Fold and Unfold for Program Semantics

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Abstract

In this paper we explain how recursion operators can be used to structure and reason about *program semantics* within a functional language. In particular, we show how the recursion operator *fold* can be used to structure denotational semantics, how the dual recursion operator *unfold* can be used to structure operational semantics, and how algebraic properties of these operators can be used to reason about program semantics. The techniques are explained with the aid of two main examples, the first concerning arithmetic expressions, and the second concerning Milner's concurrent language CCS. The aim of the paper is to give functional programmers new insights into recursion operators, program semantics, and the relationships between them.

1 Introduction

Many computations are naturally expressed as recursive programs defined in terms of themselves, and properties proved of such programs using some form of inductive argument. Not surprisingly, many programs will have a similar recursive structure, and many proofs will have a similar inductive structure. To avoid repeating the same patterns of program and proof again and again, special recursion operators and proof principles that abstract out the common patterns can be introduced, allowing us to concentrate on the details that are specific to each different application.

In the functional programming community, much previous work in this area has focussed on a recursion operator called *fold*, and on its associated proof principle called *universality*. Fold captures a common programming pattern in which a list of values is *processed* in a certain recursive manner, and universality captures a common pattern of inductive proof concerning programs that process lists. Fold and universality have proved useful in a variety of application areas, including algorithm construction [1, 11, 2], hardware construction [7, 6], compiler construction [12], and automatic program transformation [20, 3, 8]. Using ideas from category theory, fold has been uniformly generalised from lists to a large class of recursive datatypes [10, 14].

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In this paper we are concerned with the application of recursion operators in the area of program semantics. One of the most popular styles of semantics is the *denotational* approach [19], in which the meaning of programs is defined using a valuation function that maps programs into values in an appropriate semantic domain. The valuation function is defined using a set of recursion equations, and must be compositional in the sense that the meaning of a program is defined purely in terms of the meaning of its syntactic subcomponents. In fact, the pattern of recursion required by compositionality is precisely the pattern of recursion captured by fold. Hence, a denotational semantics can be characterised as a semantics defined by folding over program syntax. Although widely known in certain circles, many functional programmers are still not aware of this connection.

The recursion operator fold has a natural dual, called *unfold*, which captures a common programming pattern in which a list of values is *produced* (as opposed to processed) in a certain recursive manner. The dual proof principle, again called universality, captures a common pattern of coinductive proof concerning programs that produce lists. Unfold has also been generalised from lists to a large class of recursive datatypes [10, 14]. While applications of fold abound, relatively little attention has been given to unfold in the functional programming community.

Another popular style of semantics is the *operational* approach [17], in which the meaning of programs is defined using a transition relation that captures single execution steps in an appropriate abstract machine. The transition relation is defined using a set of inference rules, and the meaning of a program is given by repeatedly applying the relation to generate a transition tree that captures all possible execution paths of the program. In fact, the pattern of recursion used to construct transition trees is precisely the pattern of recursion captured by unfold. Hence, an operational semantics can be characterised as a semantics defined by unfolding to transition trees. This connection has been developed using category theory [18, 21], but most functional programmers are not aware of this connection.

In this paper we explain how recursion operators can be used to structure and reason about program semantics within the functional language Haskell [16]. In particular, we show how fold can be used to structure denotational semantics, how unfold can be used to structure operational semantics, and how algebraic properties of these operators can be used to reason about program semantics.

The techniques are explained with the aid of two main examples, the first concerning arithmetic expressions, and the second concerning Milner's concurrent language CCS [15].

As the paper proceeds we adopt an increasingly categorical approach to semantics, to give a deeper understanding of the issues. However, previous knowledge of category theory is not required. The aim of the paper is to give functional programmers new insights into recursion operators, program semantics, and the relationships between them.

2 Denotational semantics

In denotational semantics [19], the meaning of terms is defined using a valuation function that maps terms into values in an appropriate semantic domain. In this section we explain how a denotational semantics can be characterised as a semantics defined by folding over syntactic terms.

Formally, a denotational semantics for a language T of syntactic terms comprises two components: a set V of semantic values, and a valuation function $[\![\;]\!] : T \rightarrow V$ that maps terms to their meaning as values. The valuation function must be compositional in the sense that the meaning of a compound term is defined purely in terms of the meaning of its T-subterms. When the set of semantic values is clear, a denotational semantics is often identified with a compositional valuation function.

2.1 Arithmetic expressions

As an example, let us consider a language of simple arithmetic expressions, built up from the set $\mathbf{Z}$ of integer values using the addition operator $+$. The language $\mathbf{E}$ of such expressions is defined by the following grammar:

$$\mathbf{E} ::= \mathbf{Z} \mid \mathbf{E} + \mathbf{E}$$

We assume that parentheses can be used to disambiguate expressions if required. The grammar for expressions can be directly translated into a Haskell datatype definition, parameterised over the type of values for flexibility:

```
data Expr a = Val a | Add (Expr a) (Expr a)
```

For example, the expression $1+(2+3)$ is represented by the value `Add (Val 1) (Add (Val 2) (Val 3))`. From now on, we mainly consider expressions represented in Haskell.

