

When is a function a fold or an unfold?

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Abstract

We give a necessary and sufficient condition for when a set-theoretic function can be written using the recursion operator `fold`, and a dual condition for the recursion operator `unfold`. The conditions are simple, practically useful, and generic in the underlying datatype.

1 Introduction

The recursion operator `fold` encapsulates a common pattern for defining programs that *consume* values of a *least* fixpoint type such as finite lists. Dually, the recursion operator `unfold` encapsulates a common pattern for defining programs that *produce* values of a *greatest* fixpoint type such as streams (infinite lists). Theory and applications of `fold` abound — see [11,4] for recent surveys — while in recent years it has become increasingly clear that the less well-known concept of `unfold` is just as useful [5,6,10,13,15].

Given the interest in `fold` and `unfold`, it is natural to ask when a program can be written using one of these operators. Surprisingly little is known about this question. This article gives a complete answer for the special case in which programs are total functions between sets. In particular, we give a necessary and sufficient condition for when a set-theoretic function can be written using `fold`, and a dual condition for `unfold`. The conditions are simple, practically useful, and generic in the underlying datatype. However, our proofs are set-theoretic, and make essential use of classical logic and the Axiom of Choice; hence our results do not generalize to categories of constructive functions¹.

¹ Such as the effective topos or the category of ω -sets.

2 Fold and unfold

In this section we review the categorical treatment of **fold** and **unfold** in terms of initial algebras and final coalgebras; for further details see [18,20,14,1].

Suppose that we fix a category \mathcal{C} and a functor $F : \mathcal{C} \rightarrow \mathcal{C}$. An *algebra* is pair (A, f) comprising an object A and an arrow $f : F A \rightarrow A$, and a *homomorphism* $h : (A, f) \rightarrow (B, g)$ from one such algebra to another is an arrow $h : A \rightarrow B$ such that the following square commutes:

$$\begin{array}{ccc} F A & \xrightarrow{F h} & F B \\ f \downarrow & & \downarrow g \\ A & \xrightarrow{h} & B \end{array}$$

An *initial algebra* is an initial object in the category with algebras as objects and homomorphisms as arrows. We write $(\mu F, \text{in})$ for an initial algebra, and **fold** f for the unique homomorphism $h : (\mu F, \text{in}) \rightarrow (A, f)$ from the initial algebra to any other algebra (A, f) . That is, **fold** f is defined as the unique arrow that makes the following square commute:

$$\begin{array}{ccc} F(\mu F) & \xrightarrow{F(\text{fold } f)} & F A \\ \text{in} \downarrow & & \downarrow f \\ \mu F & \xrightarrow{\text{fold } f} & A \end{array}$$

The dual notions of *coalgebra*, *cohomomorphism*, and *terminal coalgebra* are defined similarly. We write $(\nu F, \text{out})$ for a terminal coalgebra, and **unfold** f for the unique cohomomorphism $h : (A, f) \rightarrow (\nu F, \text{out})$ from any coalgebra (A, f) to the terminal coalgebra. That is, **unfold** f is defined as the unique arrow that makes the following square commute:

$$\begin{array}{ccc} A & \xrightarrow{\text{unfold } f} & \nu F \\ f \downarrow & & \downarrow \text{out} \\ F A & \xrightarrow{F(\text{unfold } f)} & F(\nu F) \end{array}$$

In the literature, **fold** f and **unfold** f are sometimes written as $\langle f \rangle$ and $\llbracket f \rrbracket$, and called *catamorphisms* and *anamorphisms* respectively.

2.1 Example: finite lists

Suppose that we define a functor $L : \mathcal{SET} \rightarrow \mathcal{SET}$ by $L A = \mathbf{1} + (\mathbb{N} \times A)$ and $L f = \text{id}_{\mathbf{1}} + (\text{id}_{\mathbb{N}} \times f)$, where \mathbb{N} is the set of natural numbers. Then an algebra is

a pair (A, f) comprising a set A and a function $f : \mathbf{1} + (\mathbb{N} \times A) \rightarrow A$. Functions of this type can always be uniquely decomposed into the form $f = [g, h]$ for some other functions $g : \mathbf{1} \rightarrow A$ and $h : \mathbb{N} \times A \rightarrow A$. A homomorphism $f : (A, [g, h]) \rightarrow (B, [i, j])$ is a function $f : A \rightarrow B$ such that $f \cdot g = i$ and $f \cdot h = j \cdot (\text{id}_{\mathbb{N}} \times f)$.

