Homotopy Type Theory and Hedberg's Theorem

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Overview

This talk:

Introduction to Homotopy Type Theory

Generalizations of Hedberg's Theorem, based on joint work with T. Altenkirch, T. Coquand, M. Escardo

Reminder: Type Theory

Intensional Type Theory

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...for proof assistants and (dependently typed) programming
...as used for Coq and Agda

e.g.

$$\lambda f \rightarrow \lambda a \rightarrow f\, a\, a : (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$$

Reminder: Equality

Definitional Equality

Decidable equality for typechecking & computation; e. g.

$$(\lambda a.b)x =_\beta b[x/a]$$

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Propositional Equality

Equality needing a proof, i. e. a term of the identity type, e. g.

$$\forall\, m\, n\, .\, (m + n) \equiv (n + m)$$

Reminder: Identity Types

Propositional equality

. . . is just an inductive type

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Formation

$$\frac{a, b \; : \; A}{a \equiv b \; : \; type}$$

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$$\frac{a \;:\; A}{refl_a \;:\; a \equiv a}$$

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Elimination (J)

$$\frac{P \;:\; (a, b : A) \rightarrow a \equiv b \rightarrow Set \qquad m : \forall a \,.\, P(a, a, refl_a)}{J_{(a,b,q)} \;:\; P(a, b, q)}$$

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Computation (β)

$$J_{(a,a,refl_a)} =_\beta m a$$

Uniqueness of Identity Proofs (UIP)

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Simple,
Good computational properties,
More powerful Pattern Matching

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Advantages

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Disadvantages

Intuitively wrong,
impossible to express statements about equality,
isomorphic sets can not (really) be treated as equal

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Voevodsky (and Awodey, independently, and others):

Without UIP: new model of Type Theory

(types as weak ω-groupoids)

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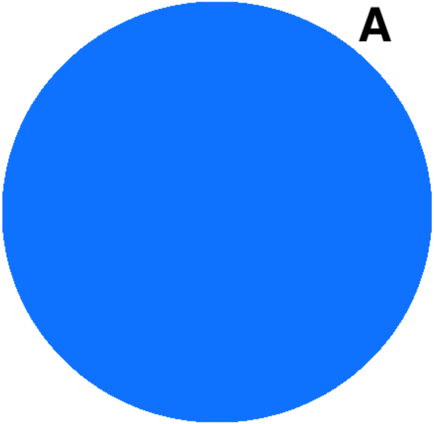
- R is a *Quillen equivalence* of model categories
- $\Rightarrow$ (more or less) a model that uses topological spaces as types

Homotopic Model

Topological Space

Set with structure

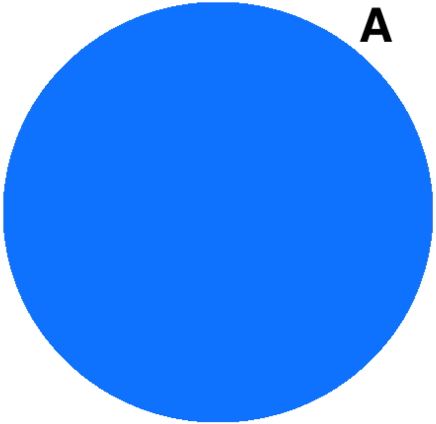


Homotopic Model

Topological Space

Set with structure

$A : type$



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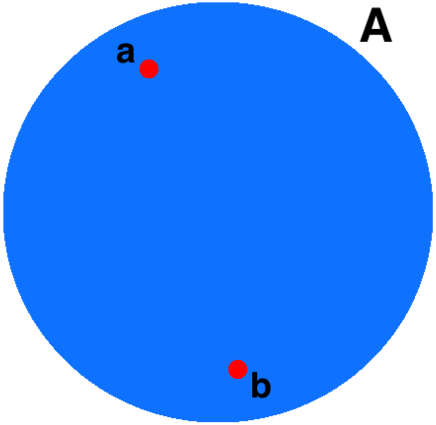

