Describing OR

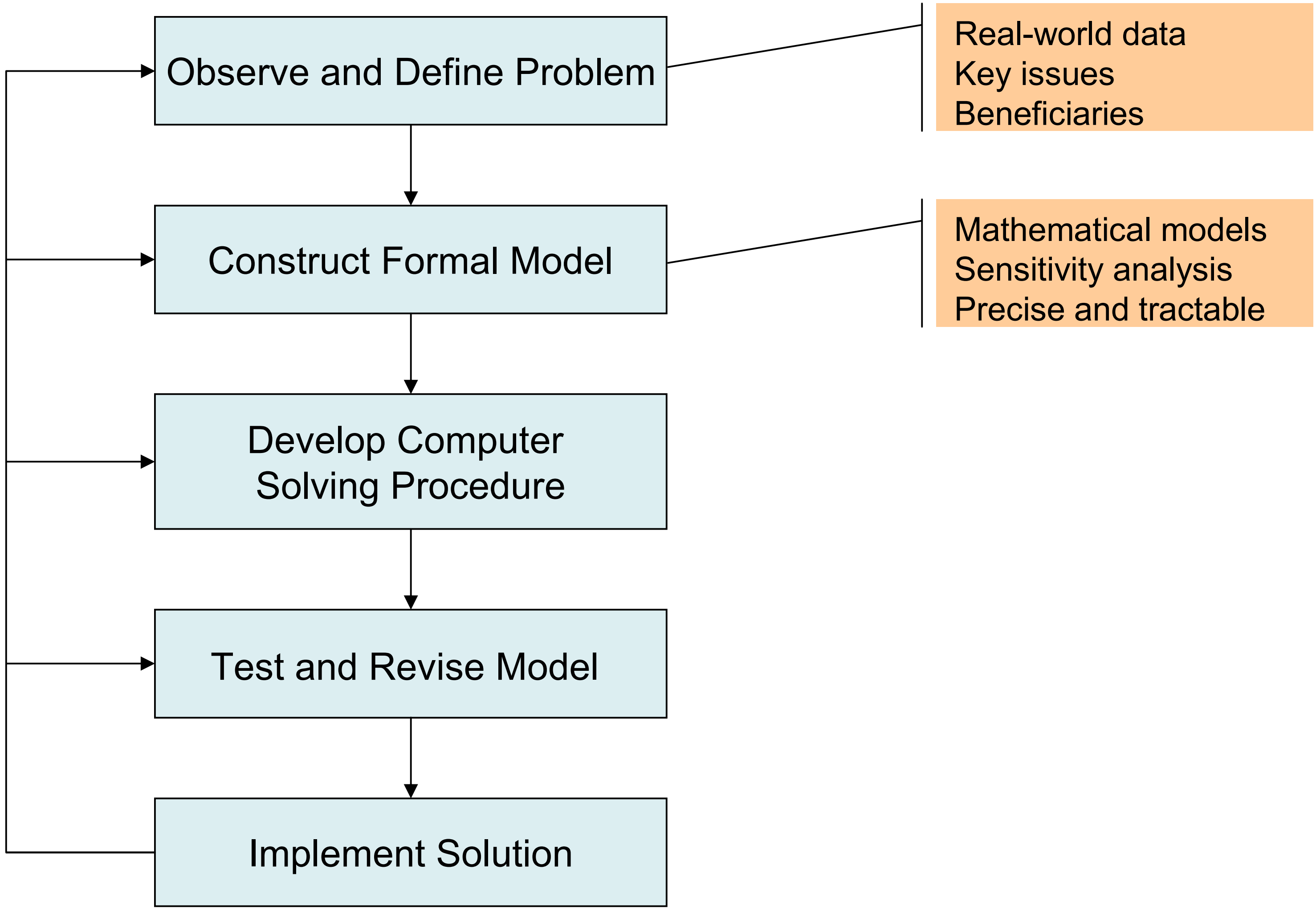
- A scientific approach to improve operations for better management of business and other organisations