The functor L has an initial algebra $(\mu L, \text{in}) = (List(\mathbb{N}), [nil, cons])$, where $List(A)$ is the set of all finite lists with elements drawn from A , and $nil : \mathbf{1} \rightarrow List(\mathbb{N})$ and $cons : \mathbb{N} \times List(\mathbb{N}) \rightarrow List(\mathbb{N})$ are constructors for this set. Given any other set A and two functions $i : \mathbf{1} \rightarrow A$ and $j : \mathbb{N} \times A \rightarrow A$, the function $\text{fold } [i, j] : List(\mathbb{N}) \rightarrow A$ is uniquely defined by the following two equations:

$$\begin{aligned} \text{fold } [i, j] \cdot nil &= i \\ \text{fold } [i, j] \cdot cons &= j \cdot (\text{id}_{\mathbb{N}} \times \text{fold } [i, j]) \end{aligned}$$

That is, $\text{fold } [i, j]$ processes a list by replacing the nil constructor at the end of the list by the function i , and each $cons$ constructor within the list by the function j . For example, the function $sum : List(\mathbb{N}) \rightarrow \mathbb{N}$ that sums a list of naturals can be defined by $sum = \text{fold } [zero, plus]$, where $zero : \mathbf{1} \rightarrow \mathbb{N}$ and $plus : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ are given by $zero () = 0$ and $plus (x, y) = x + y$.

We will use this datatype in examples later. For notational simplicity, we will write ‘ $[]$ ’ for $nil ()$, and ‘ $x : xs$ ’ for $cons (x, xs)$. Thus, we might have written the above definition of fold more perspicuously as:

$$\begin{aligned} (\text{fold } [i, j]) [] &= i \\ (\text{fold } [i, j]) (x : xs) &= j (x, (\text{fold } [i, j]) xs) \end{aligned}$$

2.2 Example: streams

Suppose that we define a functor $S : SET \rightarrow SET$ by $SA = \mathbb{N} \times A$ and $Sf = \text{id}_{\mathbb{N}} \times f$. Then a coalgebra is a pair (A, f) comprising a set A and a function $f : A \rightarrow \mathbb{N} \times A$. Functions of this type can always be uniquely decomposed into the form $f = \langle g, h \rangle$ for some other functions $g : A \rightarrow \mathbb{N}$ and $h : A \rightarrow A$. A cohomomorphism $f : (A, \langle g, h \rangle) \rightarrow (B, \langle i, j \rangle)$ is a function $f : A \rightarrow B$ such that $i \cdot f = g$ and $j \cdot f = f \cdot h$.

The functor S has a terminal coalgebra $(\nu S, \text{out}) = (Stream(\mathbb{N}), \langle head, tail \rangle)$, where $Stream(A)$ is the set of all streams with elements drawn from A , and $head : Stream(\mathbb{N}) \rightarrow \mathbb{N}$ and $tail : Stream(\mathbb{N}) \rightarrow Stream(\mathbb{N})$ are destructors for this set. Given any other set A and two functions $g : A \rightarrow \mathbb{N}$ and $h : A \rightarrow A$, the function $\text{unfold } \langle g, h \rangle : A \rightarrow Stream(\mathbb{N})$ is uniquely defined by the following two equations:

$$\begin{aligned} head \cdot \text{unfold } \langle g, h \rangle &= g \\ tail \cdot \text{unfold } \langle g, h \rangle &= \text{unfold } \langle g, h \rangle \cdot h \end{aligned}$$

That is, $\text{unfold } \langle g, h \rangle$ produces a stream by using the function g to produce

the *head* of the stream, and the function h to generate another value that is then itself unfolded in the same way to produce the *tail* of the stream. For example, the function $from : \mathbb{N} \rightarrow Stream(\mathbb{N})$, which produces a stream of naturals ascending in steps of one, can be defined by $from = \mathbf{unfold}(\mathbf{id}_{\mathbb{N}}, succ)$ where $succ : \mathbb{N} \rightarrow \mathbb{N}$ is given by $succ\ x = x + 1$.