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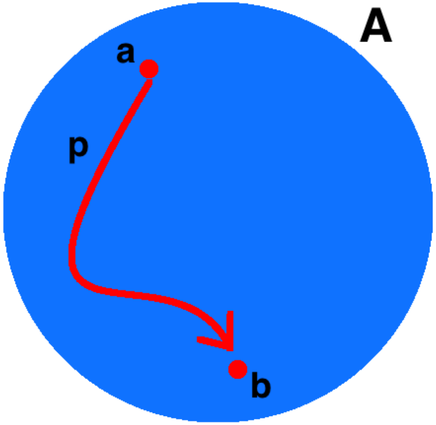

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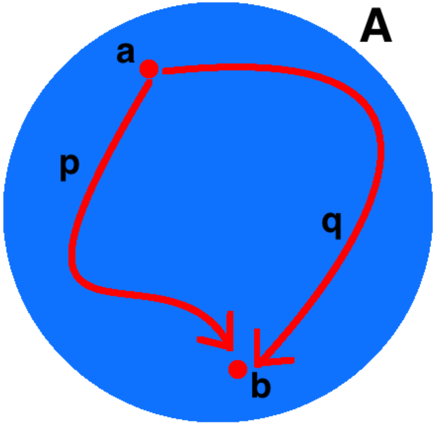

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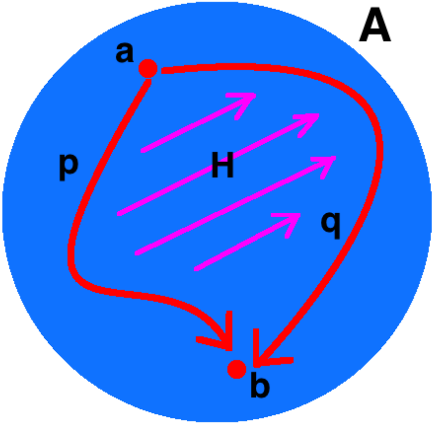

UIP in the Homotopic Model

Example: want to prove

$\forall p \,.\, p \equiv refl_a$

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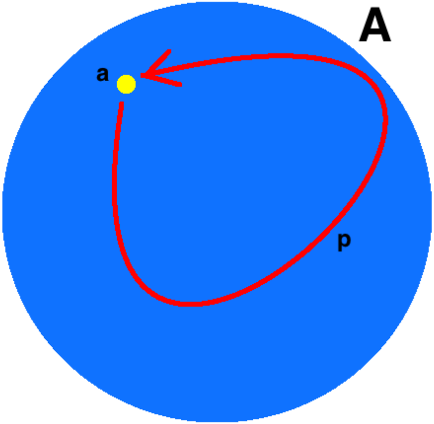

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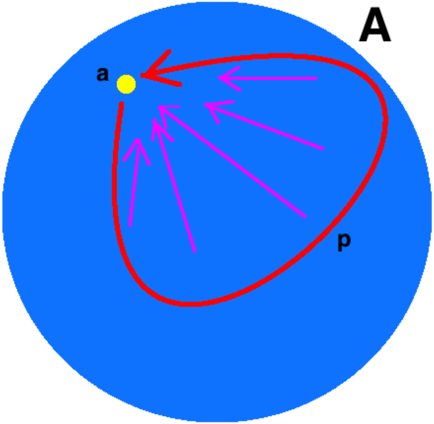

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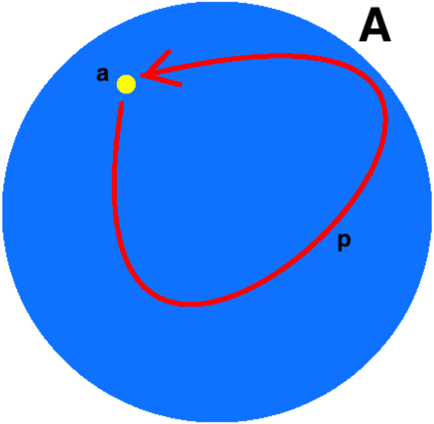

