A Lambda Term Representation Inspired by Ordered Linear Logic

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Outline

Motivation

$$(\lambda x.\lambda y.\, a\, b\, y)\, g\, f$$

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\longrightarrow & a\, [g/x, f/y] \quad b\, [g/x, f/y] \quad f \\
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Observed Problem

- space leak:

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- space leak:

$$\begin{aligned} & (a\,b)\,[g/x, f/y] \\ \longrightarrow\quad & a\,[g/x, f/y]\;\; b\,[g/x, f/y] \end{aligned}$$

- similar for other representation (like De Bruijn indices)
- possible solution: make all relevant information available at the λ

Our suggestion: Ordered Terms

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t, u & ::= & x & \text{free variable (named } x\text{)} \\
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 & | & t \,^{m}\, u & \text{application (} m \in \mathbb{N}\text{)}
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t, u & ::= & & x & \text{free variable (named } x\text{)} \\
& & | & \bullet & \text{bound variable (nameless)} \\
& & | & t \ {}^{m}\ u & \text{application } (m \in \mathbb{N}) \\
& & | & \lambda^{\vec{k}}.\, t & \text{abstraction } (\vec{k} \in \mathbb{N}^*)
\end{array}
$$

$$\vec{k} = [k_1, k_2, \ldots, k_n]$$

Abstraction and Application

Meaning of $\lambda^{[k_1,\dots,k_n]}.\,t$: Bind the unbound $\bullet$ at positions

$$k_1 + 1$$
$$k_1 + k_2 + 2$$
$$k_1 + k_2 + k_3 + 3$$
$$\dots$$

Meaning of $t\,^m\,u$: number of unbound $\bullet$ in t equals m

Example: The S combinator

$$\lambda x.\, \lambda y.\, \lambda z.\, x\, z\, (y\, z)$$

$$\lambda^{[0]}.\, \lambda^{[1]}.\, \lambda^{[1,1]}.\, \bullet^{1} \bullet^{2} \left(\bullet^{1} \bullet\right)$$

Example: Evaluation

$$\left(\lambda^{[0]}.\,\lambda^{[1]}.\,\lambda^{[1,1]}.\,\bullet\,^{1}\,\bullet\,^{2}\left(\bullet\,^{1}\,\bullet\right)\right)\,^{0}\,g\,^{0}\,f\,^{0}\,n$$

Example: Evaluation

$$
\begin{array}{ll}
 & \left(\lambda^{[0]}.\,\lambda^{[1]}.\,\lambda^{[1,1]}.\,\bullet\,^{1}\bullet\,^{2}\left(\bullet\,^{1}\bullet\right)\right)\,^{0}g\,^{0}f\,^{0}n \\
\longrightarrow & \left(\lambda^{[1]}.\,\lambda^{[1,1]}.\,\bullet\,^{1}\bullet\,^{2}\left(\bullet\,^{1}\bullet\right)\right)[g]\quad f\ n
\end{array}
$$

Example: Evaluation

$$
\begin{array}{rl}
 & \big(\lambda^{[0]}.\,\lambda^{[1]}.\,\lambda^{[1,1]}.\,\bullet\,{}^1\,\bullet\,{}^2\,(\bullet\,{}^1\,\bullet)\big)\,{}^0\,g\,{}^0\,f\,{}^0\,n \\
\longrightarrow & \big(\lambda^{[1]}.\,\lambda^{[1,1]}.\,\bullet\,{}^1\,\bullet\,{}^2\,(\bullet\,{}^1\,\bullet)\big)[g]\;\; f\; n \\
\longrightarrow & \big(\lambda^{[1,1]}.\,\bullet\,{}^1\,\bullet\,{}^2\,(\bullet\,{}^1\,\bullet)\big)[g, f]\;\; n
\end{array}
$$