3 When is an arrow a fold or an unfold?

The **fold** operator encapsulates a common pattern for defining an arrow of type $\mu F \rightarrow A$. It is natural then to ask when an arrow of this type can be written using **fold**. More precisely, when can an arbitrary arrow $h : \mu F \rightarrow A$ be written in the form $h = \mathbf{fold}\ f$ for some other arrow $f : FA \rightarrow A$?

A technically complete, but nonetheless unsatisfactory, answer to this question is provided by the universal property of the **fold** operator [18], which can be stated as the following equivalence:

$$h = \mathbf{fold}\ f \quad \Leftrightarrow \quad h \cdot \mathbf{in} = f \cdot Fh$$

The \Rightarrow direction of this equivalence states that $\mathbf{fold}\ f$ is a homomorphism from the initial algebra $(\mu F, \mathbf{in})$ to another algebra (A, f) , while the \Leftarrow direction states that any other homomorphism h between these two algebras must be equal to $\mathbf{fold}\ f$. Taken as a whole, the universal property expresses the fact that $\mathbf{fold}\ f$ is the *unique* homomorphism from $(\mu F, \mathbf{in})$ to (A, f) .

The universal property provides a complete answer to our question — h can be written in the form $\mathbf{fold}\ f$ precisely when $h \cdot \mathbf{in} = f \cdot Fh$ — but is less helpful than it might be because it requires that we already know f . Given a specific h , however, the universal property can often be used to guide the construction of an appropriate f [11], but we do not consider this a completely satisfactory answer either, because this approach is only a heuristic, and it is sometimes difficult to apply in practice.

The problem with the universal property is that it concerns an *intensional* aspect of h , namely the function f that forms part of its implementation. Often a condition based on purely *extensional* aspects is more useful. A partial answer to our question with purely extensional concerns is that every left invertible arrow $h : \mu F \rightarrow A$ can be written using **fold** [20]. Formally, if we assume that there exists an arrow $g : A \rightarrow \mu F$ such that $g \cdot h = \mathbf{id}_{\mu F}$, then the equation $h = \mathbf{fold}\ f$ can be solved for f as follows:

$$\begin{aligned} h &= \mathbf{fold}\ f \\ \Leftrightarrow & \quad \{ \text{universal property} \} \\ h \cdot \mathbf{in} &= f \cdot Fh \\ \Leftrightarrow & \quad \{ \text{identities} \} \\ h \cdot \mathbf{in} \cdot \mathbf{id}_{F(\mu F)} &= f \cdot Fh \end{aligned}$$

$$\begin{aligned}
 &\Leftrightarrow \{ \text{functors} \} \\
 &\quad h \cdot \text{in} \cdot F(\text{id}_{\mu F}) = f \cdot F h \\
 &\Leftrightarrow \{ \text{assumption} \} \\
 &\quad h \cdot \text{in} \cdot F(g \cdot h) = f \cdot F h \\
 &\Leftrightarrow \{ \text{functors} \} \\
 &\quad h \cdot \text{in} \cdot F g \cdot F h = f \cdot F h \\
 &\Leftarrow \{ \text{substitutivity} \} \\
 &\quad f = h \cdot \text{in} \cdot F g
 \end{aligned}$$

In summary, we have derived the following implication:

$$g \cdot h = \text{id}_{\mu F} \Rightarrow h = \text{fold}(h \cdot \text{in} \cdot F g)$$

As an example, the function $\text{rev} : \text{List}(\mathbb{N}) \rightarrow \text{List}(\mathbb{N})$ that reverses a list is its own inverse, and hence it is immediate that rev can be written using fold by the above implication. Note, however, that this implication only provides a partial answer to our question, because the converse is not true in general. That is, not every arrow $h : \mu F \rightarrow A$ that can be written using fold is left invertible. For example, the function $\text{sum} : \text{List}(\mathbb{N}) \rightarrow \mathbb{N}$ was written using fold in the previous section, but is not left invertible.