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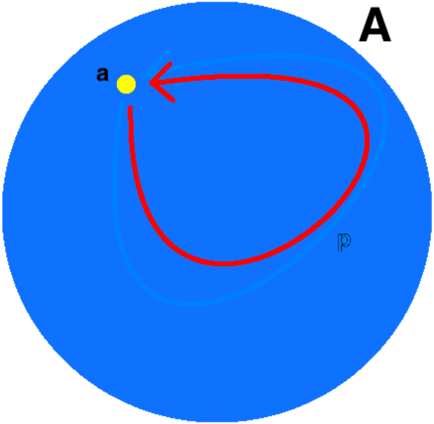

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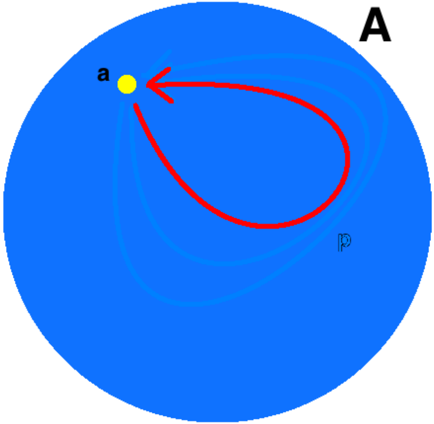

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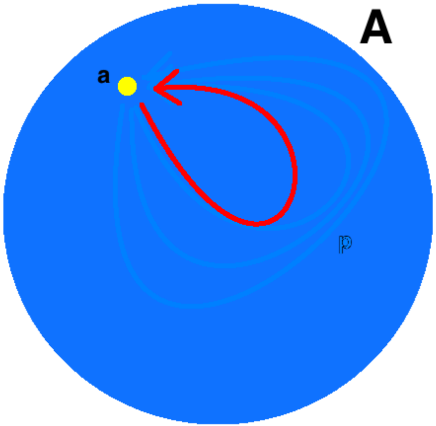

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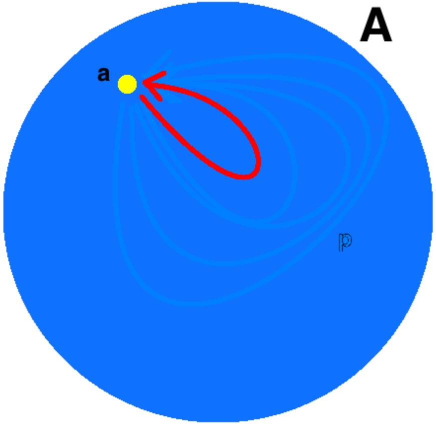

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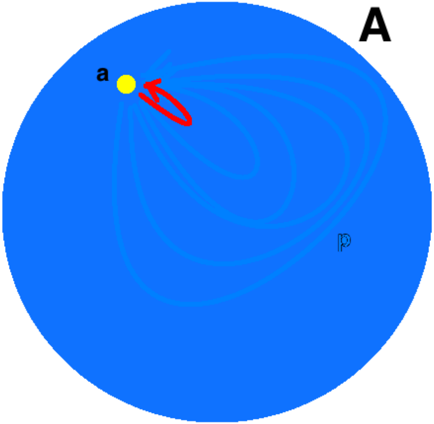