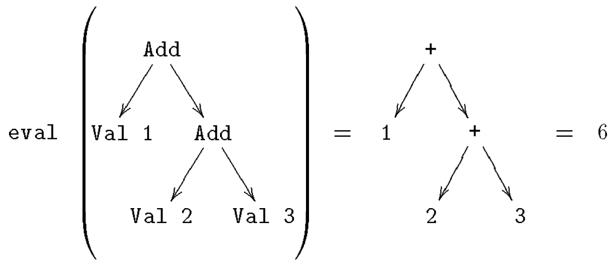
Arithmetic expressions have an the obvious denotational semantics, given by taking V as the Haskell type `Int` of integers and $[\![\;]\!]$: `Expr Int` $\rightarrow$ `Int` as the evaluation function for expressions defined recursively as follows:

$$\begin{array}{lcl} [\![\mathtt{Val}\ n]\!] & = & n \\ [\![\mathtt{Add}\ x\ y]\!] & = & [\![x]\!] + [\![y]\!] \end{array}$$

This definition satisfies the compositionality requirement, because the meaning of compound expressions of the form `Add` x y is defined purely by applying $+$ to the meanings of the subexpressions x and y. The evaluation function can be translated directly into a Haskell function definition:

```
eval          :: Expr Int -> Int
eval (Val n)   = n
eval (Add x y) = eval x + eval y
```

For example, `eval (Add (Val 1) (Add (Val 2) (Val 3)))` = `1+(2+3)` = 6, or drawing expressions as trees:



Looking at this example, we see that an expression is evaluated by removing each constructor `Val` (or equivalently, replacing each constructor `Val` by the identity function `id` on integers), and replacing each constructor `Add` by the addition function `(+)` on integers. That is, even though `eval` was defined recursively, its behaviour can be understood non-recursively as simply replacing the two constructors for expressions by the functions `id` and `(+)`.

2.2 Fold for expressions

Abstracting from the specific case of `eval`, we can consider the general case of a denotational semantics `deno` that gives meaning to arithmetic expression by replacing each `Val` by a function `f`, and each `Add` by a function `g`. By definition, a semantics defined in this manner will be compositional, because the meaning of addition is defined purely by applying `g` to the meanings of the two argument expressions:

```
deno (Val n)   = f n
deno (Add x y) = g (deno x) (deno y)
```

Since the behaviour of such functions can be understood non-recursively, why don't we actually define them in this manner? This is precisely what `fold` allows us to do. Using `fold` for arithmetic expressions, we can define denotational semantics for expressions simply by supplying the function `f` that replaces each `Val` and the function `g` that replaces each `Add`. For example, using `fold` the denotational semantics `eval` can be simply defined as follows:

```
eval = fold id (+)
```

As another example, using `fold` we can define an alternative semantics `comp` that doesn't evaluate expressions directly, but rather compiles expressions into a list of instructions for execution using a stack. As for `eval`, defining the semantics using `fold` makes it compositional by definition:

```
data Inst = PUSH Int | ADD

comp :: Expr Int -> [Inst]
comp  = fold f g
        where
           f n     = [PUSH n]
           g xs ys = xs ++ ys ++ [ADD]
```

For example, `comp (Add (Val 1) (Add (Val 2) (Val 3)))` = `[PUSH 1, PUSH 2, PUSH 3, ADD, ADD]`.

The `fold` function itself can be defined simply by abstracting on the free variables `f` and `g` in the general definition of a denotational semantics `deno` for expressions:

```
fold f g (Val n)   = f n
fold f g (Add x y) = g (fold f g x) (fold f g y)
```

The type of `fold` is given by the following inference rule:

$$\frac{\texttt{f :: a -> b} \qquad \texttt{g :: b -> b -> b}}{\texttt{fold f g :: Expr a -> b}}$$

2.3 Generalising

Of course, the use of `fold` to define denotational semantics is not specific to our language of arithmetic expressions, but can be generalised to many other languages. For example, consider extending expressions with integer variables of the form `Var c` for any character `c`. Then the `fold` operator would simply be generalised to take an extra argument function `h` to replace each constructor `Var` in an expression:

```
fold f g h (Val n)   = f n
fold f g h (Add x y) = g (fold f g h x)
                         (fold f g h y)
fold f g h (Var c)   = h c
```

In turn, the denotational semantics `eval` would be generalised to give the meaning of expressions as functions from stores (containing the value of each variable) to integers. Assuming a type `Store` for stores and a function `find` for looking up the value of a variable in a store, `eval` is defined using the generalised `fold` as follows:

```
eval :: Expr Int -> (Store -> Int)
eval  = fold f g h
        where
           f n     = \s -> n
           g fx fy = \s -> fx s + fy s
           h c     = \s -> find c s
```

Again, defining the semantics using `fold` makes it compositional by definition. As in this example, the set of semantic values for most non-trivial languages will usually involve functions in some way. For a more general discussion on the use of `fold` to return functions, see [4].