- The term operations research (OR) is attributed to research in military operations

- Operational Research and Operations Research refer to the same discipline (often also Management Science)

- Progress and Applicability of OR:
    - Improvement in OR techniques
    - Advances in computing technology
    - Development of powerful algorithms
    - Applicable to all sorts of businesses and industries

Phases on the OR Approach



Data gathered by the OR team

	units of material required to produce one unit of product		
	product A	product B	max availability
Material M1	6	4	24
Material M2	1	2	6
Profit per unit	5	4	

The demand of product B cannot exceed demand of product A by more than 1 unit

The maximum daily demand of product B is 2

Construct Formal Model

Define mathematical linear expressions

- Decision variables

$x_1 = \text{units produced of product A}$

$x_2 = \text{units produced of product B}$

- Objective function

$\text{Maximise} \quad Z = 5x_1 + 4x_2$

- Constraints

$6x_1 + 4x_2 \quad \text{use of material M1 for product A and B}$

$x_1 + 2x_2 \quad \text{use of material M2 for products A and B}$

$6x_1 + 4x_2 \leq 24 \quad \text{availability of material M1}$

$x_1 + 2x_2 \leq 6 \quad \text{availability of material M2}$

$x_2 - x_1 \leq 1 \quad \text{difference in demand between products}$

$x_2 \leq 2 \quad \text{maximum demand of product B}$

$x_1 \geq 0 \text{ and } x_2 \geq 0 \quad \text{production cannot be negative}$

Example 2.2 (cont.) Apply the graphical method

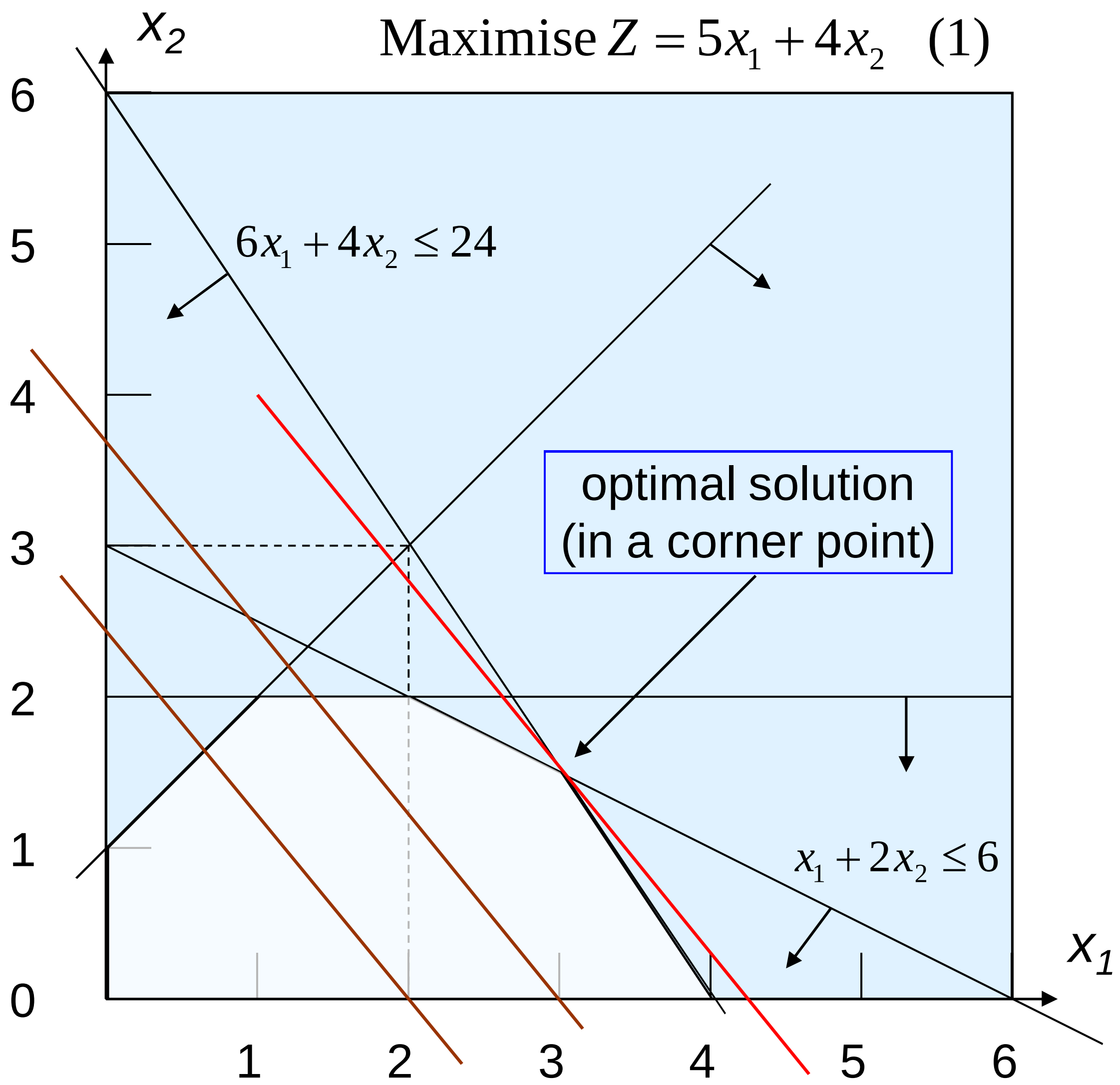
Find the line containing the optimal feasible solution