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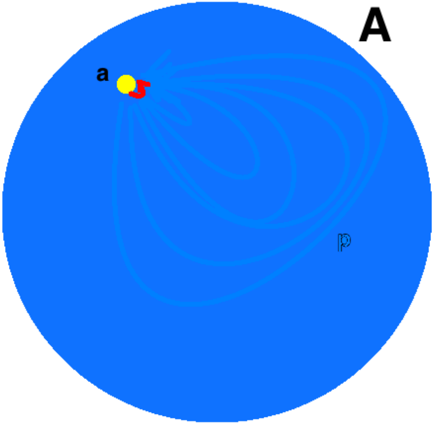
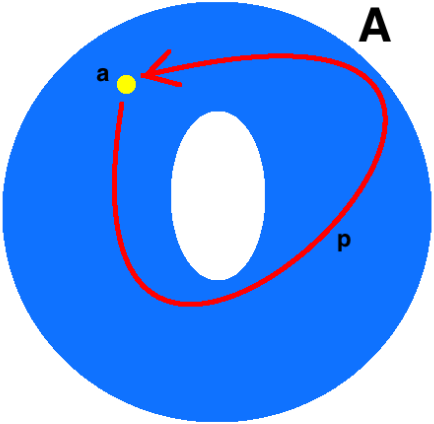


UIP in the Homotopic Model

Okay, but what now?



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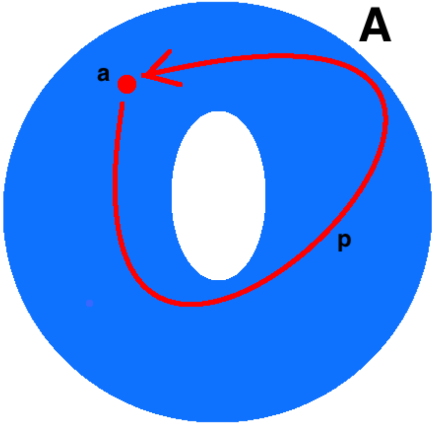

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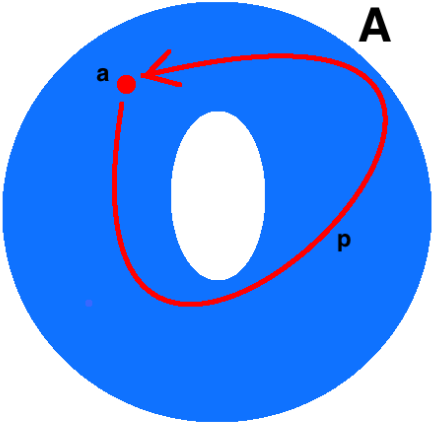

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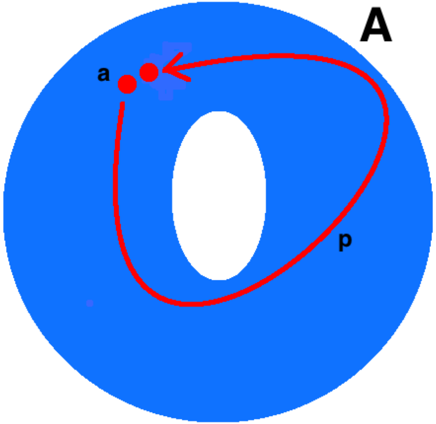

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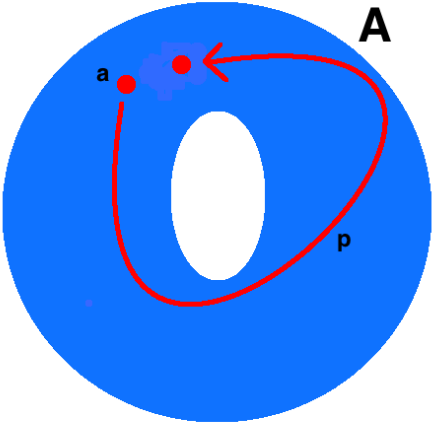

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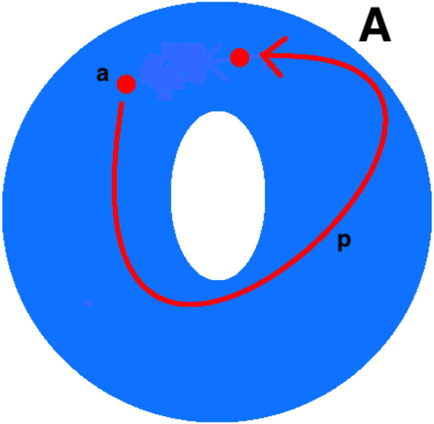