In general, we have the following simple connection between denotational semantics and `fold` operators:

Denotational semantics
$\Updownarrow$
Folding over syntax trees

3 Operational semantics

In operational semantics [17], the meaning of terms is defined using a transition relation that captures execution steps in an appropriate abstract machine. In this section we explain how an operational semantics can be characterised as a semantics defined by unfolding to transition trees.

Formally, an operational semantics for a language T of syntactic terms comprises two components: a set S of states, and a transition relation $\rightarrow \subseteq S \times S$ that relates states to all the states that can be reached by performing a single execution step. (For some applications, a more general notion of transition relation may be appropriate, but this simple notion suffices here.) If $\langle s, s' \rangle \in \rightarrow$, we say that there is a transition from state s to state s', and usually write this as $s \rightarrow s'$. When the set of states is clear, an operational semantics is often identified with a transition relation.

3.1 Arithmetic expressions

Returning to our example from the previous section, simple arithmetic expressions have an obvious operational semantics, given by taking S as the Haskell type `Expr` of expressions, and $\rightarrow \subseteq \texttt{Expr} \times \texttt{Expr}$ as the transition relation defined by the following three inference rules:

$$\frac{}{\texttt{Add (Val } n\texttt{) (Val } m\texttt{)} \;\rightarrow\; \texttt{Val } (n+m)}$$

$$\frac{x \;\rightarrow\; x'}{\texttt{Add } x\ y \;\rightarrow\; \texttt{Add } x'\ y} \qquad \frac{y \;\rightarrow\; y'}{\texttt{Add } x\ y \;\rightarrow\; \texttt{Add } x\ y'}$$

The first rule states that two values can be added together to give a single value, and the last two rules permit the first rule to be applied to either argument of an addition expression. For example, the (concrete) expression $(1+2)+(3+4)$ has two possible transitions, because the first transition rule can be applied to either argument of the top-level addition:

$$(1+2)+(3+4) \;\rightarrow\; 3+(3+4)$$

$$(1+2)+(3+4) \;\rightarrow\; (1+2)+7$$

By repeated application of a transition relation, is is possible to generate a transition tree that captures all possible execution paths for a syntactic term. For example, the expression $(1+2)+(3+4)$ gives rise to the following transition tree, which captures the two possible execution paths:

$$\begin{tikzcd}
 & (1+2)+(3+4) \arrow[dl] \arrow[dr] & \\
3+(3+4) \arrow[d] & & (1+2)+7 \arrow[d] \\
3+7 \arrow[d] & & 3+7 \arrow[d] \\
10 & & 10
\end{tikzcd}$$

The inference rules defining the transition relation for expressions can be easily translated into a Haskell function definition. The relation is represented as a list-valued function that maps expressions to lists of expressions that can be reached by performing a single execution step:

```
trans         :: Expr Int -> [Expr Int]
trans (Val n) = []
trans (Add (Val n) (Val m)) = [Val (n+m)]
trans (Add x y)
   = [Add x' y | x' <- trans x] ++
     [Add x y' | y' <- trans y]
```

In turn, we can define a Haskell datatype for transition trees, and an execution function that converts expressions into trees by repeated application of the transition function:

```
data Tree a = Node a [Tree a]

exec  :: Expr Int -> Tree (Expr Int)
exec e = Node e [exec e' | e' <- trans e]
```

Looking at the definition of `exec`, we see that an expression is executed to yield a tree by taking the expression unchanged as the root of the tree (or equivalently, applying the identity function `id` on expressions), and generating a list of residual expressions to be processed to give the subtrees by applying the `trans` function. That is, even though `exec` was defined recursively, its behaviour can be understood non-recursively as simply applying the identity function `id` to generate the root expression, and the transition function `trans` to generate a list of residual expressions to be processed to generate the subtrees.

