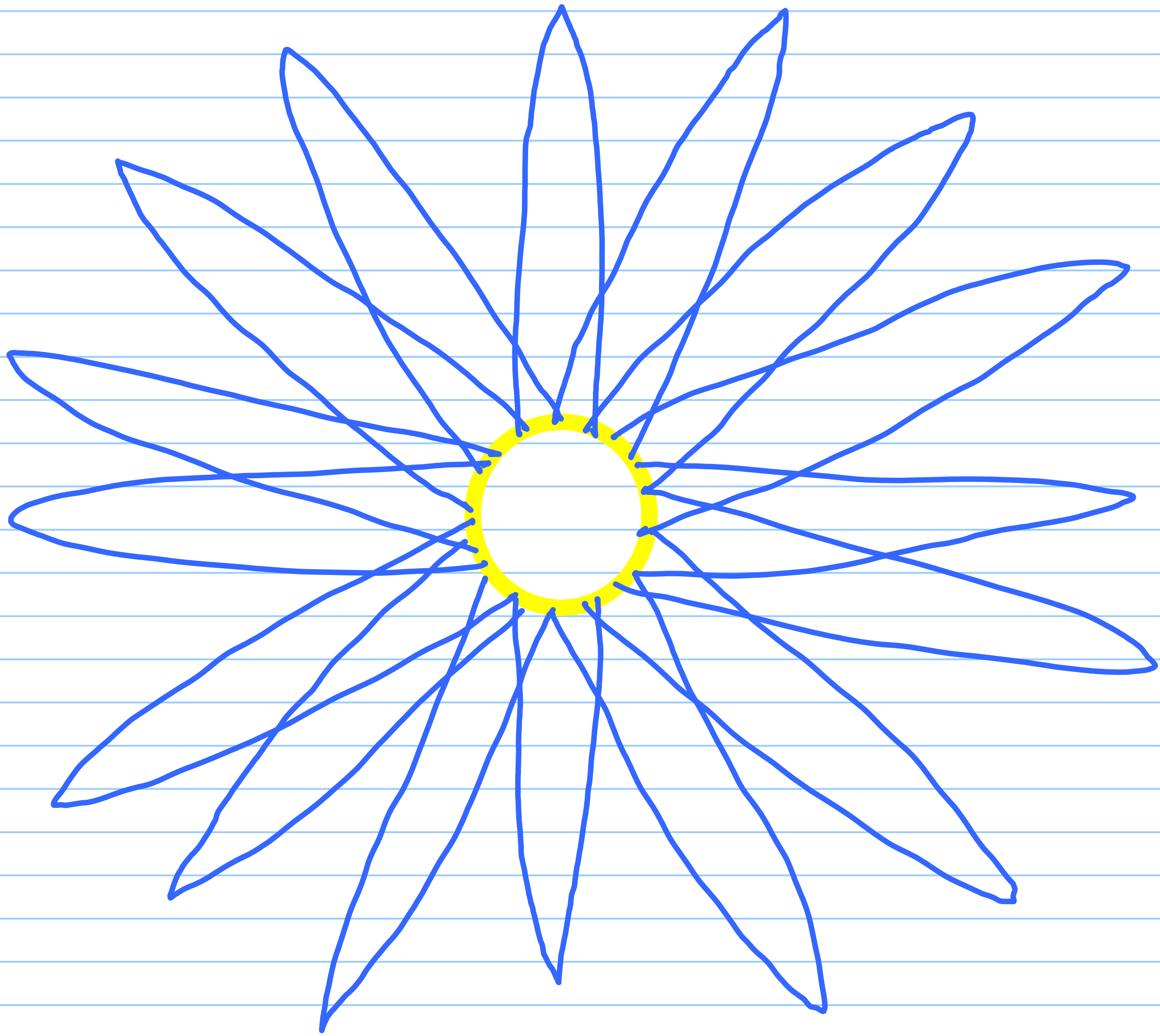
Combinatorial Games

Note Title 19/04/2007

Impartial, 2-person games with complete information.