Example: Evaluation

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\begin{array}{rr}
 & \left(\lambda^{[0]}.\,\lambda^{[1]}.\,\lambda^{[1,1]}.\,\bullet^{1}\bullet^{2}\left(\bullet^{1}\bullet\right)\right)\,^{0}g\,^{0}f\,^{0}n \\
\longrightarrow & \left(\lambda^{[1]}.\,\lambda^{[1,1]}.\,\bullet^{1}\bullet^{2}\left(\bullet^{1}\bullet\right)\right)[g]\quad f\;n \\
\longrightarrow & \left(\lambda^{[1,1]}.\,\bullet^{1}\bullet^{2}\left(\bullet^{1}\bullet\right)\right)[g,f]\quad n \\
\longrightarrow & \left(\bullet^{1}\bullet^{2}\left(\bullet^{1}\bullet\right)\right)[g,n,f,n]
\end{array}
$$

Example: Evaluation

$$
\begin{aligned}
& \left(\lambda^{[0]}.\,\lambda^{[1]}.\,\lambda^{[1,1]}.\,\bullet^{\,1}\bullet^{\,2}\left(\bullet^{\,1}\bullet\right)\right)^{\,0} g^{\,0} f^{\,0} n \\
\longrightarrow\ & \left(\lambda^{[1]}.\,\lambda^{[1,1]}.\,\bullet^{\,1}\bullet^{\,2}\left(\bullet^{\,1}\bullet\right)\right)[g]\quad f\ n \\
\longrightarrow\ & \left(\lambda^{[1,1]}.\,\bullet^{\,1}\bullet^{\,2}\left(\bullet^{\,1}\bullet\right)\right)[g, f]\quad n \\
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$$
\begin{array}{rl}
& \big(\lambda^{[0]}.\,\lambda^{[1]}.\,\lambda^{[1,1]}.\,\bullet\,{}^{1}\,\bullet\,{}^{2}\,(\bullet\,{}^{1}\,\bullet)\big)\,{}^{0}\,g\,{}^{0}\,f\,{}^{0}\,n \\
\longrightarrow & \big(\lambda^{[1]}.\,\lambda^{[1,1]}.\,\bullet\,{}^{1}\,\bullet\,{}^{2}\,(\bullet\,{}^{1}\,\bullet)\big)[g]\;\; f\; n \\
\longrightarrow & \big(\lambda^{[1,1]}.\,\bullet\,{}^{1}\,\bullet\,{}^{2}\,(\bullet\,{}^{1}\,\bullet)\big)[g,f]\;\; n \\
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\longrightarrow & (\bullet\,{}^{1}\,\bullet)\,[g,n]\;\;(\bullet\,{}^{1}\,\bullet)[f,n] \\
\longrightarrow & \bullet[g]\;\;\bullet[n]\;\;(\bullet[f]\;\;\bullet[n])
\end{array}
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Example: Evaluation

$$
\begin{array}{rc}
 & \big(\lambda^{[0]}.\,\lambda^{[1]}.\,\lambda^{[1,1]}.\,\bullet^{\,1}\bullet^{\,2}\big(\bullet^{\,1}\bullet\big)\big)\;{}^{0}g\;{}^{0}f\;{}^{0}n \\
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\longrightarrow & \big(\bullet^{\,1}\bullet\big)\,[g,n]\quad\big(\bullet^{\,1}\bullet\big)[f,n] \\
\longrightarrow & \bullet[g]\;\;\bullet[n]\;\;(\bullet[f]\;\;\bullet[n]) \\
\longrightarrow & g\;n\,(f\;n)
\end{array}
$$

Invariant:

length of substitution list = number of unbound $\bullet$

Insertion Operation

$$[g, f] \leadsto [g, n, f, n]$$

Operation:

$$\vec{v}^{\,w/\vec{k}} = [v_1, v_2, \ldots, v_n]^{\,w/[k_1,\ldots,k_m]}$$

$$:= [v_1, \ldots, v_{k_1}, w, v_{k_1+1}, \ldots, v_{k_1+k_2}, w, \ldots]$$

Values

$$
\begin{array}{rcrl}
v, w & ::= & x\,\vec{v} & \text{large application} \\
 & | & (\lambda^{\vec{k}}.\,t)^{\vec{v}} & \text{closure}
\end{array}
$$