3.2 Unfold for trees

Abstracting from the specific case of `exec`, we can consider the general case of an operational semantics `oper` that gives meaning as trees by using a function `f` to generate the root of the tree, and a function `g` to generate a list of residual values to be processed to generate the subtrees:

```
oper  ::  a -> Tree b
oper x = Node (f x) [oper x' | x' <- g x]
```

Since the behaviour of such functions can be understood non-recursively, why don't we actually define them in this manner? This is precisely what `unfold` allows us to do. Using `unfold` for trees, we can define operational semantics as trees simply by supplying the function `f` that generates the root of the tree, and the function `g` that generates the residual values. For example, using `unfold` the operational semantics `exec` can be simply defined as follows:

```
exec = unfold id trans
```

The `unfold` function itself is defined simply by abstracting on the free variables `f` and `g` in the general definition of an operational semantics `oper` as trees:

```
unfold f g x =
   Node (f x) [unfold f g x' | x' <- g x]
```

The type of `unfold` is given by the following inference rule:

$$\frac{\texttt{f :: a -> b} \qquad \texttt{g :: a -> [a]}}{\texttt{unfold f g :: a -> Tree b}}$$

3.3 Generalising

Of course, the use of `unfold` to define operational semantics is not specific to our language of arithmetic expressions, but can be generalised to many other languages. That is, we have the following simple connection between operational semantics and `unfold` operators:

Operational semantics $\Updownarrow$ Unfolding to transition trees

This is precisely dual to the connection for denotational semantics given in the previous section. Hence, taking a structured approach to program semantics using recursion operators has revealed a duality between denotational and operational semantics that might otherwise have been missed. In the next section we will see that using recursion operators also brings benefits when proving properties of semantics.

4 Reasoning about semantics

One of the main reasons for defining the formal semantics of programming languages is to support formal reasoning about languages and programs written in them. In this section we explain how properties of semantics can be proved using the *universality* of the recursion operator `fold` [13, 14], rather than explicit structural induction.

4.1 Arithmetic expressions

Consider the following three equations concerning our semantics for (finite) simple arithmetic expressions:

(1) `and [deno e' = deno e | e' <- trans e]` $=$ `True`

(2) `and [size e' < size e | e' <- trans e]` $=$ `True`

(3) `and [n == deno e | n <- vals (oper e)]` $=$ `True`

The first equation states that the transition function preserves the denotational semantics of expressions. The second equation states that the transition function decreases the size of expressions, where the size is defined as the number of `Add` constructors. The last equation states that the denotational and operational semantics are equivalent, in the sense that the integer values in the transition tree generated by the operational semantics are all equal to the value obtained from the denotational semantics. The auxiliary functions `size` and `vals` are easy to define.

Equations (1) and (2) above can be proved by induction on the structure of `e`. In turn, by making use of these two auxiliary results, (3) can be proved by induction on the size of `e`. However, the two proofs using structural induction can also be proved using the universality of `fold`, which avoids the need for explicit use of induction.

4.2 Universality for expressions

For simple arithmetic expressions, the universality of `fold` is captured by the following equivalence:

$$\begin{array}{lcl} \texttt{h (Val n)} & = & \texttt{f n} \\ \texttt{h (Add x y)} & = & \texttt{g (h x) (h y)} \end{array}$$

$$\Updownarrow$$

$$\texttt{h} \quad = \quad \texttt{fold f g}$$

This equivalence states that `fold f g` is the *unique* solution to the first two equations, and can itself be proved using a simple structural induction. Indeed, the two equations are precisely the assumptions required to show that `h` $=$ `fold f g` using structural induction. For specific cases then, by verifying the two assumptions (which can typically be done without the need for induction), we can then appeal to universality to complete the inductive proof that `h` $=$ `fold f g`. In this manner, universality captures a common pattern of inductive proof, just as `fold` itself captures a common pattern of recursive definition.

To prove equation (1) above using the universality of `fold`, it must first be expressed in the form `h` $=$ `fold f g`. In this case, `h` can be defined simply by abstracting over `e` on the left-hand side of the equation:

```
h e = and [deno e = deno e' | e' <- trans e]
```

Abstracting on the right-hand side of the equation gives the constant function `\e -> True`, which can be expressed in the form `fold f g` by defining `f n = True` and `g x y = x && y`. Hence, by appealing to the universality of `fold` for expressions, we can conclude that equation (1) is equivalent to the following two equations:

```
h (Val n)    =  True
h (Add x y)  =  (h x) && (h y)
```

These equations can now be verified by routine calculations, without the need for an explicit induction. Universality can also be used to prove (2) in a similar way, again without the need for an explicit induction.

5 Concurrent processes in CCS

Up to this point, all our examples have been concerned with arithmetic expressions. For the remainder of the paper we show how our techniques apply to a real-life example, Milner's language CCS (Calculus of Concurrent Systems) for describing concurrent processes [15]. In this section we consider the Haskell datatypes required for the syntax and semantics of CCS processes, and show how they can be defined in an abstract manner as *least fixpoints* of *functors*. As we shall see in subsequent sections, this approach will permit a more abstract treatment of the semantics of processes.