- 2 players take turns to move
- valid moves are the same for both players
- the full state of the game is known to both
- play is guaranteed to terminate
- play ends when no move is possible, loser is person whose turn it is,



Move

Remove one or two adjacent petals

Strategy

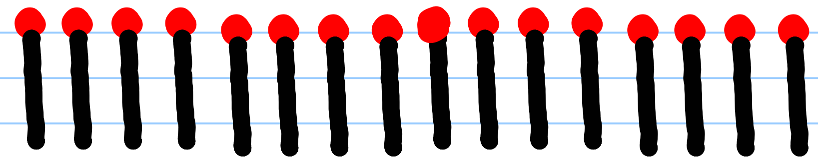
Maintain symmetry

How to win.

Identify a property of positions (the *strategy invariant*) such that

- all end positions satisfy the property.
- *every* move from a position satisfying the property *falsifies* the property.
- for every position that does not satisfy the property *there is* a move that *truthifies* the property.

Matchstick Game



Move: remove at least 1 and at most M matches (where M is fixed in advance).

Strategy: truthify $m \bmod (M+1) = 0$ where m is the number of matches.

Summary. Single-pile matchstick games.

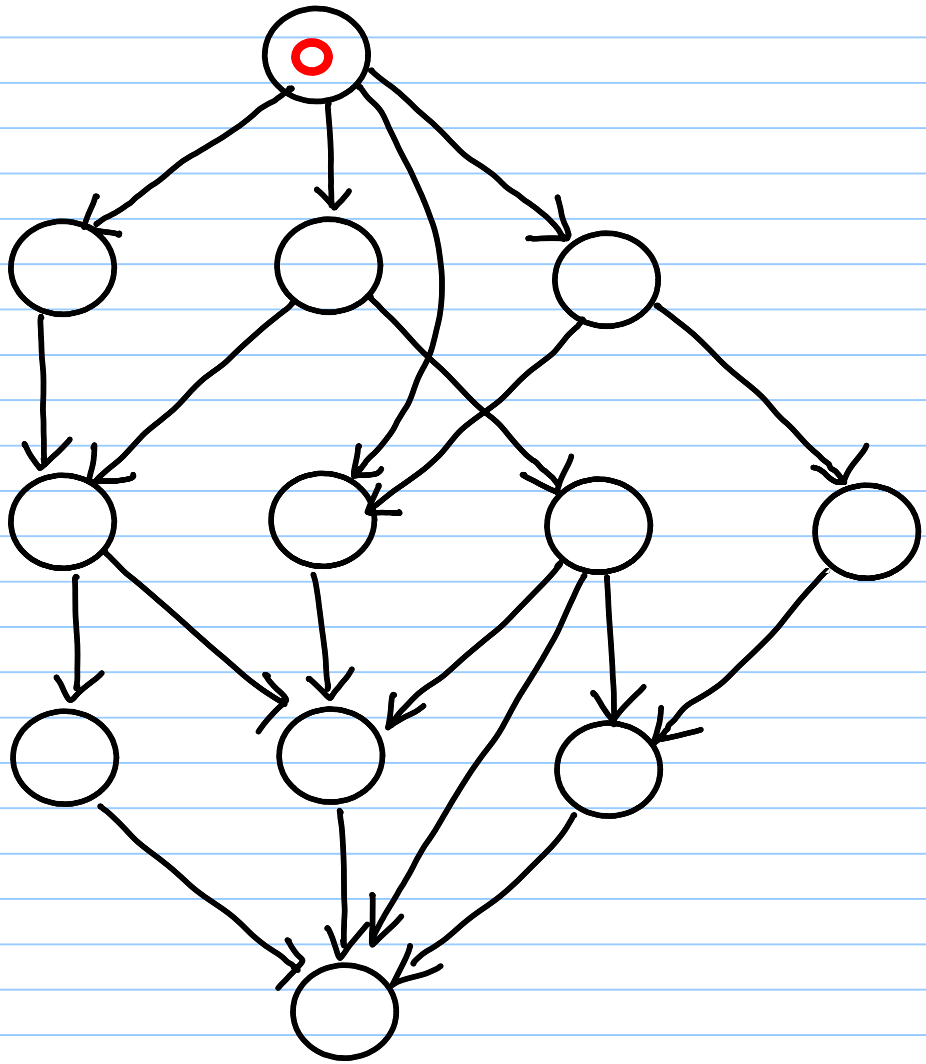
Moves	Losing Position	Strategy
1 or 2 matches	$m \bmod 3 = 0$	take $m \bmod 3$ matches
1,2 or 3 matches	$m \bmod 4 = 0$	—"— $m \bmod 4$ —"—
1 thru M matches	$m \bmod (M+1) = 0$	—"— $m \bmod (M+1)$ —"—

