Equality in the Dependently Typed Lambda Calculus: An Introduction to Homotopy Type Theory

or: Connecting Topology and Logic with Category Theory

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Typed λ Calculus

$$\textit{Natural Deduction} \quad \overset{\textit{Curry-Howard}}{\cong} \quad \textit{Type Theory}$$

$$\frac{A \to B \quad A}{B} \qquad\qquad \frac{\Gamma \vdash f : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash f\ u : B}$$

$$\frac{B}{A \to B} \qquad\qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}$$

Dependently Typed λ Calculus

Types may depend on terms:

Vec A n

are Lists over A with length n.

Dependently Typed λ Calculus

Natural Deduction	$\overset{\text{Curry-Howard}}{\cong}$	*Type Theory*	*special case*
$\exists_{x \in A} B$		$\Sigma(x{:}A).B$	$A \times B$
$\forall_{x \in A} B$		$\Pi(x{:}A).B$	$A \to B$

Usage, e.g. Agda & Epigram:
proof assistants, formal verification, proof-carrying code

Problems...

- Typechecking requires Computation.
- Equality is no longer decidable in general.
- We want decidable typechecking.

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...and Answers

Two kinds of Equality!

Definitional Equality

“Real” decidable equality such as $(\lambda a.b)x =_\beta b[x/a]$

Propositional Equality

Equality needing a proof

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Propositional Equality

$$\frac{\Gamma \vdash \ x, y \ : \ A}{\Gamma \vdash \ Id_A \, x \, y \ : \ type} \ Form$$

$$\frac{\Gamma \vdash \ x : \ A}{\Gamma \vdash \ refl_x : Id_A \, x \, x} \ Intro$$

Propositional Equality

$$\frac{\begin{array}{l} \Gamma \vdash A:\ type \\ \Gamma, x,\ y\ :\ A,\ p\ :\ Id_A\, x\, y \vdash M(x,y,p):\ type \\ \Gamma, r\ :\ A \vdash m\ :\ M(r, r, refl_r) \\ \Gamma \vdash a,\ b\ :\ A \\ \Gamma \vdash q\ :\ Id_A\, a\, b \end{array}}{\Gamma \vdash J\ M\ m\ a\ b\ q : M(a,b,q)}\ Elim\ (J)$$

$$\frac{\ldots}{J\ M\ m\ a\ a\ refl\ a = m\ a}\ Comp$$

Subst from J

- $P : A \to Set$ and $a, b : A$.
- $q : Id_A\ a\ b$
- $p : P\ a$
- Can we get something of type $P\ b$?

I.e. is $(P : A \to Set) \to (a, b : A) \to Id_A\ a\ b \to P\ a \to P\ b$ inhabited?

Sure! Using J with

$$M = \lambda\, x\, y\, p\, .\, Px \to Py$$

$$m = \lambda\, x . x$$

Call it *subst*.

Subst from J

- $P : A \to Set$ and $a, b : A$.
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Uniqueness of Identity Proofs

How many inhabitants can $Id_A\, a\, b$ have in general?

For some time, it was assumed that there is at most one (UIP), i.e. given $p, q : Id_A\, a\, b$, the type $Id\, p\, q$ is inhabited.

Hofmann-Streicher groupoid model: not derivable from J.

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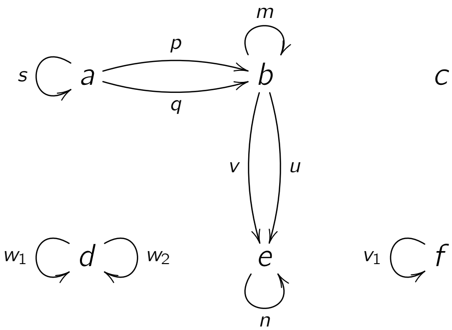
Uniqueness of Identity Proofs - Refuted



$a, b, d, e : A \qquad c, f : B$

$s : Id_A\, a\, a \qquad p, q : Id_A\, a\, b \qquad u, v : Id_A\, b\, e \qquad \ldots$

UIP is weird anyway

$$BOOL = \{true, false\}$$

isomorphisms:

$$id : BOOL \to BOOL$$
$$\neg : BOOL \to BOOL$$

So, identity equals negation?!

