Note Title 11/12/2006

Enumerating the Rationals

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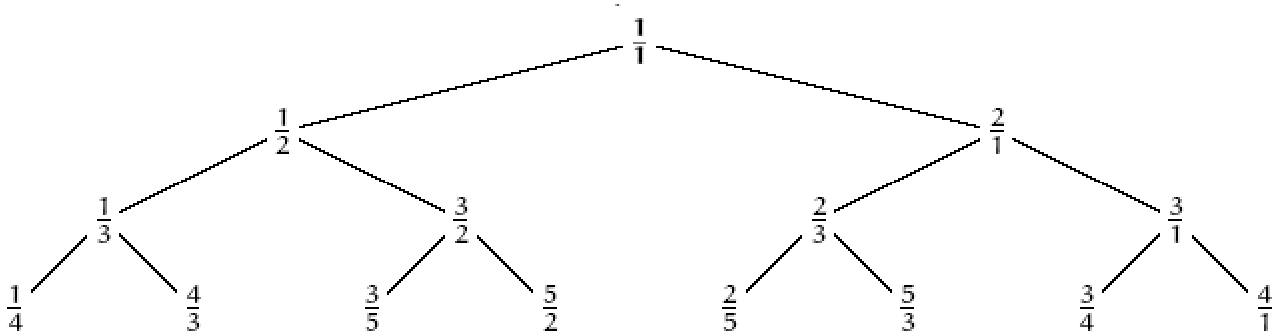
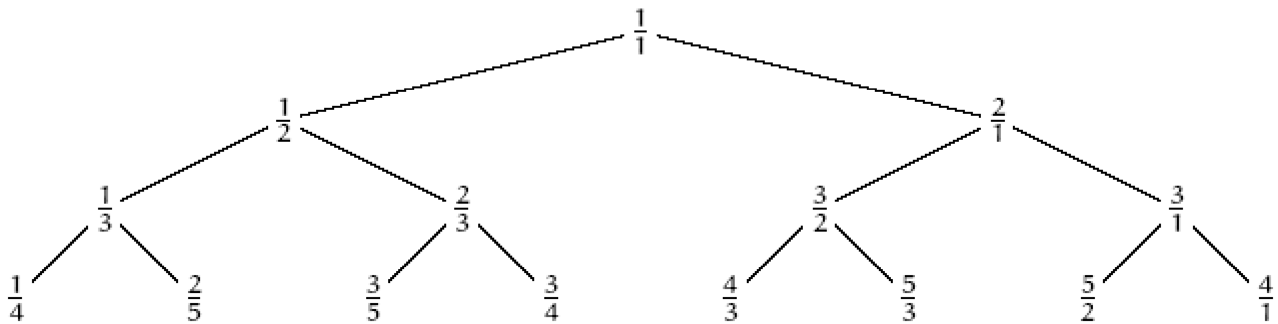




Definition The divides relation, denoted by $\backslash$, is a binary relation on integers defined by

$$[\, m \backslash n \;\equiv\; \langle \exists k :: k \times m = n \rangle \,] \qquad \square$$

Properties

$\backslash$ is a partial ordering on the natural numbers. (i.e. reflexive, transitive and anti-symmetric).

$[\, 1 \backslash m \,]$ (1 is the *least* element in the ordering)

$[\, m \backslash 0 \,]$ (0 is the *greatest* element in the ordering)

$$[\, k \backslash m \wedge k \backslash n \;\equiv\; k \backslash (m-n) \wedge k \backslash n \,] \qquad \square$$

Definition The greatest common divisor of natural numbers m and n is a solution of the equation

$$x :: \langle \forall k :: k \backslash m \wedge k \backslash n \equiv k \backslash x \rangle \; . \qquad \Box$$

Aside If $\precsim$ is a partial ordering, the *greatest lower bound (infimum)* of m and n is a solution of the equation

$$x :: \langle \forall k :: k \precsim m \wedge k \precsim n \equiv k \precsim x \rangle \; .$$

Eg. the minimum of numbers m and n is a solution of

$$x :: \langle \forall k :: k \leq m \wedge k \leq n \equiv k \leq x \rangle \; .$$

Greatest lower bounds need not exist. Eg. equality is a partial ordering, but the equation

$$x :: \langle \forall k :: k = m \wedge k = n \equiv k = x \rangle$$

has no solution when $m \neq n$. $\Box$

Definition The greatest common divisor of natural numbers m and n is a solution of the equation

$$x :: \langle \forall k :: k \backslash m \wedge k \backslash n \equiv k \backslash x \rangle \ . \qquad \square$$

Observe:

$$\langle \forall k :: k \backslash m \wedge k \backslash m \equiv k \backslash m \rangle$$

$$\langle \forall k :: k \backslash m \wedge k \backslash 0 \equiv k \backslash m \rangle$$

(m solves the equation when $m = n$ or $0 = n$).