Key:

m   no. of matches

mod   remainder after dividing

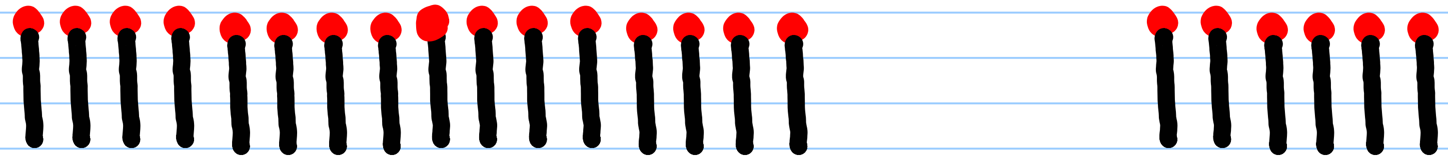
A Game



Positions are the nodes of the graph.

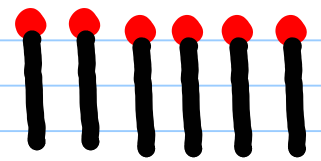
Moves are directed edges.

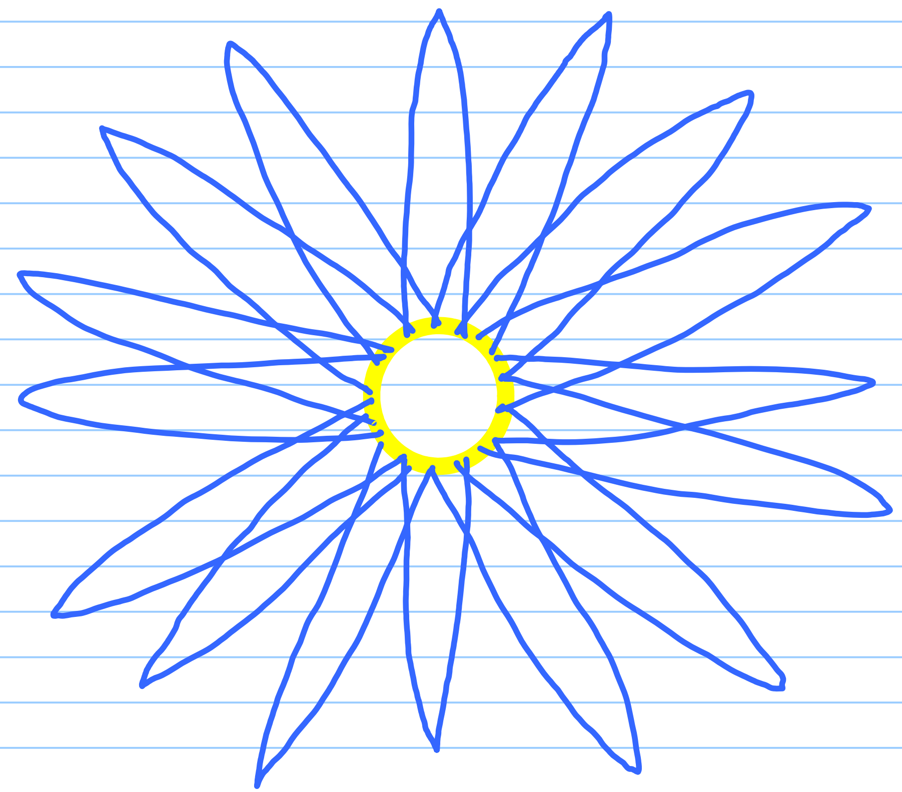
Game Sum



A game with two piles of matches is the "sum" of two single-pile matchstick games.