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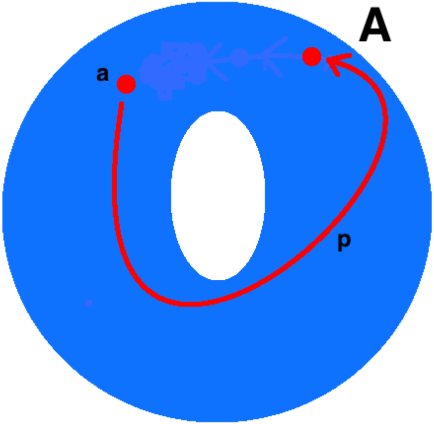

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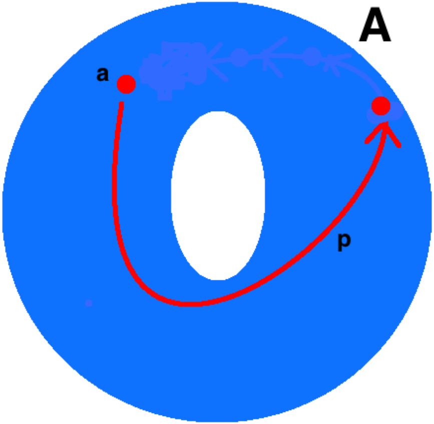

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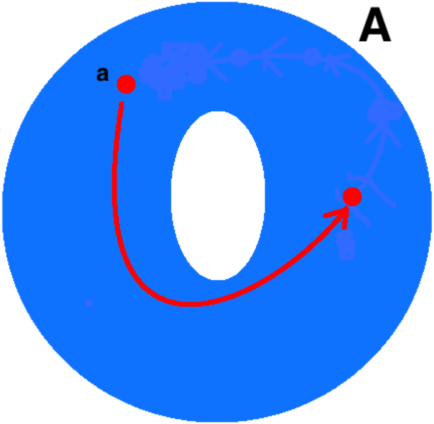

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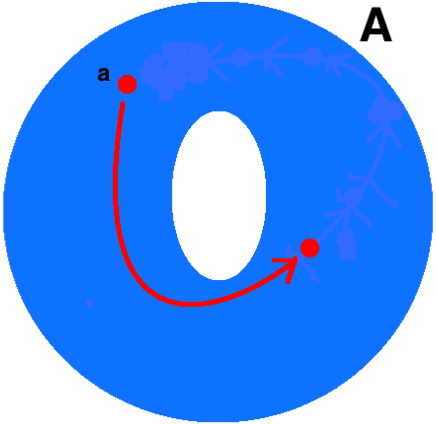

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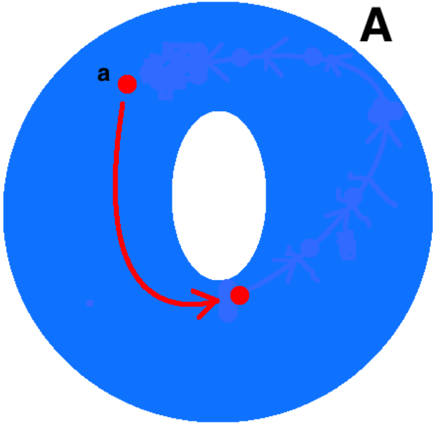

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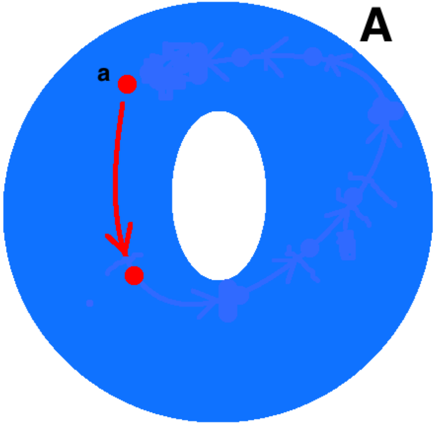