$x_1 = 3$

$x_2 = 1.5$

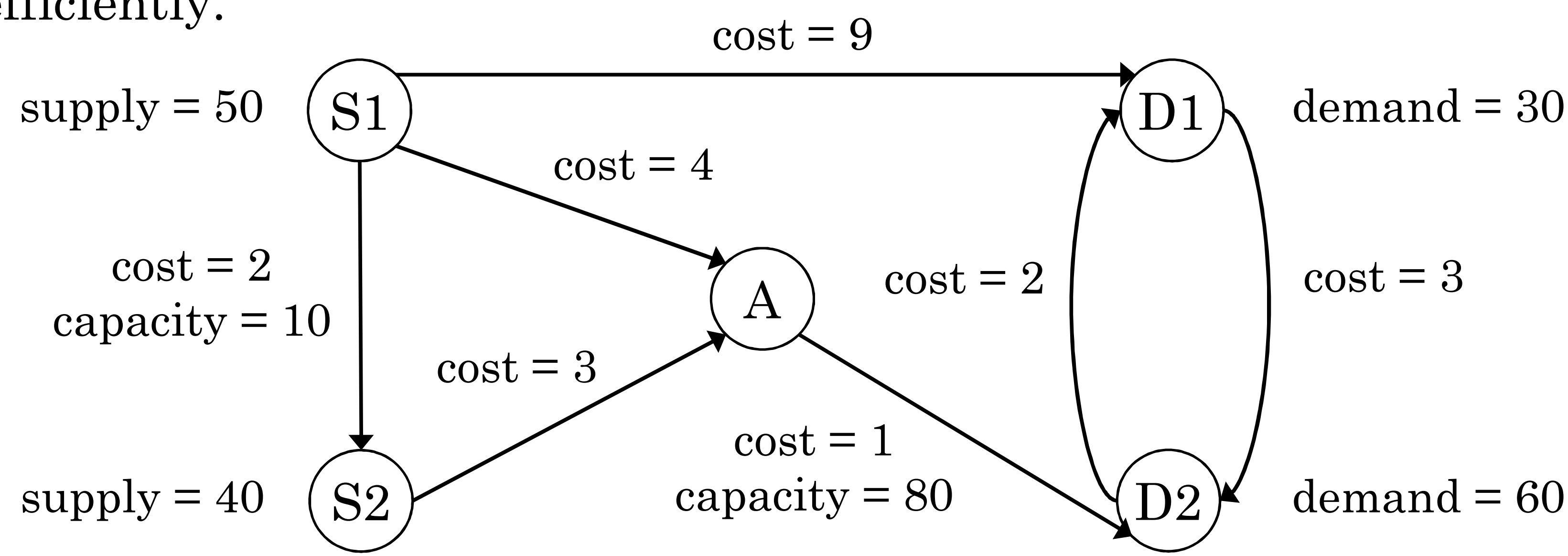
$Z = 5x_1 + 4x_2$

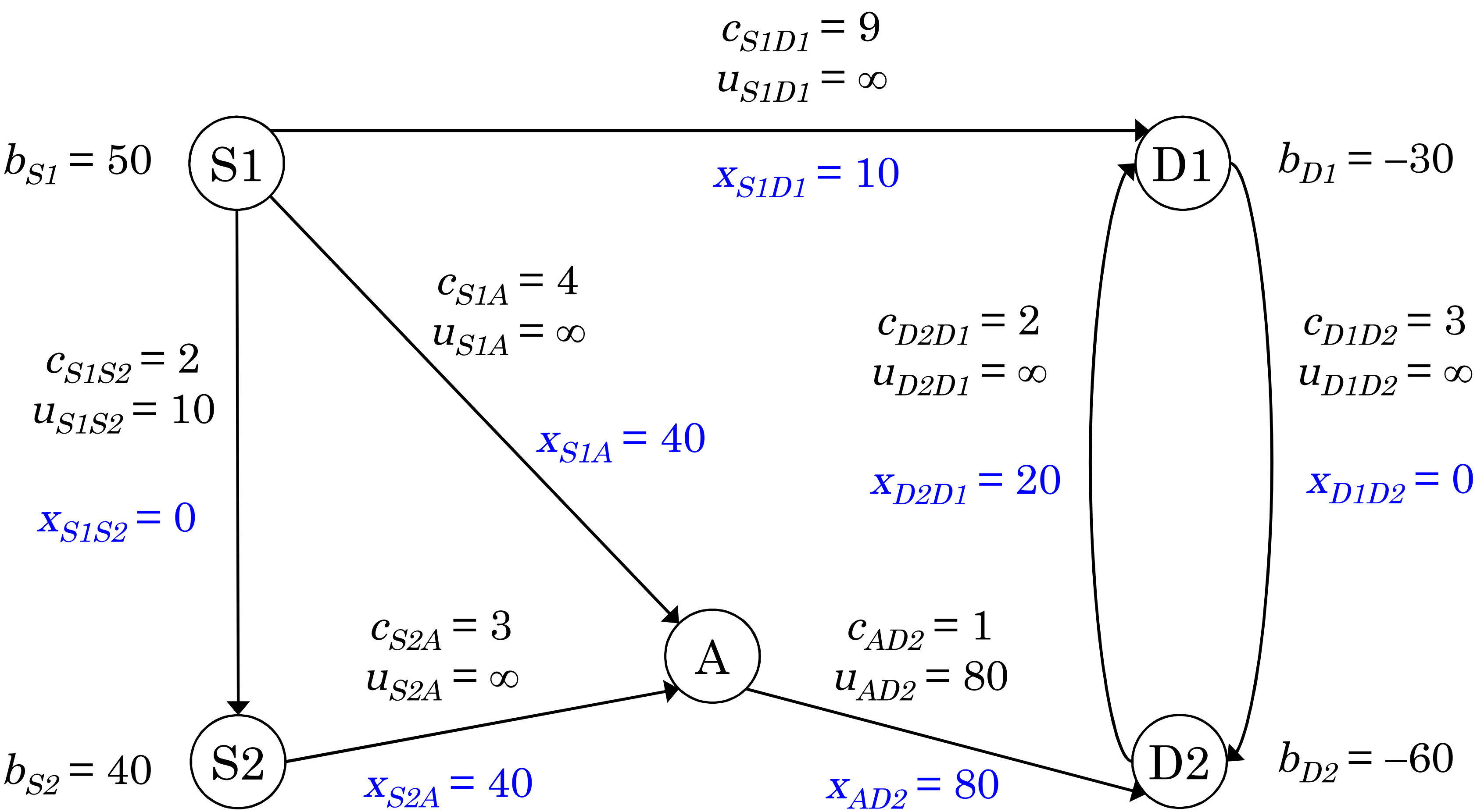
$Z = 21$



Minimum Cost Flow Problem. Consider a directed and connected network with a non-negative maximum flow capacity and a cost of flow associated to each arc, at least one supply node, at least one demand node and trans-shipment nodes. The network has enough capacity to enable all flow from the supply nodes to reach the demand nodes. The objective is to minimise the total cost of sending the available supply through the network to satisfy the demand.

Linear Programming can solve the minimum cost flow problem very efficiently.





Minimum Cost of Flow: 490

Capital Budgeting Problem

The problem is to select the best mix of competing investments given limited budget considerations and priorities in the investments.

Example 4.2 A manager must select the combination of 3-year projects between four of them so as to maximise return. The expenditures and returns for each project are given below.