Given a set $\mathbf{N}$ of process names, and a set α of process actions, the language $\mathbf{P}$ of processes in CCS is defined by the following grammar:

$$
\begin{array}{lcll}
\mathbf{P} & ::= & \mathbf{N} & -\textit{constants} \\
 & | & \alpha.\mathbf{P} & -\textit{prefixing} \\
 & | & \sum_{i \in I} \mathbf{P}_i & -(\textit{finite})\ \textit{choice} \\
 & | & \mathbf{P} \mid \mathbf{P} & -\textit{parallelism} \\
 & | & \mathbf{P} \backslash \alpha & -\textit{restriction} \\
 & | & \mathbf{P}[f] & -\textit{relabelling}
\end{array}
$$

We assume that parentheses can be used to disambiguate processes if required. Named processes are defined by (possibly recursive) equations. The set α of actions is assumed to comprise input actions $a, b, c, \ldots$, the corresponding output actions $\bar{a}, \bar{b}, \bar{c}, \ldots$, and the silent action τ used to indicate synchronisation. A relabelling function f is a function from actions to actions that preserves their underlying structure, in the sense that $f(\bar{x}) = \overline{f(x)}$ and $f(\tau) = \tau$.

As a simple example of a process, consider the recursive equation $A = a.A + b.A$. Intuitively, this equation defines the process A that can either perform the action a and then continue as A again, or perform the action b and then continue as A again. More formally, the meaning of a process can be described by a (possibly infinite) transition tree, in which the nodes represent the states of the process, and the edges are labelled with the actions that are performed in moving between states. For example, the meaning of A is given by the infinite tree pictured in Figure 1.

Assuming types `Name` and `Act` for names and actions respectively, the grammar for processes can be directly translated into a Haskell datatype definition:

```
data Proc = Con Name
          | Pre Act Proc
          | Cho [Proc]
          | Par Proc Proc
          | Res Proc Act
          | Rel Proc (Act -> Act)
```

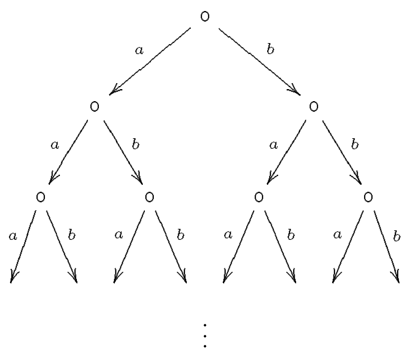


Figure 1: Transition tree for $A = a.A + b.A$

(The `Proc` type could be parameterised over the type of actions, but we use a fixed type `Act` for simplicity.) In turn, a datatype for trees can be defined as follows:

```
data Tree = Node [(Act,Tree)]
```

However, there is another approach to defining `Proc` and `Tree` that will permit the semantics of processes as trees to be defined in a more abstract manner. Rather than defining these types directly as recursive datatypes, we prefer to define them indirectly as least fixpoints of functors.

5.1 Least fixpoints

In semantics, it is common to model recursively defined values as least fixpoints of non-recursively defined functions [19]. For the special case of recursively defined types, the least fixpoint `Fix f` of a type constructor `f` (a function from types to types) can be defined in Haskell as follows:

```
newtype Fix f = In (f (Fix f))
```

For example, the recursive type `Proc` can be expressed as the least fixpoint of a non-recursive type constructor `P`, where the definition for `P` is precisely the same as for the original `Proc` type, except that each recursive call within the definition is replaced by an instance of a type parameter `p`:

```
type Proc = Fix P

data P p  = Con Name
          | Pre Act p
          | Cho [p]
          | Par p p
          | Res p Act
          | Rel p (Act -> Act)
```

Constructors for the new `Proc` type are defined simply by applying the tag `In` to the constructors for `P`:

```
con n   = In (Con n)
pre a p = In (Pre a p)
cho ps  = In (Cho ps)
par p q = In (Par p q)
res p a = In (Res p a)
rel p f = In (Rel p f)
```

In turn, the recursive type `Tree` can be expressed as the least fixpoint of a non-recursive type constructor `T`:

```
type Tree = Fix T

data T t  = Node [(Act,t)]
```

Given the above definitions, it can be shown that the Proc and Tree types defined as least fixpoints are isomorphic to the original types defined using explicit recursion. That is, the types are equivalent in the sense that there is a one-to-one correspondence between their values.