Another "sum" of two games:





Move: choose either remove some matches

or remove a petal

The *sum* of two games is a game defined as follows:

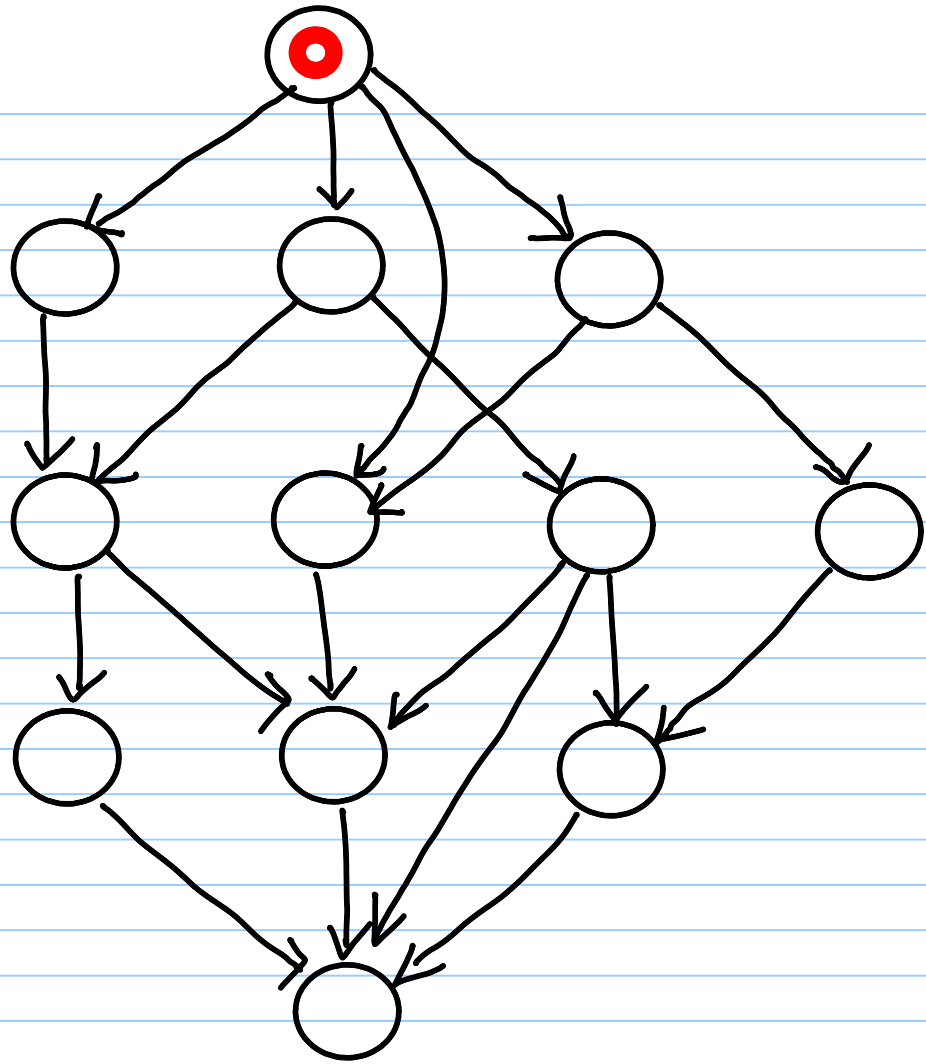
Call the two games the *left* game and the *right* game.

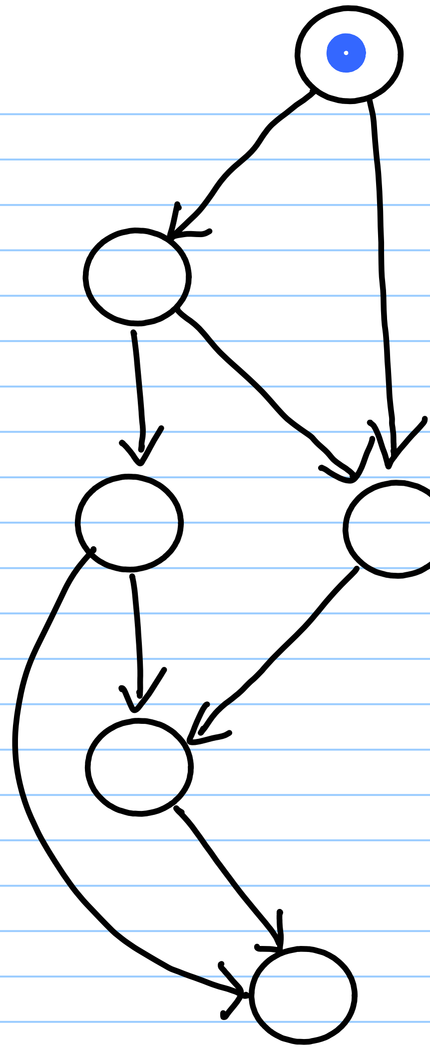
A *position* in the sum game is an ordered pair (l, r) of positions, where l is a position in the left game and r is a position in the right game.