	Expenditure (millions £) per Year			Return (millions £)
Project	**Y1**	**Y2**	**Y3**	
1	5	1	8	20
2	4	7	10	40
3	6	9	3	20
4	7	4	2	15
Budget (millions £)	18	22	20	

Capital Budgeting Problem Formulation

The decision variables are whether to choose or not each of the projects.

The corresponding BIP formulation:

$$\text{Maximise} \quad Z = 20x_1 + 40x_2 + 20x_3 + 15x_4$$

$$\begin{aligned}
\text{subject to} \quad & 5x_1 + 4x_2 + 6x_3 + 7x_4 \leq 18 \quad (1) \\
& x_1 + 7x_2 + 9x_3 + 4x_4 \leq 22 \quad (2) \\
& 8x_1 + 10x_2 + 3x_3 + 2x_4 \leq 20 \quad (3) \\
& x_1, x_2, x_3, x_4, x_5 \text{ are all binary}
\end{aligned}$$

If projects 1 and 3 cannot be chosen at the same time and that project 4 can be chosen only if project 1 is chosen:

$$x_1 + x_3 \leq 1 \quad (4)$$

$$x_4 \leq x_1 \quad (5)$$

Generalised Assignment Problem

Given a set of n tasks and a set of m workers with a certain cost for assigning each task to each worker. Each worker has a limited amount of time available, each task takes a given worker a certain amount of time. Each task can be assigned to one worker but a worker can undertake more that one task according to the time available. The problem is to assign all the tasks to the workers so that the total cost is minimised without exceeding the time available of any worker.

Minimise $Z = \sum_{j=1}^{m} \sum_{i=1}^{n} c_{ij} x_{ij}$

subject to $\sum_{j=1}^{m} x_{ij} = 1 \quad \text{for } i = 1,2,\ldots n \quad (1)$

$\sum_{i=1}^{n} t_{ij} x_{ij} \le T_j \quad \text{for } j = 1,2,\ldots m \quad (2)$

$x_{ij} = 1$ if task i is assigned to worker j, 0 otherwise

Branch and Bound

The principle underlying the Branch and Bound (B&B) solution technique is divide and conquer.

If the search tree is to large to explore, the problem is divided into smaller problems which then are solved in a similar way.