5.2 Functors

The next concept to be considered is that of a functor, which comes from category theory [9]. The notion of a functor is captured as a built-in class in Haskell, defined as follows:

```
class Functor f where
   map :: (a -> b) -> (f a -> f b)
```

This definition states that a type constructor f is a member of the class Functor if it is equipped with a map function that lifts functions of type a -> b to functions of type f a -> f b. Although not made explicit in the Haskell definition, a functor must also preserve the identity function and distribute over function composition, in the sense that:

```
map   id    =  id
map  (g.h)  =  (map g).(map h)
```

For example, the type constructor P can be made into an instance of the class Functor with the following definition:

```
instance Functor P where
   map f x = case x of
                Con n   -> Con n
                Pre a p -> Pre a (f p)
                Cho ps  -> Cho [f p | p <- ps]
                Par p q -> Par (f p) (f q)
                Res p a -> Res (f p) a
                Rel p g -> Rel (f p) g
```

It is easy to verify that this definition satisfies the equations required of a functor. In turn, the type constructor T can be made into an instance of the class Functor as follows:

```
instance Functor T where
   map f (Node xs) =
      Node [(a, f t) | (a,t) <- xs]
```

In summary, we have now expressed the recursive type Proc as the least fixpoint of a non-recursive functor P, and the recursive type Tree as the least fixpoint of the non-recursive functor T. The map functions for both functors play no rôle yet, but they will in subsequent sections.

6 Operational semantics of CCS

As for most languages involving some form of concurrency, the standard semantics for CCS is an operational semantics [15]. In this section we show how the operational semantics for processes as trees can be defined in Haskell in an abstract manner using a *polytypic* version of unfold.

The operational semantics of CCS is given by a transition relation $\rightarrow \subseteq \mathbf{P} \times \alpha \times \mathbf{P}$, where $\mathbf{P}$ is the set of processes, and α is the set of actions. If $\langle P, a, P' \rangle \in \rightarrow$, we say that the process P can perform the action a to become the process P', and usually write this as $P \xrightarrow{a} P'$. The transition relation $\rightarrow$ is defined by the following set of inference rules:

$$\frac{P \xrightarrow{a} P'}{A \xrightarrow{a} P'}\ (A = P) \qquad \frac{}{a.P \xrightarrow{a} P}$$

$$\frac{P_j \xrightarrow{a} P_j{}'}{\sum_{i \in I} P_i \xrightarrow{a} P_j{}'}\ (j \in I)$$

$$\frac{P \xrightarrow{a} P'}{P \mid Q \xrightarrow{a} P' \mid Q} \qquad \frac{Q \xrightarrow{a} Q'}{P \mid Q \xrightarrow{a} P \mid Q'} \qquad \frac{P \xrightarrow{a} P' \quad Q \xrightarrow{\bar{a}} Q'}{P \mid Q \xrightarrow{\tau} P' \mid Q'}$$

$$\frac{P \xrightarrow{b} P'}{P \backslash a \xrightarrow{b} P' \backslash a}\ (a, \bar{a} \neq b) \qquad \frac{P \xrightarrow{a} P'}{P[f] \xrightarrow{f(a)} P'[f]}$$

For example, using these rules the named process A defined by $A = a.A + b.A$ has two possible transitions:

$$A \xrightarrow{a} A \qquad A \xrightarrow{b} A$$

By repeated application of the transition relation, it is possible to generate a (possibly infinite) transition tree that captures all possible execution paths for a process. For example, the process A gives rise to the tree in Figure 1.

The inference rules defining the transition relation for processes can be easily translated into a Haskell function definition. The relation is represented as a list-valued function that maps processes to lists of (action,process) pairs that arise from single execution steps:

```
trans        :: Proc -> [(Act,Proc)]
trans (In x) = case x of
   Con n   -> trans (defn n)
   Pre a p -> [(a,p)]
   Cho ps  -> concat (map trans ps)
   Par p q -> [(a, par p' q) |
                 (a,p') <- trans p] ++
              [(b, par p q') |
                 (b,q') <- trans q] ++
              [(Tau, par p' q') |
                 (a,p') <- trans p,
                 (b,q') <- trans q,
                 synch a b]
   Res p a -> [(b, res p' a) |
                 (b,p') <- trans p,
                 strip a /= strip b]
   Rel p f -> [(f a, rel p' f) |
                 (a,p') <- trans p]
```

The auxiliary function defn maps process names to their definitions and should be defined as appropriate by the user, while synch decides if two actions can synchronise, and strip removes any bars from an action to give its underlying name. Both synch and strip are easy to define.

In turn, we can define an execution function that converts processes into trees by repeated application of the transition function using the unfold function for trees:

```
exec :: Proc -> Tree
exec  = unfold trans
```

The general purpose unfold function for our type Tree of transition trees can itself be defined as follows:

```
unfold f x =
   In (Node [(a, unfold f x') | (a,x') <- f x])
```

However, by exploiting the fact that Tree is defined as the least fixpoint of a functor, the execution function can be defined in a more abstract manner by repeated application of a transition *co-algebra* using a polytypic version of unfold that is not specific to any particular recursive datatype.

