Functions out of Higher Truncations

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Topic

In Homotopy Type Theory

(i.e. dependent type theory with Σ, Π, $=$, univalence, HITs),
how can we

Construct functions $\|\mathbf{A}\|_n \to \mathbf{B}$

if B is $(n+1)$-truncated?

Topic

In Homotopy Type Theory

✓ ✓ *(clear, I hope)* *(does not matter)*

(i.e. dependent type theory with Σ, Π, $=$, univalence, HITs),

how can we

Construct functions $\|A\|_n \to B$

? **?** *(will explain)*

if B is $(n+1)$-truncated?

Reminder: Identity/Equality Types in Martin-Löf's Dependent Type Theory

- If **A** is a type and $\mathbf{x}, \mathbf{y} : \mathbf{A}$, then $\mathbf{x} = \mathbf{y}$ is also a type (a.k.a. $x =_A y$ or $\mathsf{Id}_A(x, y)$)
- Does **UIP** hold? I.e. if $p, q : x = y$, do we automatically get $p = q$?
- Hofmann-Streicher 1994: No! *[LICS Test of Time Award 2014]*
- Types can have non-trivial higher structure (fist step of birth of Homotopy Type Theory)

Introduction: Truncation Levels and Truncations

- ⋆ "being n-truncated" is a property of types [due to VV]
- ⋆ intuition: "trivial on levels $\geq n$"
- ⋆ Def:
  1. A is (-2)-truncated iff $A \simeq \mathbf{Unit}$
  2. A is $(n+1)$-truncated iff $x = y$ is n-truncated $(\forall x, y : A)$
- ⋆ basic lemma: A is n-truncated $\Rightarrow$ A is $(n+1)$-truncated
- ⋆ examples:

  (-2)-truncated a.k.a. *contractible*: **Unit**

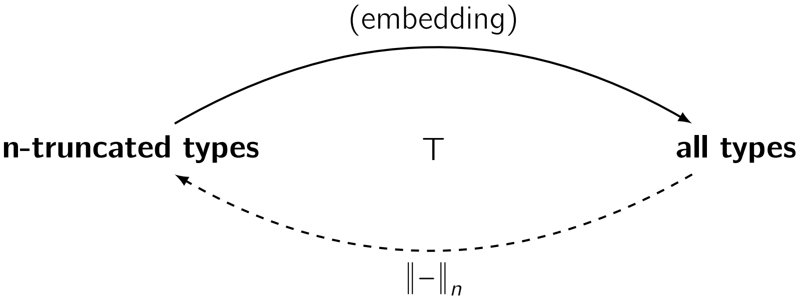
  (-1)-truncated a.k.a. *propositional*: $\varnothing$

  0-truncated a.k.a. *set*, *satisfying UIP*: $\mathbb{N}$, **Bool**, . . .

  1-truncated: universe of sets

  . . .

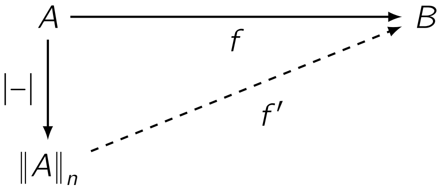
Introduction: Truncation Levels and Truncations



- $(\|\mathbf{A}\|_{\mathrm{n}} \to \mathbf{B}) \;\simeq\; (\mathbf{A} \to \mathbf{B})$ if B is n-truncated
- intuition: $\|-\|_n$ "truncates" a type (thus "loses information"!), have $|-| : A \to \|A\|_n$

Back to the topic of this talk

So, how to get a map $\|A\|_n \to B$ in general?
Or: When does $f : A \to B$ factor through $\|A\|_n$?



- ⋆ always, if B is n-truncated
- ⋆ this paper:

Theorem

If B is $(n+1)$-truncated:

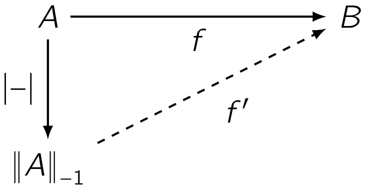
$f : A \to B$ factors through $\|A\|_n$
iff
f induces trivial maps on all $(n+1)$-st loop spaces.

Special cases of the result

Special case $n = -1$

If B is 0-truncated, i.e. has unique identity proofs:

$$f : A \to B \text{ factors through } \|A\|_{-1}$$
iff
$$f \text{ is weakly constant: } \Pi_{x,y:A}\ f(x) = f(y).$$



f must be weakly constant.

This was already known

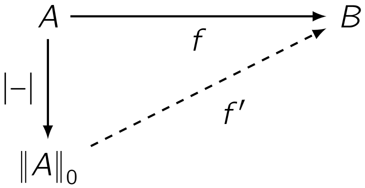
[K-Escardó-Coquand-Altenkirch 2014].

Special cases of the result

Special case $n = 0$

If B is 1-truncated:

$f : A \to B$ factors through $\|A\|_0$
iff
$\mathsf{ap}_f : (x = y) \to (f(x) = f(y))$ is weakly constant.



ap_f must be weakly constant.

Known from the *Rezk completion*
[Ahrens-Kapulkin-Shulman 2014].

Special cases of the result

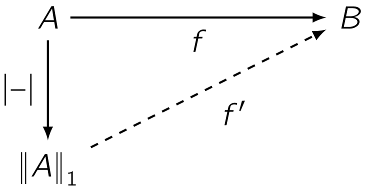
Special case $n = 1$

If B is 2-truncated:

$f : A \to B$ factors through $\|A\|_1$

iff

f introduces trivial maps on all second homotopy groups / loop spaces: $\mathsf{ap}_f^2 : \Omega^2(A, a) \to \Omega^2(B, f(a))$ weakly constant.



ap_f^2 must be weakly constant.

This (and all other cases) are new.

Two proofs

Two proofs of our result:

- ⋆ first proof:
  - given $f' : \|A\|_n \to B$, we do get $f : A \to B$ which is trivial on all $(n+1)$-st loop spaces
  - if A is *n-connected*, this map is an equivalence
  - piece together maps on the "n-connected components" of A
- ⋆ second proof:
  - construct a higher inductive type $H^{A,n}$
  - show that $H^{A,n}$ has a suitable elimination principle
  - show $H^{A,n} \simeq \|A\|_n$

Overview: Open and Solved Cases

n-tr. B / $\|A\|_?$	-1	0	1	2	3	4	$\cdots$	∞
-1		✓	✓	✓	✓	✓	$\cdots$	✓
0			✓					
1	trivial – standard universal property applicable			✓		unsolved cases		
2					✓			
3						✓		
$\cdots$								

Solved in: [K-Escardó-Coquand-Altenkirch 2014], [Ahrens-Kapulkin-Shulman 2014], [K 2015], **[HERE]**