Experiments and Optimal Results for Outerplanar Drawings of Graphs

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Outerplanar Crossing Numbers (2003-2006)

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Outerplanar Drawing Problem

- Outerplanar (also called One-Page, Circular, Convex) Drawing:

  placing vertices of a \( n \)-vertex, \( m \)-edge connected graph \( G = (V,E) \) along a circle, and the edges are drawn as straight lines.
Outerplanar Drawing Problem

- Outerplanar crossing number $\nu_1(G)$ of the graph $G$:
  - The smallest possible number of crossings in an outerplanar drawing of the graph $G$
  - (one-page, circular, convex crossing number).
An Example

an outerplanar drawing of a graph $G$

the optimal outerplanar drawing of the $G$

$\nu_1(G) = 1$

25 crossings
**Motivation**

- Outerplanar drawing problem is NP-hard problem (*Mäkinen, 1988*)
- VLSI layouts with fewer crossings are more easily realisable and consequently cheaper.
- Aesthetical drawing of cluster graphs
- The exact crossing numbers are very rare – they are of great interest, - theoretical point of view, benchmarks.
The Latest Heuristic Algorithms

BB algorithm (Baur and Brandes, WG04)

1. **Greedy-append phase:**
   - At each step a vertex with the largest number of already placed neighbours is selected, where ties are broken in favour of vertices with fewer unplaced neighbours.
   - Then appended to the end that yields fewer crossings of edges being closed with open edges. An edge is called open, if it connects a placed vertex with an unplaced one. \(O((n + m)\log n)\).

2. **Sifting phase:** Every vertex is moved along a fixed ordering of all other vertices. The vertex is then placed in its (locally) optimal position. \(O(mn)\)
The Latest Heuristic Algorithms

AVSDF+ algorithm (He and Sýkora, ITAT04):

1. **Greedy phase**: (a variation of the Depth First Search)
   - place the vertex with the smallest degree as the root
   - visit unplaced adjacent vertices of current vertex, according to the ascending degree. $O(m)$

2. **Adjusting phase**:
   - at each step a vertex with the largest crossing number created by its incident edges is selected,
   - find its best position among the current one and the ones next to its adjacent vertices. $O(m^2)$
Genetic Algorithm

(He, Netton, Sykora., SOFSEM05)

Initial Random Population

Fitness: Evaluation of Solution

New Generations
Selection
Crossover
Mutation

Is termination criterion met?

Order of vertices
popSize=16

Maximal possible edge number of generations

Crossing number

Order Crossover 40%

(p \propto 1/cr^2)

No.
Yes.
Experiments

Test suits:

- **Special graphs:** Hypercubes, Halin graphs, meshes and complete p-partite graphs with the same partition size;
- **Rome graphs:** RND_BUP and ALF_CU from GDToolkits. RND_BUP is a set of random biconnected undirected planar graphs. ALF_CU is a set of connected undirected graphs.
- **Random Connected Graphs** (RCG) with different size and different density.
**Experiments**

Test methods:

- Compare GA with BB+ on all graphs
- Compare GA with AVSDF+ on all graphs
- Compare GA, BB+, AVSDF+ on RCG with density 1%, 3%, and 5%.

For each density, 12 groups of graphs with different number of vertices were tested; and for every group 10 different graphs were generated and average running time and average number of crossings were calculated.
### Compare GA with AVSDF+

<table>
<thead>
<tr>
<th>Graphs</th>
<th>GA</th>
<th>same</th>
<th>AVSDF+</th>
</tr>
</thead>
<tbody>
<tr>
<td>hypercube (4)</td>
<td>75%</td>
<td>0%</td>
<td>25%</td>
</tr>
<tr>
<td>Halin graphs (18)</td>
<td>17%</td>
<td>17%</td>
<td>66%</td>
</tr>
<tr>
<td>meshes (28)</td>
<td>79%</td>
<td>14%</td>
<td>7%</td>
</tr>
<tr>
<td>Kn(p) (36)</td>
<td>0%</td>
<td>100%</td>
<td>0%</td>
</tr>
<tr>
<td>ALF CU (268)</td>
<td>72%</td>
<td>21%</td>
<td>7%</td>
</tr>
<tr>
<td>RND BUP (169)</td>
<td>60%</td>
<td>13%</td>
<td>27%</td>
</tr>
</tbody>
</table>
## Compare GA with BB+

