Some Families of Categories with Propositional Hom-Sets

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poset	• small • $\lvert\mathbb{C}(A,B)\rvert \leq 1$	A *partially ordered set*, or *poset*, is a small category $\mathbb{C}$ such that any Hom-Set $\mathbb{C}(A,B)$ has at most cardinality one. It is *pointed* if it has a terminal object $\perp$.
ω-**cpo**	• poset $\mathbb{C}$ • $\omega^{op} \to \mathbb{C}$ has limit	An ω*-complete partial order*, or ω*-cpo*, or just *cpo* (ambiguous!), is a poset $\mathbb{C}$ such that any diagram $D : \omega^{op} \to \mathbb{C}$ has a limit, where ω is the category $0 \to 1 \to 2 \to \ldots$. Such a diagram is also called ω*-chain*.
dcpo	• poset $\mathbb{C}$ • if $\mathbb{C}'$ has cone for $1+1 \to \mathbb{C}'$, then $\mathbb{C}' \to \mathbb{C}$ has limit	A *directed-complete partial order*, or *dcpo*, or sometimes *cpo* (ambiguous!), is a poset $\mathbb{C}$ with the following property: If $\mathbb{C}'$ has cones for all diagrams $1+1 \to \mathbb{C}'$, then any the diagram $\mathbb{C}' \to \mathbb{C}$ has a limit. Here, $\mathbb{C}'$ is (technically) any category, but it is also sufficient to consider small discrete categories (note that morphisms in $\mathbb{C}'$ do not make any difference). In particular, every dcpo is an ω-cpo. Further, an ω-cpo is a poset with the stated property for all countable $\mathbb{C}'$.
Semilattice	• poset • binary products	A *semilattice* is a poset with all binary products ("meet-semilattice") or all binary coproducts ("join-semilattice"). It is a *bounded semilattice* if it has a terminal object (if it is a meet-semilattice) resp. an initial object (join-semilattice).
Lattice	• poset • binary products and coproducts	A *lattice* is a poset which has all binary products ("meets") and coproducts ("joins"). It is a *bounded lattice* if it also has both an initial and a terminal object. A lattice is *complete*, if the underlying category is complete and cocomplete (all small limits and colimits exist). Note that a poset with all small limits already has all small colimits and vice versa. Consequently, a "complete semilattice" is already a complete lattice. A *complete lattice homomorphism* is a functor that is both continuous and cocontinuous.
Complete lattice	• small complete category	This is somewhat surprising, but not difficult to prove
Heyting algebra	• bounded lattice • exponentials	A *Heyting algebra* is a bounded lattice with exponentials. Is is *complete* iff the lattice (i. e. the category) is complete.
Boolean algebra	• Heyting algebra • $A + 0^A \cong 1$	A *Boolean algebra* is a Heyting algebra where, for each object A, the coproduct of A and the exponential 0^A is 1 (where 0, 1 are the initial/terminal object).