Dually, the unfold operator also satisfies a universal property, which can be used to show that every right invertible arrow of type $A \rightarrow \nu F$ can be written using unfold [20]. For example, the function $\text{evenpos} : \text{Stream}(\mathbb{N}) \rightarrow \text{Stream}(\mathbb{N})$ that removes every other element from a stream has a right inverse (any function that inserts an element between each adjacent pair in a stream), and hence it is immediate that evenpos can be written using unfold . However, not every arrow $h : A \rightarrow \nu F$ that can be written using unfold is right invertible. For example, the function $\text{from} : \mathbb{N} \rightarrow \text{Stream}(\mathbb{N})$ was written using unfold in the previous section, but is not right invertible.

As far as we are aware, the invertibility results above are the only known results that state when arbitrary arrows of the correct type can be written using fold or unfold . We conclude this section by noting that much more progress has been made concerning specific kinds of arrows. For example, the fusion law states that the composition of a homomorphism and a fold can always be written as a fold , while the banana split law states that two folds applied to the same argument can always be written as a single fold [20].

4 When is a function a fold?

In this section we give a necessary and sufficient condition for when an arrow can be written using fold , for the special case of the category \mathcal{SET} in which the arrows are total functions between sets. We dualize the result to unfold in

the following section.

The result depends on the following definition:

Definition 4.1 The *kernel* [17] of a function $f : A \rightarrow B$ is the set of pairs of elements that are identified by f :

$$\ker f = \{ (a, a') \in A \times A \mid f a = f a' \}$$

The main result of this section is a necessary and sufficient condition for when an arbitrary arrow $h : \mu F \rightarrow A$ in \mathcal{SET} can be written in the form $h = \text{fold } f$ for some other arrow $f : F A \rightarrow A$.

Theorem 4.2 *Suppose that $h : \mu F \rightarrow A$. Then*

$$(\exists g : F A \rightarrow A. \ h = \text{fold } g) \Leftrightarrow \ker (F h) \subseteq \ker (h \cdot \text{in})$$

(Another way of saying this is that h is a fold iff $\ker h$ is a congruence under in ; that is, writing $\text{Rel}(F)(R)$ for the *relational lifting* to a relation on $F A$ of relation R on A [12], iff $(x, y) \in \text{Rel}(F)(\ker h)$ implies $(\text{in } x, \text{in } y) \in \ker h$.)

The crux of the proof is the well-known observation that inclusion of kernels is equivalent to the existence of ‘postfactors’:

Lemma 4.3 *Suppose that $f : A \rightarrow B$ and $h : A \rightarrow C$. Then*

$$(\exists g : B \rightarrow C. \ h = g \cdot f) \Leftrightarrow (\ker f \subseteq \ker h \wedge B \rightarrow C \neq \emptyset)$$

Proof. The proof is straightforward. For the \Rightarrow direction, assume that $g : B \rightarrow C$ and $h = g \cdot f$; then clearly $B \rightarrow C \neq \emptyset$, and moreover,

$$\begin{aligned} & (a, a') \in \ker f \\ \Leftrightarrow & \{ \text{kernels} \} \\ & f a = f a' \\ \Rightarrow & \{ \text{substitutivity} \} \\ & g(f a) = g(f a') \\ \Leftrightarrow & \{ h = g \cdot f \} \\ & h a = h a' \\ \Leftrightarrow & \{ \text{kernels} \} \\ & (a, a') \in \ker h \end{aligned}$$

Conversely, assume that $\ker f \subseteq \ker h$ and $B \rightarrow C \neq \emptyset$, so that either $B = \emptyset$ or $C \neq \emptyset$. When $B = \emptyset$, let g be the unique function in $B \rightarrow C$; note that g is the ‘empty function’, and so $g \cdot f$ is empty too. Moreover, $A = \emptyset$ because of the type of f , so h is also empty and hence equal to $g \cdot f$. When $C \neq \emptyset$, we define $g b$ for b in the range of f by $g b = h a$ for some a with $f a = b$; this is a proper definition, because if there are two choices a, a' with $f a = f a' = b$,

then $h a = h a'$ also by assumption. For b outside the range of f , we define $g b$ arbitrarily. By construction, this gives $h a = g(f a)$ for every a . \square

