LECTURE 3 – MATHEMATICAL PRELIMINARIES PART 1

Overview

- Sets, Relations and Functions
- Strings and Languages

Sets, Relations and Functions

A set is a collection of objects with no repetition. The simplest way to describe a set is by listing its elements. If a set is described using a defining property, the description should clearly specify the objects and the universe of discourse:

$$A = \{x \mid x \in \aleph \wedge x < 10\}$$

Important notation for sets:

$\varnothing$, $a \in A$, $a \notin A$

$A \subset B$, $A \not\subset B$, $A \subseteq B$, $A = B$,

$A \cup B$, $A \cap B$, $A \setminus B$, $A \times B$

$\wp(A)$ or 2^A, A' or A^c , $\prod A$

For all sets A, B, and C in the universe U the following set properties hold:

Associative law: $(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Commutative law: $A \cup B = B \cup A$

$A \cap B = B \cap A$

Complement law: $A \cup A^c = U$

$A \cap A^c = \varnothing$

Idempotency law: $A \cap A = A$

$A \cup A = A$

Identity law: $A \cup \varnothing = A$

$A \cap U = A$

Zero law: $A \cup U = U$

$A \cap \varnothing = \varnothing$

Involution law: $(A^c)^c = A$

De Morgan's law: $(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

Distributive law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

A set |A| is said to be finite if A contains a finite number of elements.

A set |A| is said to be infinite if A contains an infinite number of elements.

The set A is said to be countable or enumerable is there is a way to list of the elements of A. More formally, A set A is enumerable or countable is A is finite or if there is a bijection $f:A \rightarrow \aleph^+$.

Example. The following sets are countable:

$A = \{x \mid y \in \aleph^+ \wedge x = 2 \cdot y+1\} = \{3, 5, 7, 9, 11, 13, 15,\ldots\}$

$B = \{(x,y) \mid x,y \in A\} = \{(3,3),(3,5),(5,3),(3,7),(5,5),(7,3),(3,9),(5,7),(7,5)\ldots\}$

A relation R is a subset of $A \times B$ where A is the domain and B is the range of R.

If the domain is the same as the range, A = B, then R is a relation on the set A.

Properties of relations:

R is reflexive if aRa $\forall a \in A$

R is symmetric if aRb implies bRa

R is transitive is aRb and bRc implies aRc

R is antisymmetric if aRb and bRa implies a = b

R is an equivalence relation if R is reflexive, symmetric and transitive

An equivalence class is $\{b \in A \mid aRb\}$ where R is an equivalence relation on the set A and $a \in A$

If R is a reflexive relation on A, then the reflexive closure of R is the smallest reflexive relation on A with R as a subset.

If R is a symmetric relation on A, then the symmetric closure of R is the smallest symmetric relation on A with R as a subset.

If R is a transitive relation on A, then the transitive closure of R is the smallest transitive relation on A with R as a subset.

The relation R on A is an ordering relation if R is reflexive, antisymmetric and transitive

A function f from the set A to the set B, denoted $f:A \rightarrow B$, associates with each element $a \in A$ a unique element $b \in B$ denoted by f(a).

A is the domain of f and B is the codomain of f.

The set $\{f(a) \mid a \in A\}$ is the range of f and is a subset of the codomain of f.

A function $f:A \rightarrow B$ is total if f(a) is defined for all $a \in A$, otherwise f is partial.

A function $f:A \rightarrow B$ is onto if for each $b \in B$ there is an $a \in A$ such that f(a)=b.

A function $f:A \rightarrow B$ is one-to-one if for each $b \in B$ there is $\exists! a \in A$ such that f(a)=b.

A function $f:A \rightarrow B$ is bijective if f is both onto and one-to-one.

Exercise 3.1

1. Determine the cardinality of the power set $\wp(A)$ where $A=B\times C$, $B=\{1,2\}$ and $C=\{a,b\}$. What is the cardinality of the largest element of $\wp(A)$?

2. Determine the properties for the following relations:

i) $<$ on the set of real numbers

ii) $\geq$ on the set of integer numbers

iii) "is a relative of" over the set of persons living in the UK

3. Determine the properties of function $f:\Re x\Re\rightarrow\Re x\Re$ such that $f(x,y)=(x+y,x-y)$

Strings and Languages

A string is a finite sequence of symbols $a_1a_2a_3...a_n$ where each a_i is an element of the alphabet Σ.

The length of a string x denoted $|x|$ is the number of symbols in x.

The empty string has no symbols and is denoted by ε and $|\varepsilon|=0$.

Σ^* is the set of all strings of finite length over Σ and $\varepsilon \in \Sigma^*$ for all alphabets.

For two strings $x,y \in \Sigma^*$ where $x=a_1a_2\ldots a_m$ and $y=b_1b_2\ldots b_n$ the concatenation of x and y is given by the $xy= a_1a_2\ldots a_mb_1b_2\ldots b_n$.

For a string $z \in \Sigma^*$ of the form $z=xy$ where $x,y \in \Sigma^*$, x is a prefix of y and y is a postfix of x.

For a string $z \in \Sigma^*$ of the form $z=xwy$ where $x,w,y \in \Sigma^*$, w is a substring of z.

A formal language L over the alphabet Σ is any subset of Σ^*.

The concatenation of two languages L_1 and L_2 is given by $L_1L_2=\{xy \mid x\in L_1, y\in L_2\}$.

Exercise 3.2

1. Let $\Sigma=\{0,1\}$ and $L=\{x\in\Sigma^* \text{ and } |x|=4\}$. R is a relation over L where xRy if the two first bits of x are the same as the two first bits of y. Determine if R is an equivalence relation.

2. How many prefixes, postfixes and substrings exist in a string on length n?

3. Let $\Sigma=\{a,b,c\}$ and $L=\{w \mid www=uu \text{ for some u in en } \Sigma^*\}$. Give an example of a string in L.

Reading: (Lewis&Papadimitriou, chapter 1) and (Wood, chapter 0).