LECTURE 4 – MATHEMATICAL PRELIMINARIES PART 2

Overview

- Graphs and Trees
- Induction and Recursion

Graphs and Trees

A graph is a pair (V,E) where V is a set of vertices or nodes and E is a set of edges or arcs.

A graph is complete if there is exactly one edge between each pair of distinct vertices $v_1, v_2 \in V$.

A path in a graph is a sequence of edges connecting two vertices.

A circuit is a path with the same start and end vertices.

A graph is connected if for any two vertices v_1, v_2 there is a path connecting v_1 and v_2.

A connected graph which has no circuits is called a tree.

A directed graph or digraph is a graph in which the edges are ordered pairs while if the edges are unordered pairs the graph is said to be undirected.

A subgraph of a graph G=(V,E) is any graph G'=(V',E') where $V' \subseteq V$ and $E' \subseteq E$.

A graph G=(V,E) where $V=\{v_1, v_2, v_3, \ldots, v_n\}$ can be represented by the adjacency matrix M_G which is an $n \times n$ Boolean matrix defined as follows:

$M_G[i,j]$= true if $(v_i, v_j) \in G$
false otherwise

If the graph is undirected then $(v_i,v_j)=(v_j,v_i)$ in M_G, while if the graph is directed (v_i,v_j) and (v_j,v_i) may have different value in M_G.

For a digraph G=(V,E) where $v \in V$, the number or edges from v is denoted by the outdegree(v) and the number of edges into v is denoted by indegree(v).

A labelled graph has an associated label or weight to each edge or arc.

A labelled digraph G=(V,E) where $V=\{v_1,v_2,v_3,\ldots,v_n\}$ and a label $w(v_i,v_j)$ for each (v_i,v_j) can be represented by a labelling matrix defined as follows:

$$M^w_G[i,j] = w(v_i,v_j) \text{ if } (v_i,v_j) \exists E$$
$$r \text{ otherwise, where r is any symbol not a label}$$

Induction and Recursion

Mathematical induction is a technique to prove a countable number of theorems $\{T_n \mid n \in \aleph\}$ and it follows the following format:

1. Show that the result holds for n=1, that is, show that T_1 is true.
2. Show that for any $k \in \aleph$, if T_n is true for all n<k then T_k is also true.

Example. Prove by induction that "$8^n - 3^n$ is divisible by 5" for all $n \in \aleph^+$.

Base step: "$8^n - 3^n$ is divisible by 5" holds for n=1 because $8^1 - 3^1 = 5$

Induction hypothesis: Assume that $8^n - 3^n$ is divisible by 5 if n<k

Induction step: If $8^n - 3^n$ is divisible by 5 then $8^{n+1} - 3^{n+1}$ is also divisible by 5

$$8^{n+1} - 3^{n+1} = 8 \times 8^n - 3 \times 3^n$$
$$= 5 \times 8^n + 3\,(8^n - 3^n)$$

since $8^n - 3^n$ is divisible by 5 (induction hypothesis), $3\,(8^n - 3^n)$ is also divisible by 5, and 5×8^n is also divisible by 5, then it follows that $8^{n+1} - 3^{n+1}$ is also divisible by 5.

A function f is a recursive function if f is defined with an expression involving f.

The typical example of a recursive function is the factorial:

$$fact(n) = 1 \text{ if } n = 1, n \times fact(n) \text{ if } n > 1$$

An inductive definition is based on a recursive function. For example, the length of strings is given by the following inductive definition:

Base: $|\varepsilon|=0$

Induction: If $a \in \sum$ and $x \in \sum^*$ then $|xa|=|x|+1$

Exercise 4.1

1. Consider the following recursive definition for the set A. The set A includes the initial elements 5 and 7. If a and b are in A then a+b and a – b are in A. No other element is in A. What are the elements of A?

2. Prove by mathematical induction that $1+2+3+...+n = n(n+1)/2$

3. Give an inductive definition for the inverse of a string w given by INV(w) and show by mathematical induction that |INV(w)|=|w|.

4. Show by mathematical induction that $\sum_{i=0}^{n}(3i-1)=\frac{n(2n-1)}{2}$

5. Show by mathematical induction that $\sum_{i=1}^{n}(i)(i+1)(i+2)=\frac{n(n+1)(n+2)(n+3)}{4}$

Reading: (Lewis&Papadimitriou, chapter 1) and (Wood, chapter 0).