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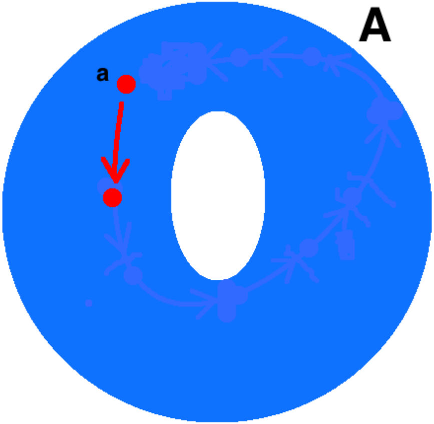

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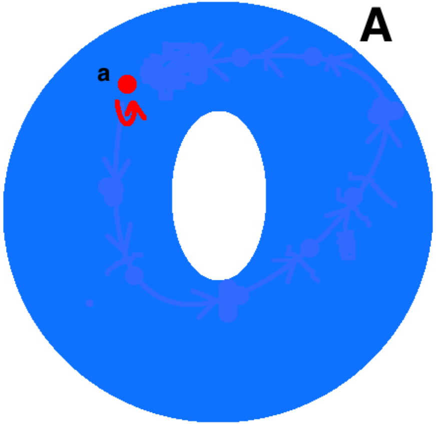

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Hedberg's theorem

Decidable Equality

$$\mathit{DecidableEquality}_A \;:=\; \forall\, a\, b\,.\; (a \equiv b \;+\; \neg\, a \equiv b)$$

Hedberg's theorem

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$$DecidableEquality_A \;:=\; \forall\, a\, b\,.\; (a \equiv b \;+\; \neg\, a \equiv b)$$

Hedberg's theorem

$$DecidableEquality_A \;\longrightarrow\; UIP_A$$

Hedberg's theorem

Constant Function

$$const(f) \;:=\; \forall\, a\, b\,.\, f\, a \equiv f\, b$$

Constant Endofunction on Path Spaces

$$g \;:\; \forall\, a\, b\,.\, a \equiv b \;\rightarrow\; a \equiv b$$

$$path\text{-}const(g) \;:=\; \forall\, a\, b\,.\, const\, g_{ab}$$

Hedberg's theorem

Lemma 1

$$\mathit{DecidableEquality} \longrightarrow \Sigma_g \forall\, a\, b\, .\, \mathit{const}\, g_{ab}$$

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$$\mathit{DecidableEquality} \longrightarrow \Sigma_g \, \forall \, a \, b \, . \, \mathit{const} \, g_{ab}$$

Proof.

- Given *dec* : $\forall \, a \, b \, . \, (a \equiv b \, + \, \neg \, a \equiv b)$.

Hedberg's theorem

Lemma 1

$DecidableEquality \longrightarrow \Sigma_g \forall a\, b\, .\, const\, g_{ab}$

Proof.

- Given $dec : \forall a\, b\, .\, (a \equiv b + \neg a \equiv b)$.
- Given a, b, we want: $g_{ab} : a \equiv b \rightarrow a \equiv b$.

Hedberg's theorem

Lemma 1

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Proof.

- Given $\mathit{dec} : \forall\, a\, b\, .\, (a \equiv b + \neg\, a \equiv b)$.
- Given a, b, we want: $g_{ab} : a \equiv b \rightarrow a \equiv b$.
- If $\mathit{dec}\, a\, b = \mathit{inr}\ \_$, then nothing to do

Hedberg's theorem

Lemma 1

$$\mathit{DecidableEquality} \longrightarrow \Sigma_g \forall\, a\, b\,.\, \mathit{const}\, g_{ab}$$

Proof.