Euclid's Algorithm

Replacing specification by

$x,y ::\quad x=y \wedge \langle\forall k :: k\backslash m \wedge k\backslash n \equiv k\backslash x \wedge k\backslash y\rangle$

suggests invariant in Euclid's Algorithm :

$\{0<m \wedge 0<n\}$

$x,y := m,n$

$;\ \{\text{Invariant: } \langle\forall k :: k\backslash m \wedge k\backslash n \equiv k\backslash x \wedge k\backslash y\rangle \wedge 0<m \wedge 0<n$

$\quad \text{Bound: } x+y \}$

$\text{do}\quad x<y \rightarrow y := y-x$

$\ \square\quad y<x \rightarrow x := x-y$

od

$\{x=y \wedge \langle\forall k :: k\backslash m \wedge k\backslash n \equiv k\backslash x \wedge k\backslash y\rangle\}$

Properties Euclid's algorithm shows, constructively, that at least one solution of equation

$$x :: \langle \forall k :: k \backslash m \wedge k \backslash n \equiv k \backslash x \rangle$$

exists when $0 < m$ and $0 < n$.
Earlier we observed solutions when $0 = m$ or $0 = n$.
It is easy to show —exercise— that, if a solution exists, it is unique. $\square$

Conclusion: There is a binary function on natural numbers, which we will denote by the infix operator ∇, such that

$$[\ k \backslash m \wedge k \backslash n \equiv k \backslash m \nabla n\ ] \quad . \qquad \square$$

Theorem $m \nabla n$ is a linear combination of m and n.

Proof $m \nabla 0 = m = m \times 1 + 0 \times 1$.

$\{ 0 < m \wedge 0 < n \}$

$x, y := m, n$; $C := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

; $\{$ Invariant: $(x, y) = (m, n) \times C$

where $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $\}$

do $x < y \rightarrow (x, y) := (x, y) \times A$; $C := C \times A$

$\square$ $y < x \rightarrow (x, y) := (x, y) \times B$; $C := C \times B$

od

$\{ x = y = m \nabla n \wedge (x, y) = (m, n) \times C \}$ $\square$

$$\{\ (x,y) = (m,n) \times C\ \}$$

$$(x,y) := (x,y) \times A\ ; \quad C := C \times A$$

$$\{\ (x,y) = (m,n) \times C\ \}$$

Verification Condition

$$[\ (x,y) = (m,n) \times C$$

$$\Rightarrow \quad (x,y) \times A = (m,n) \times (C \times A)\ ]$$

$$(x,y) \times A \;=\; (m,n) \times (C \times A)$$

$=$ { matrix multiplication is associative }

$$(x,y) \times A \;=\; ((m,n) \times C) \times A$$

$\Leftarrow$ { Leibniz }

$$(x,y) \;=\; (m,n) \times C \;.$$

$$\{\, 0 < m \wedge 0 < n \,\}$$

$$x, y := m, n \;;\quad C := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$;\ \{\, \text{Invariant:} \quad (x, y) = (m, n) \times C$$

$$\text{where} \quad A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \ \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \,\}$$

$$\begin{array}{lll} \mathbf{do} & x < y \to & (x, y) := (x, y) \times A \;;\quad C := C \times A \\ \square & y < x \to & (x, y) := (x, y) \times B \;;\quad C := C \times B \\ \mathbf{od} & & \end{array}$$

$$\{\, x = y = m \nabla n \ \wedge \ (x, y) = (m, n) \times C \,\}$$

$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A^{-1}$, $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = B^{-1}$

$\{ 0 < m \wedge 0 < n \}$

$x, y := m, n$; $D := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

; { Invariant: $(x, y) \times D = (m, n)$ }

do $x < y \rightarrow (x, y) := (x, y) \times A$; $D := R \times D$

$\square$ $y < x \rightarrow (x, y) := (x, y) \times B$; $D := L \times D$

od

$\{ (1, 1) \times D = \left(\frac{m}{m \nabla n}, \frac{n}{m \nabla n} \right) \}$

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A^{-1}, \quad L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = B^{-1}$$

$\{ 0 < m \wedge 0 < n \}$

$x, y := m, n \;;\; E := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$;\ \{ \text{Invariant: } \quad E \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} \}$

do

$\quad x < y \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} := B \times \begin{pmatrix} x \\ y \end{pmatrix} \;;\; E := E \times L$

