Note Title 26/03/2006

Exercises in Quantifier Manipulation

Given a bag of three kinds of objects, the number of objects is reduced by removing two objects of different kind and replacing them by one object of the third kind.

When can repeated application of this process reduce the number of objects to exactly one? If so, how?

Suppose $\{ n_k \mid 0 \leq k < M \}$ is a bag of integers.

Consider the bag of differences

$$\{ n_j - n_k \mid j < k \} \quad .$$

Evaluate: the number of even differences is even.

I.e. simplify

$$\boxed{\mathit{even}.\langle \Sigma j,k : \; j < k \wedge \mathit{even}.(n_j - n_k) \; : \; 1 \rangle} \quad .$$

$$\overbrace{\langle \underset{\text{quantifier}}{\oplus} \overset{\text{dummy}}{k} : \underset{\text{range}}{R} : \underset{\text{term}}{T} \rangle}^{\text{scope}}$$

Eg.

$$\langle \Sigma k : 0 \leq k < N : k^2 \rangle$$

$$\langle \forall k : even.k : even.(k^2 + N) \rangle$$

$$\langle \equiv k : 0 \leq k < N \wedge even.k : 3 \backslash k \rangle$$

Theorem. The sum of a bag of numbers is even iff the number of odd numbers in the bag is even.

Distributivity

Trading

$$\langle \oplus k : R \wedge S : T \rangle \;=\; \langle \oplus k : R : \text{if } S \to T \;\square\; \neg S \to 1_{\oplus} \rangle$$

$$\langle \forall k : R \wedge S : T \rangle \;=\; \langle \forall k : R : S \Rightarrow T \rangle$$

$$\langle \exists k : R \wedge S : T \rangle \;=\; \langle \exists k : R : S \wedge T \rangle$$

$$\langle \equiv k : R \wedge S : T \rangle \;=\; \langle \equiv k : R : S \Rightarrow T \rangle$$

$$\langle \not\equiv k : R \wedge S : T \rangle \;=\; \langle \not\equiv k : R : S \wedge T \rangle \; .$$

j,k range over $[0..M)$

$$even.\langle \Sigma j,k : \quad j<k \ \wedge\ even.(n_j - n_k) \ : \quad 1 \rangle$$

j, k range over $[0..M)$

$$\langle \Sigma j,k : j<k : n_j + n_k \rangle$$

$=$

$$
\begin{aligned}
& \quad even.\langle \Sigma j,k : \ j<k \wedge even.(n_j - n_k) : \ 1\rangle \\
= & \qquad \{ \text{above} \} \\
& \quad even.((M-1) \times \Sigma n) \ \equiv \ even.(\tfrac{1}{2} \times M \times (M-1)) \\
= & \qquad \{ \text{distributivity, case analysis on } even.M \} \\
& \quad odd.M \vee \langle \equiv j :: even.n_j \rangle \ \equiv \ even.\lfloor \tfrac{M}{2} \rfloor \\
= & \qquad \{ \text{calculus} \} \\
& \quad \textbf{if} \ odd.M \ \rightarrow \ even.\lfloor \tfrac{M}{2} \rfloor \\
& \quad \square \ even.M \ \rightarrow \ \langle \equiv j :: even.n_j \rangle \equiv even.\lfloor \tfrac{M}{2} \rfloor \\
& \quad \textbf{fi}
\end{aligned}
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