<table>
<thead>
<tr>
<th>Graphs</th>
<th>GA</th>
<th>same</th>
<th>BB+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hypercubes (4)</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Halin graphs (18)</td>
<td>50%</td>
<td>28%</td>
<td>22%</td>
</tr>
<tr>
<td>meshes (28)</td>
<td>86%</td>
<td>4%</td>
<td>10%</td>
</tr>
<tr>
<td>$Kn(p)$ (36)</td>
<td>11%</td>
<td>89%</td>
<td>0%</td>
</tr>
<tr>
<td>ALF CU (268)</td>
<td>67%</td>
<td>21%</td>
<td>12%</td>
</tr>
<tr>
<td>RND BUP (169)</td>
<td>53%</td>
<td>18%</td>
<td>29%</td>
</tr>
</tbody>
</table>
GA, BB+, AVSDF+ on RCG (5%)
GA, BB+, AVSDF+ on RCG (3%)
GA, BB+, AVSDF+ on RCG (1%)
Some Exact results for 3-row meshes

<table>
<thead>
<tr>
<th>Meshes</th>
<th>AVSDF+</th>
<th>BB+</th>
<th>GA</th>
<th>theory value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 \times 4$</td>
<td>6</td>
<td>7</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$3 \times 5$</td>
<td>7</td>
<td>9</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>$3 \times 6$</td>
<td>12</td>
<td>10</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>$3 \times 7$</td>
<td>13</td>
<td>21</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>$3 \times 8$</td>
<td>18</td>
<td>16</td>
<td>14</td>
<td>12</td>
</tr>
<tr>
<td>$3 \times 9$</td>
<td>19</td>
<td>17</td>
<td>16</td>
<td>15</td>
</tr>
</tbody>
</table>
### Some Exact Results for Halin Graphs

<table>
<thead>
<tr>
<th>Halin Graphs</th>
<th>AVSDF+</th>
<th>BB+</th>
<th>GA</th>
<th>theory value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(8,6)</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(9,6)</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(10,7)</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>(11,7)</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>(32,20)</td>
<td>19</td>
<td>19</td>
<td>21</td>
<td>18</td>
</tr>
<tr>
<td>(64,40)</td>
<td>41</td>
<td>55</td>
<td>46</td>
<td>38</td>
</tr>
</tbody>
</table>
Some Exact Results for Complete p-partite Graphs

<table>
<thead>
<tr>
<th>Kn(P)</th>
<th>AVSDF+</th>
<th>BB+</th>
<th>GA</th>
<th>theory value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_3(2)$</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$K_4(2)$</td>
<td>16</td>
<td>16</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>$K_5(2)$</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>$K_3(3)$</td>
<td>54</td>
<td>54</td>
<td>54</td>
<td>54</td>
</tr>
<tr>
<td>$K_4(3)$</td>
<td>216</td>
<td>216</td>
<td>216</td>
<td>216</td>
</tr>
<tr>
<td>$K_5(3)$</td>
<td>600</td>
<td>600</td>
<td>600</td>
<td>600</td>
</tr>
<tr>
<td>$K_3(4)$</td>
<td>279</td>
<td>283</td>
<td>279</td>
<td>279</td>
</tr>
</tbody>
</table>
The only exact known result for complete bipartite graphs was achieved by A. Riskin [2003]:

if $m$ divides $n$,

$$
\nu_1(K_{m,n}) = \frac{n(m-1)(2mn-3m-n)}{12}
$$

if $m=n$,

$$
\nu_1(K_{n,n}) = n\binom{n}{3}
$$
Our results

3-row meshes:

for any odd \( n \geq 3 \):
\[ \nu_1(P_3 \times P_n) = 2n-3 \]

for any even \( n \geq 4 \):
\[ \nu_1(P_3 \times P_n) = 2n-4 \]

For an arbitrary Halin graph \( G(d \geq 3) \),
with \( m \) leaves:
\[ \nu_1(G) = m-2 \]

For the complete p-partite graph with \( n \) vertices in each partite set, \( K_n(p) \):
\[ \nu_1(K_n(p)) = n^4\left(\frac{p}{4}\right) + n^2(n-1)(2n-1)\left(\frac{p}{3}\right)/2 + n\left(\frac{n}{3}\right)\left(\frac{p}{2}\right) \]
3-row Meshes