$\square \quad y < x \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} := A \times \begin{pmatrix} x \\ y \end{pmatrix} \;;\; E := E \times R$

od

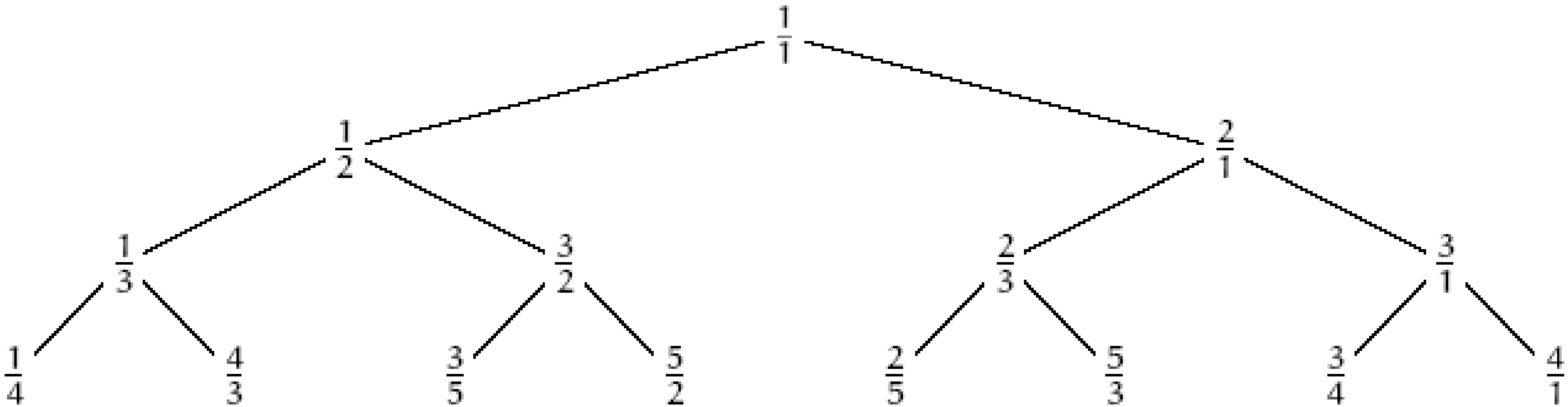
$\{ \quad E \times \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \left(\frac{m}{m \nabla n}, \frac{n}{m \nabla n} \right) \quad \}$

Tree of Matrices

$$
\begin{tikzcd}
 & & & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \arrow[dll, dash, "L"'] \arrow[drr, dash, "R"] & & & \\
 & \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \arrow[dl, dash, "L"'] \arrow[dr, dash, "R"] & & & & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \arrow[dl, dash, "L"'] \arrow[dr, dash, "R"] & \\
\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} & & \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} & & \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} & & \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}
\end{tikzcd}
$$

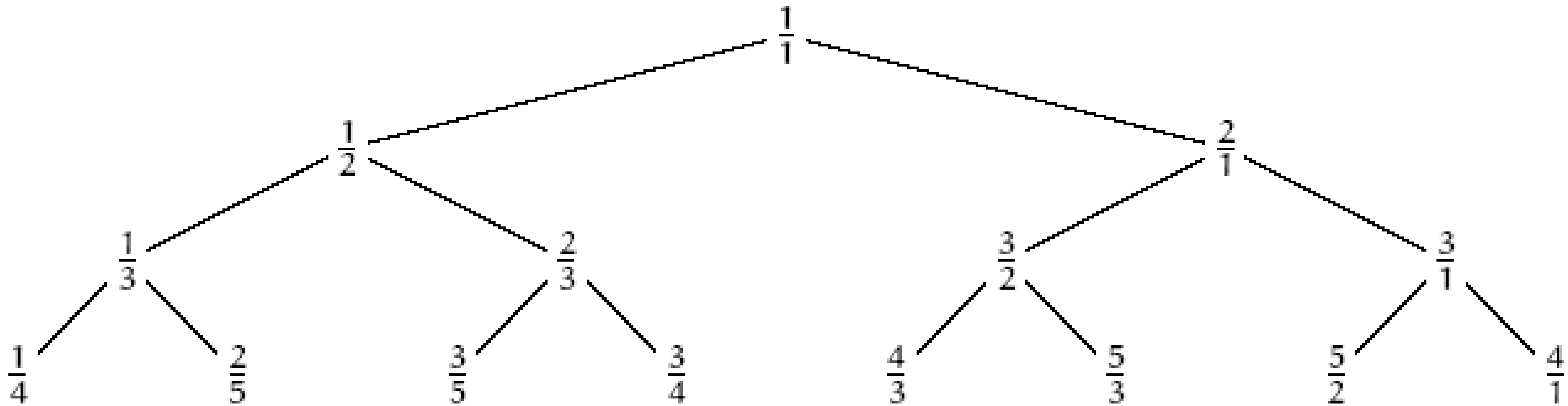
$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Calkin-Wilf Tree