- Given $dec : \forall\, a\, b\,.\, (a \equiv b + \neg\, a \equiv b)$.
- Given a, b, we want: $g_{ab} : a \equiv b \to a \equiv b$.
- If $dec\, a\, b = inr\ \_$, then nothing to do
- If $dec\, a\, b = inl\, p$, then $g_{ab}(\_) = p$ □

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Lemma 2

$$\Sigma_g \, \forall\, a\, b \,.\, const\, g_{ab} \longrightarrow UIP$$

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- Given $g : \forall\, a\, b\, .\, a \equiv b \rightarrow a \equiv b$ which is constant
- Given any $a, b : A$ and $p, q : a \equiv b$.

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- Given $g : \forall\, a\, b\,.\, a \equiv b \rightarrow a \equiv b$ which is constant
- Given any $a, b : A$ and $p, q : a \equiv b$.
- Claim: $p \equiv (g_{aa} refl_a)^{-1} \circ g_{ab}(p)$

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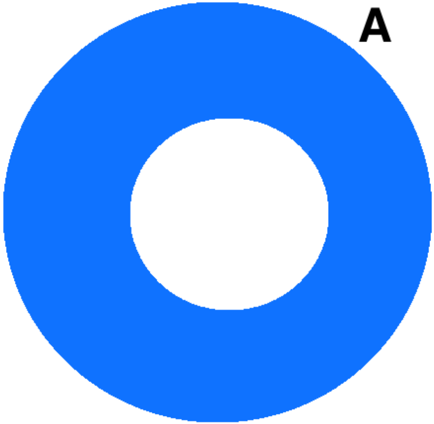
- Same for q. But g_{aa} and g_{ab} are constant. □

Corollary: The Circle type does not have decidable equality

$$dec : (a, b : A) \rightarrow (a \equiv b \; + \; \neg\, a \equiv b)$$



Generalizations of Hedberg's theorem

We have seen

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DecidableEquality is a very strong property. How about something weaker?

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DecidableEquality is a very strong property. How about something weaker? For example:

Separated

$$\forall\, a\, b\,.\, \neg\neg(a \equiv b) \rightarrow a \equiv b$$

“general”

$$\forall\, a\, b\,.\, [\mathit{propositional\ evidence\ for}\ a \equiv b] \rightarrow a \equiv b$$

Propositions

So, what is "propositional evidence"?

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Write **Prop** for this "subset" of **Type**

H-Propositional Reflection

A some type. We want to say that A is inhabited without giving away a specific inhabitant. (Awodey/Bauer)

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is defined to be the left adjoint of *emb:* $\textbf{Prop} \hookrightarrow \textbf{Type}$

This means:

- A^* is in **Prop**
- $\eta \; : \; A \rightarrow A^*$
- if P is a proposition and $A \rightarrow P$, then $A^* \rightarrow P$

Generalizations of Hedberg's Theorem

"Propositional evidence for $a \equiv b$" is now just [an inhabitant of] $(a \equiv b)^*$.

H-Separated

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Theorem

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- $h\text{-}separated_A \longrightarrow \Sigma_g \forall\, a\, b\, .\, const\, g_{ab}$

  nearly the same as Lemma 1 :

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- $\Sigma_g \forall\, a\, b\, .\, const\, g_{ab} \longrightarrow UIP_A$
  Lemma 2
- $UIP_A \longrightarrow h\text{-}separated_A$
  $a \equiv b$ is automatically propositional,
  $\Rightarrow$ use universal property of $*$

Generalizations of Hedberg's Theorem

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The third is not so interesting.

Theorem

The first and the second are equivalent, for any X.

(This is not trivial.)

Many further questions. . .

One can ask:

- What does a constant function $X \to Y$ give us?
- What does this have to do with quotients?
- What does $\forall X . X^* \to X$ imply?
- . . .

THANK YOU!