- **Theorem 1:**

  for 3-row meshes:
  
  with any odd $n \geq 3$: $\nu_1(P_3 \times P_n) = 2n - 3$

  with any even $n \geq 4$: $\nu_1(P_3 \times P_n) = 2n - 4$

- **Upper bound**

- **Lower bound**
3-row Meshes

Upper bound:

Optimal outerplanar drawing of $P_3 \times P_5$
3-row Meshes

Upper bound:

Optimal outerplanar drawing of $P_3 \times P_6$
3-row Meshes

Lower bound

By brute-force algorithm we get
\[ \nu_1(P_3 \times P_3) = 3 \] and \[ \nu_1(P_3 \times P_4) = 4 \]

Suppose odd \( n \), \( \nu_1(P_3 \times P_n) = 2n - 3 \)

Adding a comb to a mesh \( P_3 \times P_n \)
to get mesh \( P_3 \times P_{n+2} \)
The comb makes at least 4 crossings.

\[ \nu_1(P_3 \times P_{n+2}) \geq 4 + \nu_1(D(P_3 \times P_n)) \]
\[ \geq 4 + 2n - 3 \]
\[ = 2(n + 2) - 3 \]

Proof of even \( n \) similar.
Halin Graphs

- **Theorem 2:**
  For an arbitrary Halin graph $G (d \geq 3)$, with $m$ leaves: $\nu_1 (G) = m - 2$

- **Upper bound**
- **Lower bound**
Halin Graphs

Upper bound

\[ cr(4,5) = 3 \]
\[ cr(9,11) = 2 \]

\[ v_1(G) = 5 = m - 2 \]

For a Halin graph, we can always find a Hamilton cycle, which is a solution (there are more solutions)
Halin Graphs

- **Lower bound**

**Fact 1:** Number of all in-vertices (except leaves) in a tree, \( X = n - m \)

**Fact 2:** When we put any in-vertex on the circle, at most 2 edges incident to the in-vertex will be on the circle.

**Fact 3:** The remaining \( d - 2 \) edges will produce \( d - 2 \) crossings at least, where \( d \) is the degree of each in-vertex in the tree.

\[
\nu_1(G) = d_1 - 2 + d_2 - 2 + \ldots + d_x - 2 = d_1 + d_2 + d_3 + \ldots + d_x - 2X
\]

**Fact 4:** The number of edges in the tree: \( M = n - 1 \)

\[
\nu_1(G) + m = d_1 + d_2 + d_3 + \ldots + d_x - 2X + m = (d_1 + d_2 + d_3 + \ldots + d_x + m) - 2X = 2M - 2X = 2(n-1) - 2(n-m) = 2m - 2
\]

\[
\nu_1(G) = m - 2
\]
Complete p-partite graphs, $Kn(p)$

- **Denote:** $Kn(p) = K_{n,n,...,n}$
- **Theorem 3**

\[ \nu_1(K_n(p)) = n^4 \binom{p}{4} + n^2(n-1)(2n-1)\binom{p}{3}/2 + n\binom{n}{3}\binom{p}{2} \]

- **Upper bound**
- **Lower bound**
Complete p-partite graph, $K_n(p)$

- Upper bound

Optimal drawing $K_3(3)$
Complete p-partite graphs, $K_n(p)$

- Known facts: (Riskin, A. 2003)
  1. $\nu_1(K_{n,2n}) = n^2 (n-1)(4n-5)/6$
  2. $\nu_1(K_{n,n}) = n \binom{n}{3}$
Complete p-partite graphs

Lower bound

Three types of edge crossings:
1. Number of the 2-coloured crossings: \( \binom{p}{2} v_1(K_{n,n}) \)
2. Number of the 3-coloured crossings: \( p \binom{p-1}{2} v_1(K_{n,2n}) - 2v_1(K_{n,n}) \)
3. Number of the 4-coloured crossings: \( n^4 \binom{p}{4} \)

Sum of three types of edge crossings:

\[ v_1(K_n(p)) = n^4 \binom{p}{4} + n^2(n-1)(2n-1) \binom{p}{3}/2 + n \binom{n}{3} \binom{p}{2} \]
Questions?