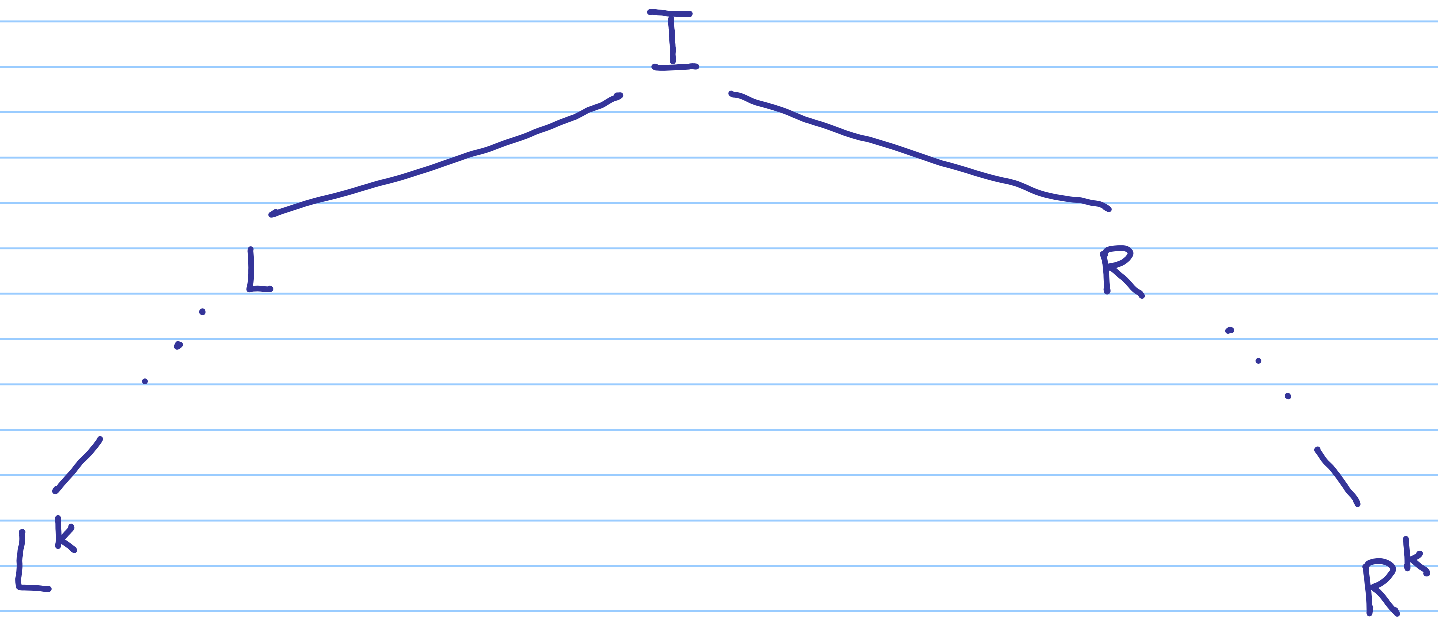
Premultiply by (1,1) .

Stern-Brocot Tree



Postmultiply by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Enumerating the Matrices



Given D, which is a k-fold product of Ls and Rs, and is not R^k, how does one determine j such that

$$\langle \exists T :: D = T \times L \times R^j \rangle \quad ?$$

Answer: revisit Euclid's algorithm.

$$
\begin{array}{lllll}
\text{do} & x < y \rightarrow & (x,y) := (x,y) \times A \; ; & D := R \times D \\
\square & y < x \rightarrow & (x,y) := (x,y) \times B \; ; & D := L \times D \\
\text{od} & & &
\end{array}
$$

$D := I\,;$

do

$$D_{10} = 0 \;\rightarrow\; D := \begin{bmatrix} 1 & 0 \\ D_{01}+1 & 1 \end{bmatrix}$$

$\square$

$$D_{10} \neq 0 \;\rightarrow\; j := \left\lfloor \frac{D_{01} + D_{11} - 1}{D_{00} + D_{10}} \right\rfloor$$

$$;\; D := D \times \begin{bmatrix} 2j+1 & 1 \\ -1 & 0 \end{bmatrix}$$

od

Calkin-Wilf — premultiply by $\begin{pmatrix} 1 & 1 \end{pmatrix}$
Stern-Brocot — postmultiply by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.