We also use the following simple fact concerning initial algebras:

Lemma 4.4

$$\mu F \rightarrow A \neq \emptyset \quad \Rightarrow \quad F A \rightarrow A \neq \emptyset$$

Proof. We note that $F A \rightarrow A \neq \emptyset$ is equivalent to $A = \emptyset \Rightarrow F A = \emptyset$, which implication can then be verified as follows:

$$\begin{aligned} & A = \emptyset \\ \Rightarrow & \quad \{ \mu F \rightarrow A \neq \emptyset \} \\ & \mu F = \emptyset \\ \Rightarrow & \quad \{ \text{in} : F(\mu F) \rightarrow \mu F \} \\ & F(\mu F) = \emptyset \\ \Rightarrow & \quad \{ \mu F = \emptyset = A \} \\ & F A = \emptyset \end{aligned}$$

\square

Proof of Theorem 4.2 Given the two lemmata above, the proof of the theorem is almost embarrassingly simple:

$$\begin{aligned} & \exists g : F A \rightarrow A. \quad h = \text{fold } g \\ \Leftrightarrow & \quad \{ \text{universal property} \} \\ & \exists g : F A \rightarrow A. \quad h \cdot \text{in} = g \cdot F h \\ \Leftrightarrow & \quad \{ \text{Lemma 4.3} \} \\ & \ker(F h) \subseteq \ker(h \cdot \text{in}) \wedge F A \rightarrow A \neq \emptyset \\ \Leftrightarrow & \quad \{ \text{Lemma 4.4, } h : \mu F \rightarrow A \} \\ & \ker(F h) \subseteq \ker(h \cdot \text{in}) \end{aligned}$$

\square

Remark 4.5 For the type $List(A)$ of finite lists with elements drawn from A , with constructors $nil : \mathbf{1} \rightarrow List(A)$ and $cons : A \times List(A) \rightarrow List(A)$, Theorem 4.2 reduces to stating that an arbitrary function $h : List(A) \rightarrow B$ can be written directly as a fold precisely when the lists that are identified by h are closed under $cons$, in the sense that for all x, xs, ys ,

$$h xs = h ys \quad \Rightarrow \quad h(x : xs) = h(x : ys)$$

Example 4.6 If we define $sum : List(\mathbb{N}) \rightarrow \mathbb{N}$ by the equations

$$\begin{aligned} sum [] &= 0 \\ sum (x : xs) &= x + sum xs \end{aligned}$$

then it is easy to show that the lists identified by sum are closed under $cons$:

$$\begin{aligned} sum (x : xs) &= sum (x : ys) \\ \Leftrightarrow & \{ \text{definition of } sum \} \\ x + sum xs &= x + sum ys \\ \Leftarrow & \{ \text{substitutivity} \} \\ sum xs &= sum ys \end{aligned}$$

Hence, sum can be written directly using **fold**.

Example 4.7 In contrast, if we define a function $stail : List(\mathbb{N}) \rightarrow List(\mathbb{N})$ (for ‘safe tail’) by the equations

$$\begin{aligned} stail [] &= [] \\ stail (x : xs) &= xs \end{aligned}$$

then a simple counterexample verifies that the lists identified by $stail$ are not closed under $cons$: for example, with $xs = []$ and $ys = 0 : []$, we have $stail xs = [] = stail ys$, but $stail (1 : xs) = [] \neq 0 : [] = stail (1 : ys)$. Therefore $stail$ cannot be written directly as a **fold**.

Example 4.8 For the type $List(\mathbb{R})$ of finite lists of reals, consider the problem of computing $floorsum = floor \cdot rsum$, where $rsum : List(\mathbb{R}) \rightarrow \mathbb{R}$ sums a list of reals and $floor : \mathbb{R} \rightarrow \mathbb{Z}$ rounds a real r down to the largest integer at most r . Because the result is an integer, one might wonder whether $floorsum$ can be carried out as a fold to integers, thereby avoiding the computationally more expensive real arithmetic. It cannot: we have $floorsum (0.3 : []) = floorsum (0.6 : [])$, but $floorsum (0.5 : 0.3 : []) \neq floorsum (0.5 : 0.6 : [])$.