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Extensionality

Given:

- $f : A \to B$
- $g : A \to B$
- $p : \Pi(x : A).Id_B\,(f x)\,(g x)$

Can we construct something of type $Id_{A \to B}\, f\, g$ (Leibniz)? No!

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Extensionality

Idea: Adding extensionality as additional axiom.

But then, assume p is a (nontrivial) equality proof using this axiom.

Consequence:

$$\mathit{subst}\ (\lambda h \to \mathbb{N})\ p\ 0$$

Non-canonical natural numbers!

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Vladimir Voevodsky

Voevodsky's suggestion

Do not use *UIP*

...because it is weird and has undesirable consequences!

Do not use the Extensionality Axiom!

... because of the same reason!

Use Univalence instead!

... because it is better - as we will see in a moment!

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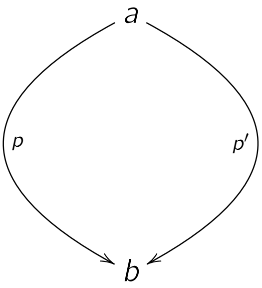
Lumsdaine's and v.d.Berg's result



Weak ω groupoid

for example:

- $a := b := x$
- $p := p' := \mathrm{refl}_x$
- $H := H' := \mathrm{refl}_{\mathrm{refl}_x}$
- $\mathrm{refl}_{\mathrm{refl}_{\mathrm{refl}_x}}$
- ...

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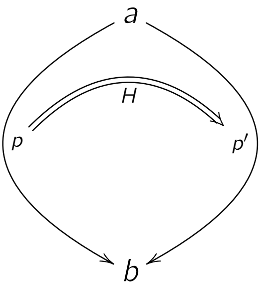

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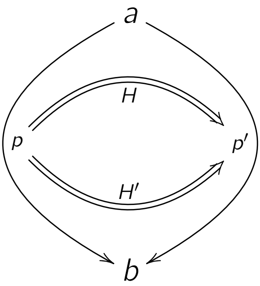

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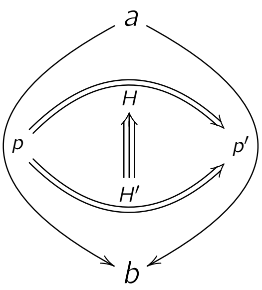

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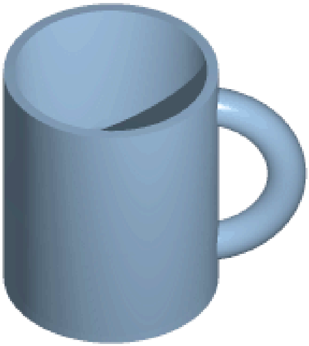
- ...

A very well-known structure. . .

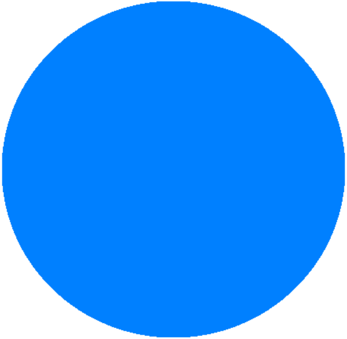
. . . in Topology!



(source: Wikipedia)

A disc

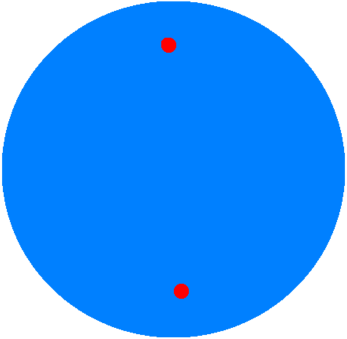
a (nondependent!) type - we call it X



a topological space - we call it X

A disc

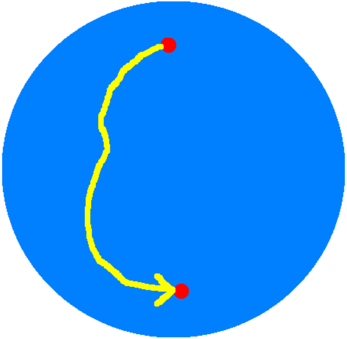
two terms



two points

A disc

?

