\,(\lambda x.x)\,(\mathsf{mrec}\,M)\,M' \\
\mathsf{mcvit}\,M\,(\mathsf{in}\,M') &\longrightarrow M\,\mathsf{out}\,(\mathsf{mcvit}\,M)\,M' \\
\mathsf{mcvrec}\,M\,(\mathsf{in}\,M') &\longrightarrow M\,(\lambda x.x)\,\mathsf{out}\,(\mathsf{mcvrec}\,M)\,M' \\
\mathsf{out}\,(\mathsf{in}\,M') &\longrightarrow M' \\
\mathsf{msfit}\,M\,(\mathsf{in}\,M') &\longrightarrow M\,\mathsf{inv}\,(\mathsf{msfit}\,M)\,M' \\
\mathsf{msfit}\,M\,(\mathsf{inv}\,M') &\longrightarrow M'
\end{aligned}
$$

$$
\frac{M \longrightarrow M'}{E[M] \longrightarrow E[M']}
$$

$E ::= \ldots$

Outline

- Background & Motivation
- Preliminary Concepts
    - Recursive types (equi/iso, positive/negative)
    - Mendler style iteration/recursion
- Current design of Nax
- Future work

Future Work

- Try to write more interesting examples involving negative datatypes using msfiter
  (e.g., Normalization by Evaluation for a simple calculus)
- Extend the language from F rather than F_ω
  (non-regular datatypes, or datatypes of higher kinds)
- Add indexed types (in spirit of GADTs) and user defined kinds lifted from user defined types
- Concrete syntax and type checking/inference

- Dependent types and Induction principle?
  (i.e., dependent version of recursion combinators)
- Explore more iteration/recursion combinators

Current status of Nax

```
data Tag = E | O -- values of 1st order type can be lifted to index

flip E = O
flip O = E

gadt P : (Tag -> Nat -> *) -> Tag -> Nat -> * where
  Base : P r {E} {zero}
  StepO : r {O} {i} -> P r {E} {succ i}
  StepE : r {E} {i} -> P r {O} {succ i}

type Proof t n = Mu (Tag -> Nat -> *) P t n

type Even n = Proof {E} n
base = In (Tag -> Nat -> *) Base
stepO x = In (Tag -> Nat -> *) (StepO x)

type Odd n  = Proof {O} n
stepE x = In (Tag -> Nat -> *) (StepE x)

-- stepProof : Proof {t} {i} -> Proof {flip t} {succ i}
stepProof pf = miter {t i . Proof {flip t} {succ i}} pf
                 where   phi  Base       = stepE base
                         phi  (StepO p) = stepE(phi p)
                         phi  (StepE p) = stepO(phi p)

-- evenORodd : Vec a {n} -> Either (Even {n}) (Odd {n})
```

Mendler style Induction for positive datatypes

$$\text{mprec} :: (\forall r.\ (r \to \text{Mu } f) \to (r \to a) \to f\ r \to a) \to \text{Mu } f \to a$$
$$\text{mprec}\ \varphi\ (\text{In } x) \quad = \varphi\ \text{id}\ \ (\text{mprec}\ \varphi)\ x$$

$$\text{mind} :: (\ \forall r.\ (\text{cast}: r \to \text{Mu } f) \to ((x{:}r) \to a\ (\text{cast } x))$$
$$\to (y{:}f\ r) \to a\ (\text{fmap cast } y)\ ) \to (z{:}\text{Mu } f) \to a\ z$$
$$\text{mind}\ \varphi\ (\text{In } x) = \varphi\ \text{id}\ \ (\text{mprec}\ \varphi)\ x$$

- Just as the conventional style, we can define induction as a dependent version of primitive recursion on positive datatypes (note the use of fmap)
- We don't know yet how to formulate induction for negative datatypes

Conclusion

- We want a core language with both normalization and arbitrary recursive types
- We know that this is possible by discovering a a new family of Mendler style iteration combinator $\mathrm{msfiter}$
- We are designing Nax to realize this idea