On the other hand, the reverse composition $sum \cdot map floor$, which floors every element of the list before summing, can be written as a fold: an argument similar to Example 4.6 applies. This is an instance of *deforestation* [24], an optimisation whereby two computations are combined into one and the intermediate data structure (here of type $List(\mathbb{Z})$) is eliminated.

Remark 4.9 For the type $Tree(A)$ of binary trees with constructors $leaf : A \rightarrow Tree(A)$ and $node : Tree(A) \times Tree(A) \rightarrow Tree(A)$, Theorem 4.2 reduces to stating that an arbitrary function $h : Tree(A) \rightarrow B$ can be written directly as a **fold** precisely when the trees that are identified by h are closed under $node$, in the sense that for all t, u ,

$$h t = h t' \wedge h u = h u' \quad \Rightarrow \quad h (node (t, u)) = h (node (t', u'))$$

Example 4.10 For another deforestation example, consider $flatsum = sum \cdot flatten$, where $flatten : Tree(A) \rightarrow List(A)$ generates a list of the elements of a tree. The intermediate list in $flatsum$ can be eliminated, because

$$\begin{aligned}
 & flatsum (node (t, u)) \\
 = & \{ \text{definition of } flatsum \} \\
 & sum (flatten (node (t, u))) \\
 = & \{ \text{definition of } flatten \} \\
 & sum (flatten t ++ flatten u) \\
 = & \{ \text{sum distributes over } ++ \} \\
 & sum (flatten t) + sum (flatten u) \\
 = & \{ \text{definition of } flatsum \} \\
 & flatsum t + flatsum u
 \end{aligned}$$

from which we conclude that trees identified under $flatsum$ are closed under $node$. (Here, ‘++’ concatenates two lists.)

Example 4.11 The predicate $bal : Tree(A) \rightarrow \mathbb{B}$ that holds of tree iff it is *balanced* (all the leaves at the same depth) is not a fold: with tree t being balanced and of depth 1, and tree u being balanced and of depth 2, both t and u are identified by bal (both yielding true), yet $bal (node (t, t)) \neq bal (node (t, u))$.

Example 4.12 However, the function $dbal : Tree(A) \rightarrow \mathbb{N} \times \mathbb{B}$ that computes a pair, the depth of the tree and whether it is balanced, is a fold. Because

$$\begin{aligned}
 depth (node (t, u)) &= 1 + \max (depth t, depth u) \\
 bal (node (t, u)) &= bal t \wedge bal u \wedge depth t = depth u
 \end{aligned}$$

trees identified by $dbal$ are closed under $node$. This is an example of a *mutomorphism* [7] or *almost homomorphism* [3,8]; transforming a function into such a form is an important step towards constructing an efficient data-parallel algorithm for computing it.

5 When is a function an unfold?

Dualising Theorem 4.2 to unfold is straightforward. The appropriate dual to the notion of the kernel of a function is simply its *image*:

Definition 5.1 The *image* of a function $f : A \rightarrow B$ is the set of elements that are produced by f :

$$\text{img } f = \{ b \in B \mid \exists a \in A. f a = b \}$$

The duality between kernels and images is perhaps not immediately evident, but is revealed by thinking relationally. In particular, if functions are viewed as relations in the obvious way, then the relational composition $f^\circ \cdot f$ of a function f with its converse f° is precisely the kernel of f , while the dual composition $f \cdot f^\circ$ is (the identity relation on) the image of f .

We can now present our result for `unfold`, which gives a necessary and sufficient condition for when an arbitrary arrow $h : A \rightarrow \nu F$ in \mathcal{SET} can be written in the form $h = \text{unfold } g$ for some other arrow $g : A \rightarrow F A$.

Theorem 5.2 *Suppose that $h : A \rightarrow \nu F$. Then*

$$(\exists g : A \rightarrow F A. \quad h = \text{unfold } g) \quad \Leftrightarrow \quad \text{img } (F h) \supseteq \text{img } (\text{out} \cdot h)$$

(Another way of saying this is that h is an `unfold` iff $\text{img } h$ is an invariant of `out`; that is, writing $\text{Pred}(F)(P)$ for the *predicate lifting* to a predicate on $F A$ of predicate P on A [12], iff $\text{Pred}(F)(\in \text{img } h)$ (`out` x) follows from $(\in \text{img } h) x$.)