A *move* $(l, r) \mapsto (l', r')$ in the sum game satisfies

either: $l \mapsto l'$ is a move in the left game, and $r = r'$,

or : $l = l'$ and $r \mapsto r'$ is a move in the right game.





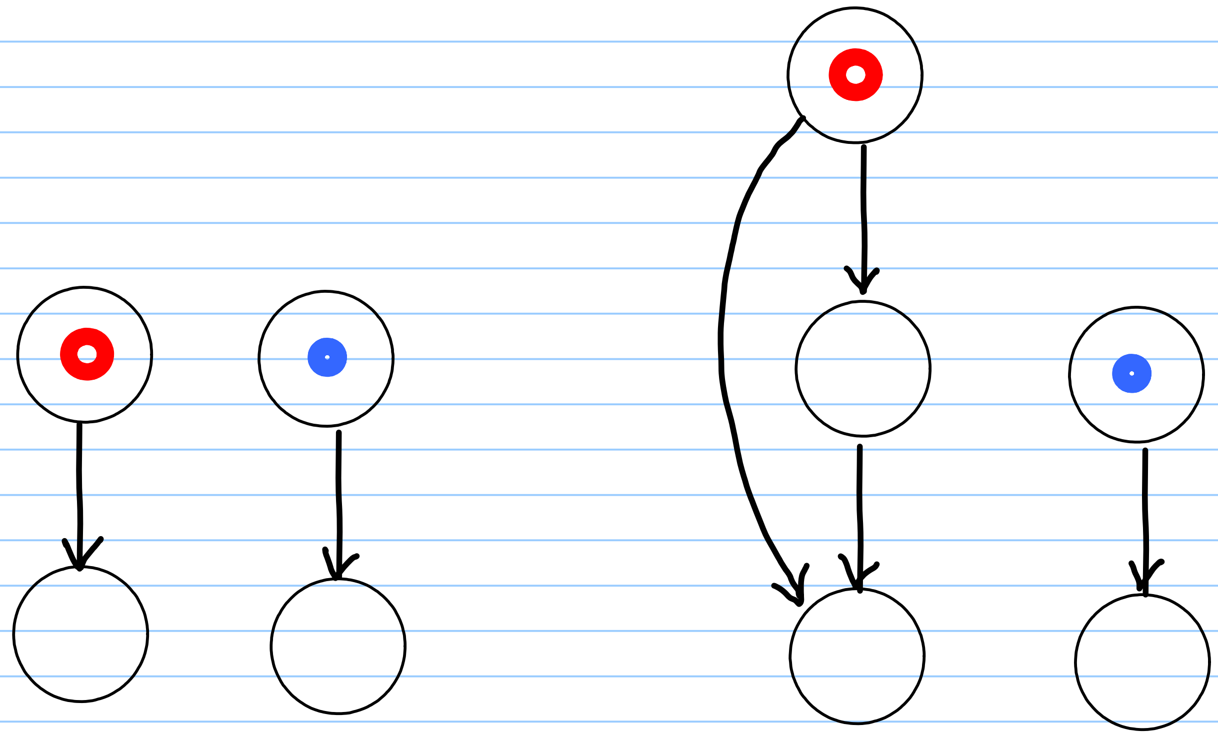
How to win.

Identify a property of positions (the *strategy invariant*) such that

- all end positions satisfy the property.
- *every* move from a position satisfying the property *falsifies* the property.
- for every position that does not satisfy the property *there is* a move that *truthifies* the property.

Conjecture: there is a function $\oplus$ such that

$$W.(\ell + r) = W.\ell \oplus W.r \quad .$$



Sum game 1 Sum game 2

m matches   n matches

Move in left or right game: remove any +ve no. of matches.

Strategy: truthify $m = n$.

Key:

l position in left game

r position in right game

v "value" of position

Mex number

The mex (minimal excludant) number of a position is defined to be:

the minimum natural number that is not included in the mex numbers of the position's successors.

Winning strategy:

Truthify $\quad mex.l = mex.r \quad$.