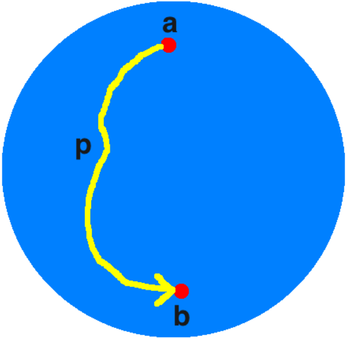


a path

A disc

$a, b : X$

$p : Id\, a\, b$



$a, b \in X$

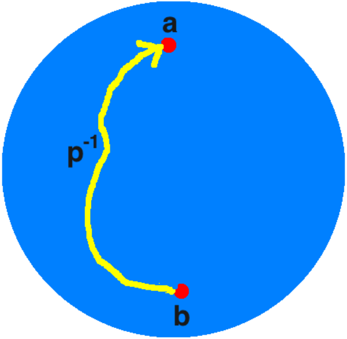
$p : [0,1] \to X$

$p(0) = a$

$p(1) = b$

A disc

$p^{-1} \ : \ Id\, b\, a$



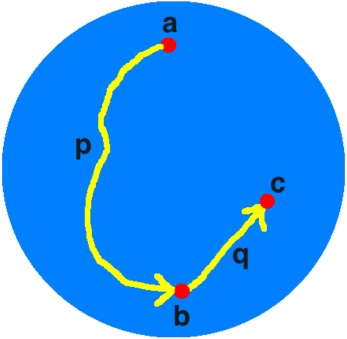
$p^{-1} \ : \ [0,1] \to X$

$p^{-1}(t) = p(1-t)$

A disc

$p\ :\ a \equiv b$

$q\ :\ Id\, b\, c$



$a, b\ \in\ X$

$p\ :\ [0,1] \to X$

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$q\ :\ [0,1] \to X$

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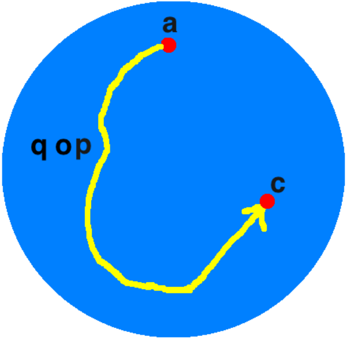
A disc

$q \circ p \ : \ Id\, a\, c$



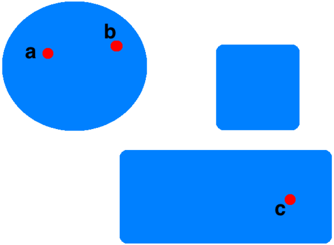
$q \circ p \ :$

$[0,1] \to X$

$x \mapsto$
$$\begin{cases} p(2x), x < 0.5 \\ q(2x-1), \text{else} \end{cases}$$

Another set

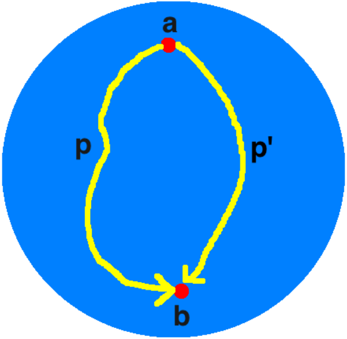
$Id\,a\,c$ not inhabited



not path-connected

A disc

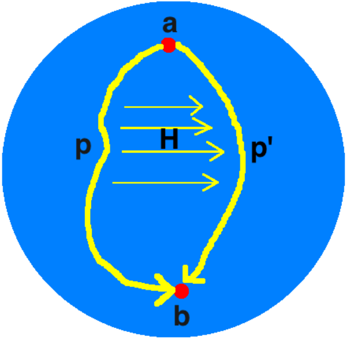
$p, p' \ : \ Id\, a\, b$



$p, p' \ : \ [0,1] \rightarrow X$

A disc

$H : Id\, p\, p'$



$$H : [0,1]^2 \to X$$
$$H(0,\cdot) = p$$
$$H(1,\cdot) = p'$$
$$H(t,0) = a$$
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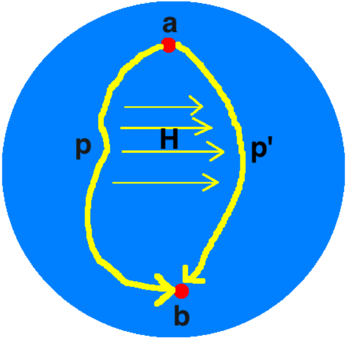
A disc