The B&B techniques uses:

- Branching: the problem is divided by partitioning the entire set of feasible solutions into smaller and smaller subsets.
- Bounding: The bound (best possible solution) of a subset of solutions is computed to determine the location of optimal solutions.
- Fathoming: When a sub-problem is conquered (fathomed), there is no need to further branch that sub-tree.

B&B Algorithm for BIP
(assuming maximisation)

Initialise
$Z^* = -\infty$, Bound, Conquer-Test, Optimality-Test

⇓

Branch
Select most recently created unconquered sub-problem and larger bound†
Divide sub-problem by fixing the branching variable to 0 or 1

⇓

Bound
Solve each relaxed sub-problem rounding down value of Z

⇓

Conquer-Test
Check if sub-problem can be conquered in one of the 3 ways
Discard conquered sub-problems

⇓

Optimality-Test
Stop if no more unconquered sub-problems, if incumbent solution exists then is optimal
Otherwise go to Branch step

Goal Programming

Given a MOO problem, the goal programming approach consists in transforming the multi-objective formulation to a single-objective model by setting goals for each objective and minimising the deviation from these goals.

1. Establish a goal or target value for each $f_i(x)$ of the k objectives in the problem
2. Convert the objective expressions to constraints using the goal for each objective
3. Introduce one deficiency variable d_i for each of the k objectives to model the deviation from the goal
4. Minimise the sum of deviation variables and solve as a single-objective problem

Example 7.6 Goal programming for a MOO problem.

Maximise $Z_1 = 3x_1 + x_2$

Minimise $Z_2 = x_1 - x_2$

Subject to $x_1 + x_2 \leq 4$

$x_1 \leq 3$

$x_2 \leq 3$

$x_1, x_2 \geq 0$

$\Rightarrow$ Set goals

$\Omega_1 = 8$

$\Omega_2 = 0$

$\Rightarrow$

Subject to $3x_1 + x_2 \geq 8$

$x_1 - x_2 \leq 0$

$x_1 + x_2 \leq 4$

$x_1 \leq 3$

$x_2 \leq 3$

$x_1, x_2 \geq 0$

$\Downarrow$

Introduce deficiency variables d_1 and d_2

$\Leftarrow$

Minimise $Z = d_1 + d_2$

Subject to $3x_1 + x_2 + d_1 \geq 8$

$x_1 - x_2 - d_2 \leq 0$

$x_1 + x_2 \leq 4$

$x_1 \leq 3$

$x_2 \leq 3$

$x_1, x_2, d_1, d_2 \geq 0$

Note: the deficiency variables are introduced with positive or negative sign (or both) depending on the inequality or equality in the constraint.

Basics of Nonlinear Programming

Nonlinear functions might have:

- terms in which the decision variables have exponents $\neq 1$
- terms involving multiplications between the decision variables
- terms with other general transformations of the decision variables

Some examples:

$$f(x) = 10x - x^2$$

$$f(x_1, x_2) = x_1 x_2 - 3x_1^2 - x_2^2$$

$$f(x) = \ln x - 2x$$

$$f(x) = 8x + e^x$$

$$f(x) = \sin(x_1 + x_2) + 5x_1 - x_2^3$$

$$f(x) = 0.5x^5 - 6x^4 - 12x^2 + 10x$$

The nonlinear programming (NLP) problem is then to:

Find $x = (x_1, x_2, \cdots, x_n)$

to maximise $f(x)$

subject to $g_i(x) \leq b_i$ for $i = 1, \ldots, m$

and $x \geq 0$

$f(x)$ and $g_i(x)$ are not restricted to be linear

When the objective function or the constraints involve nonlinear terms, there is no longer a guarantee that optimal solutions will be located in corner points or in the boundary of the bounded feasible region.

In nonlinear programming problems, optimal solutions can occur in the interior of the feasible region.

Local Optima and Global Optima

Depending on its shape, a nonlinear function can have a <u>single optimum or multiple optima</u>.

Local optima may occur at a point in which the slope of the function is zero or at a boundary of the feasible region.