The crux of the proof is the dual of Lemma 4.3, namely that inclusion of images is equivalent to the existence of ‘prefactors’:

Lemma 5.3 *Suppose that $f : B \rightarrow C$ and $h : A \rightarrow C$. Then*

$$(\exists g : A \rightarrow B. \quad h = f \cdot g) \quad \Leftrightarrow \quad (\text{img } f \supseteq \text{img } h \wedge A \rightarrow B \neq \emptyset)$$

Proof. For the \Rightarrow direction, assume that $g : A \rightarrow B$ and $h = f \cdot g$; then clearly $A \rightarrow B \neq \emptyset$, and moreover,

$$\begin{aligned} & c \in \text{img } h \\ \Leftrightarrow & \quad \{ \text{images} \} \\ & \exists a. \quad h a = c \\ \Leftrightarrow & \quad \{ h = f \cdot g \} \\ & \exists a. \quad f (g a) = c \\ \Rightarrow & \quad \{ g : A \rightarrow B \} \\ & \exists b. \quad f b = c \\ \Leftrightarrow & \quad \{ \text{images} \} \\ & c \in \text{img } f \end{aligned}$$

Conversely, assume that $\text{img } f \supseteq \text{img } h$ and $A \rightarrow B \neq \emptyset$, so that either $A = \emptyset$ or $B \neq \emptyset$. When $A = \emptyset$, then h is the empty function; let g be the empty function too, so $f \cdot g$ is also empty and hence equal to h . When $B \neq \emptyset$, we define $g a$ for $a \in A$ as follows. Let $c = h a$; by assumption, $c \in \text{img } f$ too, so there exists $b \in B$ with $f b = c$, and we define $g a$ to be such a b . If there is more than one such b , it doesn’t matter which one that we choose. By construction, this gives $h a = f (g a)$ for every a . \square

We also use the dual of Lemma 4.4:

Lemma 5.4

$$A \rightarrow \nu F \neq \emptyset \quad \Rightarrow \quad A \rightarrow F A \neq \emptyset$$

Proof. We note that $A \rightarrow F A \neq \emptyset$ is equivalent to $A \neq \emptyset \Rightarrow F A \neq \emptyset$, which implication can then be verified by combining the two calculations:

$$\begin{aligned} & A \neq \emptyset \\ \Rightarrow & \quad \{ A \rightarrow \nu F \neq \emptyset \} \\ & \nu F \neq \emptyset \\ \Rightarrow & \quad \{ \text{out} : \nu F \rightarrow F(\nu F) \} \\ & F(\nu F) \neq \emptyset \end{aligned}$$

and

$$\begin{aligned} & A \neq \emptyset \\ \Rightarrow & \quad \{ \text{functions} \} \\ & \nu F \rightarrow A \neq \emptyset \\ \Rightarrow & \quad \{ \text{functors} \} \\ & F(\nu F) \rightarrow F A \neq \emptyset \end{aligned}$$

That is, $A \neq \emptyset$ implies that $F(\nu F) \neq \emptyset$ and $F(\nu F) \rightarrow F A \neq \emptyset$, which conjunction in turn implies that $F A \neq \emptyset$, as required. \square

Proof of Theorem 5.2 Again, the proof is simple:

$$\begin{aligned} & \exists g : A \rightarrow F A. \quad h = \text{unfold } g \\ \Leftrightarrow & \quad \{ \text{universal property} \} \\ & \exists g : A \rightarrow F A. \quad \text{out} \cdot h = F h \cdot g \\ \Leftrightarrow & \quad \{ \text{Lemma 5.3} \} \\ & \text{img}(F h) \supseteq \text{img}(\text{out} \cdot h) \wedge A \rightarrow F A \neq \emptyset \\ \Leftrightarrow & \quad \{ \text{Lemma 5.4, } h : A \rightarrow \nu F \} \\ & \text{img}(F h) \supseteq \text{img}(\text{out} \cdot h) \end{aligned}$$

\square

Remark 5.5 For the type $\text{Stream}(A)$ of streams with elements drawn from A , with destructors $\text{head} : \text{Stream}(A) \rightarrow A$ and $\text{tail} : \text{Stream}(A) \rightarrow \text{Stream}(A)$, Theorem 5.2 reduces to stating that an arbitrary function $h : B \rightarrow \text{Stream}(A)$ can be written directly as an `unfold` precisely when the *tail* of every stream producible by h is itself producible by h , in the sense that: $\text{img}(\text{tail} \cdot h) \subseteq \text{img } h$.