$H : Id\, p\, p'$



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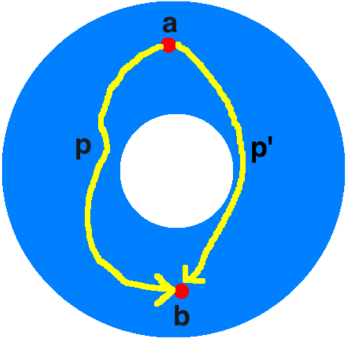
$$p : [0,1]^1 \to X$$
$$a : [0,1]^0 \to X$$

A ring

$H : Id\, p\, p'$



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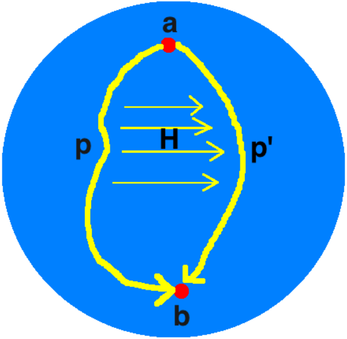

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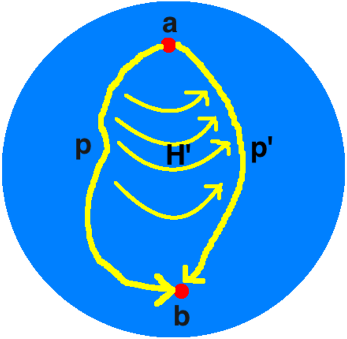
$H' : Id\, p\, p'$



$$H' : [0,1]^2 \to X$$
$$H'(0,\cdot) = p$$
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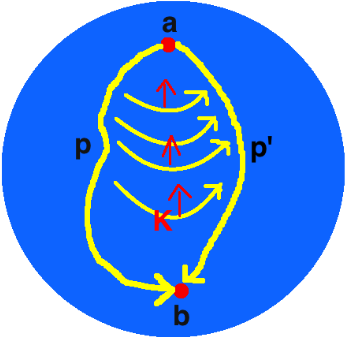
A disc

$K : Id\, H'\, H$

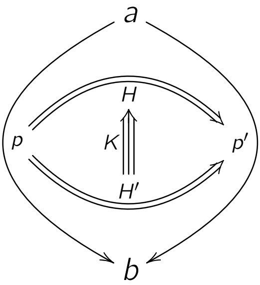


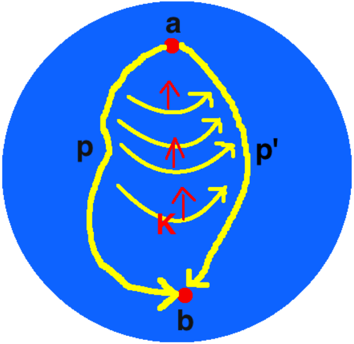
$K : [0,1]^3 \to X$

$K(0, \cdot, \cdot) = H'$

...

Putting it together





Voevodsky again

Univalence Axiom

The (canonical) mapping from equalities to weak equivalences is a weak equivalence.

- No need for *UIP*
- Extensionality
- Only canonical members of $\mathbb{N}$
- a "completely natural axiom" so that everything works as in homotopical intuition

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Summary

Hopes:

Homotopic Models:

- new results and intuition in both type and homotopy theory
- better understanding of the connection between logic and topology

Univalence:

- avoiding a couple of problems in a natural way

  UIP

  Extensionality

  Canonicity of natural numbers
- better foundation than Set Theory for (constructive) mathematics
- at the same time, natively supported by proof assistants

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(Other) People I want to mention

- Thorsten Altenkirch
- Peter Arndt
- Steve Awodey
- Thierry Coquand
- Nicola Gambino
- Richard Garner
- Chris Kapulkin
- Dan Licata
- Mike Shulman
- Thomas Streicher
- Michael Warren
- . . . and many more

Even more people I want to **Thank**

You.