6.1 Co-algebras

The concept of a co-algebra that we use comes from category theory, and generalises the idea of a transition function. In Haskell, a co-algebra for a functor `f` is a function of type

```
a -> f a
```

for some specific type `a`. For example, the transition function for processes can be converted into a transition co-algebra for the functor `T` by the following simple definition:

```
trans'  :: Proc -> T Proc
trans' p = Node (trans p)
```

A more general example of a co-algebra concerns the fixpoint type `Fix f`. In particular, the inverse function `out` of the tag `In` is a co-algebra for any functor `f`:

```
out       :: Fix f -> f (Fix f)
out (In x) = x
```

6.2 Polytypic unfold

The co-algebra `out` is special among all co-algebras for a functor `f`, being in fact the *final* co-algebra. Technically, this means that for any other co-algebra `g :: a -> f a`, there is a unique function `unfold g :: a -> Fix f` such that the following diagram commutes [13, 14]:

$$
\begin{tikzcd}
\texttt{a} \arrow[r, dashed, "\texttt{unfold g}"] \arrow[d, "\texttt{g}"'] & \texttt{Fix f} \arrow[d, "\texttt{out}"] \\
\texttt{f a} \arrow[r, "\texttt{map (unfold g)}"'] & \texttt{f (Fix f)}
\end{tikzcd}
$$

Using this diagram and the fact that `out` is the inverse to `In`, the `unfold` function itself can be defined as follows:

```
unfold g = In . map (unfold g) . g
```

That is, the function `unfold g` first applies the co-algebra `g` to break down an argument of type `a` into a structured value of type `f a`, then applies the function `map (unfold g)` to recursively process each of the `a` components to give a value of type `f (Fix f)`, and finally applies the tag `In` to give a value of the recursive type `Fix f`. In this manner, `unfold` is a general purpose function for producing values of a recursive type using a simple pattern of recursion.

While previously we defined `unfold` functions that were specific to particular recursive datatypes (for example, trees) the above version of `unfold` is *polytypic* [5], in the sense that it can be used with any recursive datatype that can be expressed as the least fixpoint of a functor.

In the case of processes, because the datatype `Tree` is expressed as the least fixpoint of the functor `T`, and the transition function is expressed as a co-algebra `trans'` for `T`, the execution function that maps processes to trees can now be defined using the polytypic version of `unfold`:

```
exec :: Proc -> Tree
exec  = unfold trans'
```

In summary, we have now expressed the operational semantics of processes as trees as the unique function `unfold trans'` that makes the following diagram commute:

$$
\begin{tikzcd}
\texttt{Proc} \arrow[r, dashed, "\texttt{unfold trans'}"] \arrow[d, "\texttt{trans'}"'] & \texttt{Tree} \arrow[d, "\texttt{out}"] \\
\texttt{T Proc} \arrow[r, "\texttt{map (unfold trans')}"'] & \texttt{T Tree}
\end{tikzcd}
$$

7 Denotational semantics of CCS

In the previous section we defined an operational semantics for processes as trees by unfolding a transition function expressed as a co-algebra. In this section we consider the less well-known denotational semantics for processes as trees, and show how it can be defined in a dual manner by folding a combining function expressed as an *algebra*.

7.1 Algebras

In the spirit of category theory, the notion of a co-algebra is dual to that of an algebra. In Haskell, an algebra for a functor `f` is a function of type

```
f a -> a
```

for some specific type `a`. For example, the tag function `In :: f (Fix f) -> Fix f` is an algebra for any functor `f`. A more specific example of a co-algebra concerns the semantics of processes as trees. In particular, it is natural to define an algebra for the functor `P` as follows:

```
comb  :: P Tree -> Tree
comb x = In (Node (case x of
   Con n   -> denode (eval (defn n))
   Pre a t -> [(a,t)]
   Cho ts  -> concat (map denode ts)
   Par t u -> [(a, comb (Par t' u)) |
                 (a,t') <- denode t] ++
              [(b, comb (Par t u')) |
                 (b,u') <- denode u] ++
              [(Tau, comb (Par t' u')) |
                 (a,t') <- denode t,
                 (b,u') <- denode u,
                 synch a b]
   Res t a -> [(b, comb (Res t' a)) |
                 (b,t') <- denode t,
                 strip a /= strip b]
   Rel t f -> [(f a, comb (Rel t' f)) |
                 (a,t') <- denode t]))
```

The auxiliary function `eval` will be defined shortly, while `denode` is the destructor function for trees:

```
denode               :: Tree -> [(Act,Tree)]
denode (In (Node xs)) = xs
```

We refer to `comb` as a *combining* function, because it takes a value built by applying a CCS operator to trees rather than to processes, and combines the trees into a single tree by interpreting the operator in the appropriate manner for trees. For example, the third case for parallel composition `Par t u` states that if the tree `t` has an `a`-labelled branch to a subtree `t'`, the tree `u` has a `b`-labelled branch to a subtree `u'`, and the actions `a` and `b` can synchronise, then the resulting combined tree has a `Tau`-labelled branch to the recursively computed subtree `comb (Par t' u')`.