Example 5.6 Consider the function $from : \mathbb{N} \rightarrow Stream(\mathbb{N})$ defined in Section 2.2. Then $(tail \cdot from) n$ is the stream $[n + 1, n + 2, \dots]$, and in general, $img(tail \cdot from)$ is the set of streams $\{ [n + 1, n + 2, \dots] \mid n \in \mathbb{N} \}$, which is included in $img\,from$, the set of streams $\{ [n, n + 1, \dots] \mid n \in \mathbb{N} \}$. Hence, $from$ can be written directly using **unfold**.

Example 5.7 In contrast, if we define a function $mults : \mathbb{N} \rightarrow Stream(\mathbb{N})$ such that $mults\,n$ produces the stream of multiples $[0, n, n \times 2, n \times 3, \dots]$ of a natural n , then $(tail \cdot mults) n$ is the stream $[n, n \times 2, \dots]$, and so $img(tail \cdot mults)$ is not included in $img\,mults$, which only includes streams whose *head* is 0. Therefore $mults$ cannot be written directly as an **unfold**.

Remark 5.8 For the type $CoTree(A)$ of infinite binary trees with elements drawn from A , with destructors $root : CoTree(A) \rightarrow A$ and $left, right : CoTree(A) \rightarrow CoTree(A)$, Theorem 5.2 reduces to stating that an arbitrary function $h : B \rightarrow CoTree(A)$ can be written as an **unfold** precisely when the *left* and *right* of every tree producible by h are themselves producible by h :

$$img(left \cdot h) \subseteq img\,h$$

$$img(right \cdot h) \subseteq img\,h$$

Example 5.9 Consider the infinite binary tree with every node labelled by its *path*, a finite list of booleans recording the left and right turns from the root in order to reach that node. The function $paths : \mathbf{1} \rightarrow CoTree(List(\mathbb{B}))$ that produces this tree is not an **unfold**, because $img(left \cdot paths)$ and $img(right \cdot paths)$ contain trees with singleton lists at their roots, which are not included in $img\,paths$, which contains a tree with the empty list at its root.

Example 5.10 In contrast, the more general function $pathsfrom : List(\mathbb{B}) \rightarrow CoTree(List(\mathbb{B}))$ that generates the tree of paths starting from a given path is an **unfold**, because $(left \cdot pathsfrom) bs = pathsfrom(false : bs)$ implies that $img(left \cdot pathsfrom)$ is included in $img\,pathsfrom$, and similarly for *right*.

6 Conclusion

We have given the first complete results for when an arbitrary arrow can be written directly as a **fold** or **unfold**, for the special case of the category SET . In future work we will investigate whether the results can be generalised to other categories, and to other patterns of recursion, such as primitive (co-)recursion [19,22] and course-of-value (co-)iteration [23].

As well as being interesting from a theoretical point of view, we also expect the results to have practical applications in program optimisation. A well-structured program is typically factored into several phases, each phase generating a data structure that is consumed by the subsequent phase; *deforestation* [9,16,21] fuses adjacent phases and eliminates the intermediate data structures. When performed as a compiler optimisation, it yields efficient ob-

ject code without sacrificing the structure and clarity of the source code. Our results can be used to determine when two phases cannot be fused to a fold or an unfold. It might be possible to use an automatic testing system such as QuickCheck [2] to find counterexamples to the appropriate inclusions.

Acknowledgements

We are very grateful to Lambert Meertens, whose suggestions lead to a substantial simplification of our proofs. We also thank the anonymous referees for their useful comments. Graham Hutton was supported by EPSRC grant *Structured Recursive Programming*, and together with Thorsten Altenkirch by ESPRIT Working Group *Applied Semantics*.

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