

Exercises in Quantifier Manipulation

Note Title

26/03/2006

Given a bag of three kinds of objects, the number of objects is reduced by removing two objects of different kind and replacing them by one object of the third kind.

When **can** repeated application of this process reduce the number of objects to exactly one?
If so, **how**?

Suppose $\{n_k \mid 0 \leq k < M\}$ is a bag of integers.

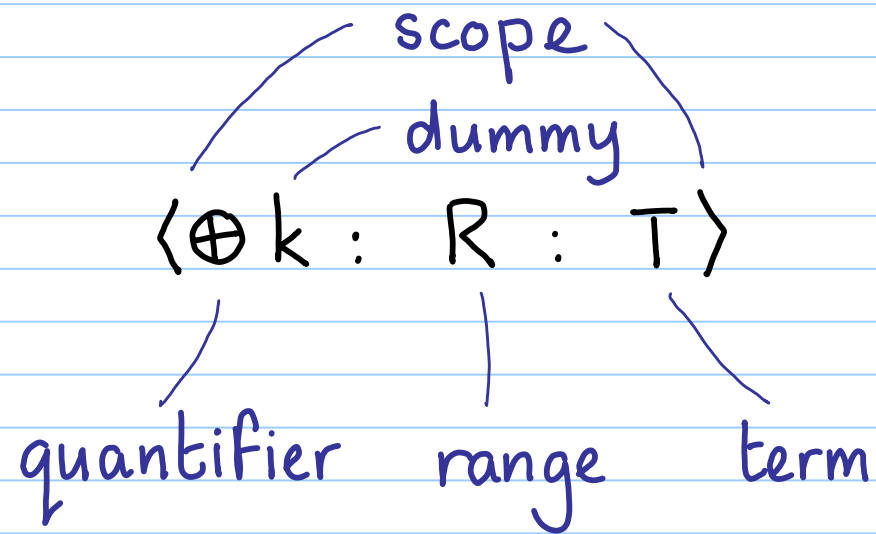
Consider the bag of differences

$$\{n_j - n_k \mid j < k\}.$$

Evaluate: the number of even differences is even.

I.e. simplify

$$\text{even.} \langle \sum_{j, k: j < k \wedge \text{even.}(n_j - n_k) : 1 \rangle.$$



Eg.

$$\langle \sum k : 0 \leq k < N : k^2 \rangle$$

$$\langle \forall k : \text{even}.k : \text{even}.(k^2 + N) \rangle$$

$$\langle \exists k : 0 \leq k < N \wedge \text{even}.k : 3 \nmid k \rangle$$

Theorem. The sum of a bag of numbers is even iff the number of odd numbers in the bag is even.

Distributivity

Trading

$$\langle \oplus k : R \wedge S : T \rangle = \langle \oplus k : R : \text{if } S \rightarrow T \square \neg S \rightarrow 1_{\oplus} \rangle$$

$$\langle \forall k : R \wedge S : T \rangle = \langle \forall k : R : S \Rightarrow T \rangle$$

$$\langle \exists k : R \wedge S : T \rangle = \langle \exists k : R : S \wedge T \rangle$$

$$\langle \equiv k : R \wedge S : T \rangle = \langle \equiv k : R : S \Rightarrow T \rangle$$

$$\langle \neq k : R \wedge S : T \rangle = \langle \neq k : R : S \wedge T \rangle .$$

j, k range over $[0..M)$

$$\text{even.} \langle \sum_{j, k : j < k \wedge \text{even.}(n_j - n_k) : 1 \rangle$$

j, k range over $[0..M)$

$$\langle \sum_{j, k: j < k} n_j + n_k \rangle$$

=

$$= \text{even.} \langle \sum_{j,k: j < k \wedge \text{even.}(n_j - n_k) : 1 \rangle$$

{ above }

$$\text{even.}((M-1) \times \sum n) \equiv \text{even.}(\frac{1}{2} \times M \times (M-1))$$

$$= \text{{ distributivity, case analysis on even.M }}$$

$$\text{odd.M} \vee \langle \equiv j :: \text{even.}n_j \rangle \equiv \text{even.} \lfloor \frac{M}{2} \rfloor$$

$$= \text{{ calculus }}$$

$$\text{if odd.M} \rightarrow \text{even.} \lfloor \frac{M}{2} \rfloor$$

$$\square \text{ even.M} \rightarrow \langle \equiv j :: \text{even.}n_j \rangle \equiv \text{even.} \lfloor \frac{M}{2} \rfloor$$

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