Evaluation

$$[\![x]\!]_{[]} \quad = \quad x$$

Evaluation

$$\llbracket x \rrbracket_{[]} = x$$

$$\llbracket \bullet \rrbracket_{[v_1]} = v_1$$

Evaluation

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\begin{aligned}
[\![x]\!]_{[]} &= x \\
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[\![\lambda^{\vec{k}}.\, t]\!]_{\vec{v}} &= (\lambda^{\vec{k}}.\, t)^{\vec{v}}
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[\![t \,{}^k\, u]\!]_{[v_1,\ldots,v_n]} &= [\![t]\!]_{[v_1,\ldots,v_k]} \mathbin{@} [\![u]\!]_{[v_{k+1},\ldots,v_n]} \\
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[\![\lambda^{\vec{k}}.\, t]\!]_{\vec{v}} &= (\lambda^{\vec{k}}.\, t)^{\vec{v}} \\
\\
(x\, \vec{v}) \mathbin{@} w &= x\, [\vec{v}, w] \\
(\lambda^{\vec{k}}.\, t)^{\vec{v}} \mathbin{@} w &= [\![t]\!]_{\vec{v}\, w / \vec{k}}
\end{aligned}
$$

Correspondence Relation

Γ : Set of variable names

$\cdot \lhd \Gamma \rhd \cdot[\cdot] \ \subset$ Terms $\times$ orderedTerms $\times$ SubstitutionLists

$M \lhd \Gamma \rhd u[\vec{x}]$: M corresponds to $u[\vec{x}]$ (in context Γ)

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$$\frac{x \in \Gamma}{x \lhd \Gamma \rhd \bullet[x]}$$

$$\frac{x \notin \Gamma}{x \lhd \Gamma \rhd x[]}$$

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$$\frac{M \lhd \Gamma \rhd t[\vec{x}] \qquad N \lhd \Gamma \rhd u[\vec{y}] \qquad \mathrm{length}(\vec{x}) = m}{M\,N \lhd \Gamma \rhd (t\ {}^{m}\ u)[\vec{x}, \vec{y}]}$$

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$$\frac{M \lhd (\Gamma \cup \{z\}) \rhd t[\vec{x}^{\,z/\vec{k}}] \qquad z \notin \vec{x}}{\lambda z.\, M \lhd \Gamma \rhd (\lambda^{\vec{k}}.\, t)[\vec{x}]}$$

$\Rightarrow$ computable bijection

Experiments

- implementation of the described representation in Haskell

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- implementation of the described representation in Haskell
- substitution lists: Lists and Trees tested
- two standard evaluation algorithms for comparison:
  - Simple closures: contexts as maps
  - Beta normal values
- experiments: large terms of the Edinburgh Logical Framework typechecked
  - proof terms kindly provided by Andrew W. Appel, Princeton University
  - 8 GB, AMD Phenom II X4 B95 (3 GHz, one core used)

Comparison

6000 lines (3.8 MB)

	time (sec)	space (MB)
Ordered (trees)	18.9	1111
Ordered (lists)	18.6	1114
Simple Closures	18.5	1152
Beta Normal Values	27.6	2034

Comparison

10000 lines (12.9 MB)

	time (sec)	space (MB)
Ordered (trees)	61.0	3230
Ordered (lists)	60.6	3237
Simple Closures	60.0	3302
Beta Normal Values	98.7	5878

Comparison

12000 lines (17.8 MB)

	time (sec)	space (MB)
Ordered (trees)	84.3	5096
Ordered (lists)	83.8	5103
Simple Closures	83.6	5226
Beta Normal Values	137.7	8513

Comparison

whole file - ca. 12500 lines (20.9 MB)

	time (sec)	space (MB)
Ordered (trees)	108.4	8877
Ordered (lists)	94.8	4948
Simple Closures	94.3	5068
Beta Normal Values	169.8	9044
(Twelf r1697)	(ca. 22)	(2720)

Related Work

1. Fernández, Mackie, Sinot (2005): *Lambda-Calculus with Director Strings*
2. Sinot (2005): *Director Strings Revisited: A Generic Approach to the Efficient Representation of Free Variables in Higher-order Rewriting*
3. Gopalan Nadathur (2001): *A Fine-Grained Notation for Lambda Terms and Its Use in Intensional Operations*
4. Liang, Nadathur, Qi (2005): *Choices in representation and reduction strategies for lambda terms in intensional contexts*