7.2 Polytypic fold

The algebra `In` is special among all algebras for a functor `f`, being in fact the *initial* algebra. Technically, this means that for any other algebra `g :: f a -> a`, there is a unique strict function `fold g :: Fix f -> a` such that the following diagram commutes [13, 14]:

$$\begin{tikzcd}
\texttt{f (Fix f)} \arrow[r, "\texttt{map (fold g)}"] \arrow[d, "\texttt{In}"'] & \texttt{f a} \arrow[d, "\texttt{g}"] \\
\texttt{Fix f} \arrow[r, dashed, "\texttt{fold g}"'] & \texttt{a}
\end{tikzcd}$$

Using this diagram and that fact that `In` is the inverse to `out`, the `fold` function itself can be defined as follows:

```
fold g = g . map (fold g) . out
```

That is, the function `fold g` first applies the function `out` to break down an argument of the recursive type `Fix f` into a structured value of type `f (Fix f)`, then applies the function `map (fold g)` to recursively process each of the `Fix f` components to give a value of type `f a`, and finally applies the algebra `g` combine all the `a` components into a single result value of type `a`. In this manner, `fold` is a general purpose function for processing values of a recursive type using a simple pattern of recursion.

While previously we defined `fold` functions that were specific to particular recursive datatypes (for example, expressions), the above version of `fold` is polytypic. Moreover, the definition for `fold` is precisely dual to that for `unfold`. Hence, taking an abstract approach to recursion operators has revealed an explicit duality between `fold` and `unfold` that might otherwise have been missed.

Returning to processes, because the datatype `Proc` is expressed as the least fixpoint of the functor `P`, and the combining function is expressed as an algebra `comb` for `P`, the denotational semantics of processes as trees can now be defined using the polytypic version of `fold`:

```
eval :: Proc -> Tree
eval  = fold comb
```

In summary, we have now expressed the denotational semantics of processes as trees as the unique strict function `fold comb` that makes the upper square in the diagram below commute, and dually, the operational semantics of processes as trees as the unique function `unfold trans'` that makes the lower square commute.

$$\begin{tikzcd}
\texttt{P Proc} \arrow[r, "\texttt{map (fold comb)}"] \arrow[d, "\texttt{In}"'] & \texttt{P Tree} \arrow[d, "\texttt{comb}"] \\
\texttt{Proc} \arrow[r, dashed, shift left, "\texttt{fold comb}"] \arrow[r, dashed, shift right, "\texttt{unfold trans'}"'] \arrow[d, "\texttt{trans'}"'] & \texttt{Tree} \arrow[d, "\texttt{out}"] \\
\texttt{T Proc} \arrow[r, "\texttt{map (unfold trans')}"'] & \texttt{T Tree}
\end{tikzcd}$$

8 Reasoning about CCS

We have now defined both operational and denotational semantics for CCS. It is natural to ask how the two semantics are related. In this section we show that they are equal by exploiting the universality of the recursion operator `fold`.

8.1 Universality of fold

For arbitrary recursive datatypes expressed as the least fixpoints of functors, the universal property of `fold` is captured by the following equivalence (for strict `h`): [13, 14]:

$$\texttt{g . map h} = \texttt{h . In} \;\Leftrightarrow\; \texttt{h} = \texttt{fold g}$$

Because `fold` is a polytypic operator, so this universal property is a polytypic proof principle [5]. Returning to our semantics for processes, it is easy to verify that `unfold trans'` is strict, using the definitions of the functions concerned, together with the strictness of tags defined using `newtype`. Hence, applying the universal property gives:

$$\texttt{unfold trans'} = \texttt{fold comb}$$

$$\Updownarrow$$

$$\texttt{comb . map (unfold trans')} = \texttt{unfold trans' . In}$$

The final equation above can now be verified by a routine induction on the size of an argument `p :: P Proc`, where the size is defined as the number of `In` tags in `p`. Hence our operational and denotational semantics for CCS are equal for all processes of finite size. Further work is still required to extend the proof to processes of infinite size.

The two semantics can also be proved equal using the dual universal property of `unfold` rather than that of `fold`, but the proof works out simpler using the later.

9 Summary and future work

In this paper, we have shown how fold and unfold can be used to structure and reason about program semantics within Haskell. The paper is based upon categorical work on semantics, but explaining the ideas using Haskell makes them simpler, accessible to a wider audience, and executable. Interesting topics for future work include:

- The application of the techniques to further examples, including languages in different paradigms;
- The use of monadic fold operators to structure the denotational semantics of programming languages with imperative effects such as mutable state;
- Exploring recursion operators and algebraic properties that correspond to non-structural patterns of induction, such as induction on the size of values;
- Further applications of fold and unfold.

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