Constructions with Non-Recursive Higher Inductive Types

LiCS'16, New York

Nicolai Kraus

University of Nottingham, EU

6 July 2016

Introduction

Setting: Martin-Löf style type theory with Σ, Π, identity types ($=$), univalent universe(s), higher inductive types ("homotopy type theory")

What are these?

ordinary inductive type: nat

$\mathbb{N}$ is a type with constructors

$$\mathsf{zero} : \mathbb{N}$$
$$\mathsf{suc} : \mathbb{N} \to \mathbb{N}$$

higher inductive type: circle

$\mathbb{S}^1$ is a type with constructors

$$\mathsf{base} : \mathbb{S}^1$$
$$\mathsf{loop} : \mathsf{base} =_{\mathbb{S}^1} \mathsf{base}$$

Think of CW complexes. . .

Introduction, II

Should we **really** think of CW complexes?

Propositional Truncation $\|A\|$

$$|-| : A \to \|A\|$$

$$h : (x, y : \|A\|) \to x =_{\|A\|} y$$

Pseudo-truncation $\langle\!\langle A \rangle\!\rangle$

$$\langle - \rangle : A \to \langle\!\langle A \rangle\!\rangle$$

$$t : (x, y : A) \to \langle x \rangle =_{\langle A \rangle} \langle y \rangle$$

universal property $\|A\|$

$$\frac{\|A\| \to B}{A \to B}$$

if B is propositional

universal property $\langle\!\langle A \rangle\!\rangle$

$$\frac{\langle\!\langle A \rangle\!\rangle \to B}{\Sigma\,(f : A \to B)\,,\ \mathrm{wconst}(f)}$$

for any B

note: $\mathrm{wconst}(f) :\equiv \Pi_{x,y:A}\ f\,a = f\,b$

"non-recursive"

Topic: Do we need recursive higher constructors?

Idea for constructing the propositional truncation as non-recursive HIT:

- start with a type A
- apply $\langle\!\langle - \rangle\!\rangle$ $\Rightarrow$ type is ("conditionally") 0-connected
- apply $\langle\!\langle - \rangle\!\rangle_0$ $\Rightarrow$ 1-connected
- apply $\langle\!\langle - \rangle\!\rangle_1$ $\Rightarrow$ 2-connected
- in every step: "connectedness-level" increased

Finally: Take the homotopy colimit of

$$A \to \langle\!\langle A \rangle\!\rangle \to \langle\!\langle \langle\!\langle A \rangle\!\rangle \rangle\!\rangle_0 \to \langle\!\langle \langle\!\langle \langle\!\langle A \rangle\!\rangle \rangle\!\rangle_0 \rangle\!\rangle_1 \to \left\langle\!\!\left\langle \langle\!\langle \langle\!\langle \langle\!\langle A \rangle\!\rangle \rangle\!\rangle_0 \rangle\!\rangle_1 \right\rangle\!\!\right\rangle_2 \to \ldots\ldots$$

All used HITs are non-recursive!

How NOT to prove this

Hard part: the colimit is propositional. Idea:

n-th homotopy group is trivial from step $(n+2)$ onwards

$\Rightarrow$ For the colimit: all homotopy groups are trivial

~~$\Rightarrow$ The colimit must be proositional.~~

Wrong because: Whitehead's theorem does not hold

But: some nice consequences, e.g. generalizes *Functions out of higher truncations* [Capriotti, K, Vezzosi, CSL'15]

How to actually prove it

Lemma 1

Given a chain $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \ldots$. If every f_i is weakly constant, then the homotopy colimit A_ω is propositional.

This explains/generalizes [van Doorn, CPP'16]

Lemma 2

Every function in the sequence

$$A \to \langle\!\langle A \rangle\!\rangle \to \langle\!\langle \langle\!\langle A \rangle\!\rangle \rangle\!\rangle_0 \to \langle\!\langle \langle\!\langle \langle\!\langle A \rangle\!\rangle \rangle\!\rangle_0 \rangle\!\rangle_1 \to \left\langle\!\!\left\langle \langle\!\langle \langle\!\langle \langle\!\langle A \rangle\!\rangle \rangle\!\rangle_0 \rangle\!\rangle_1 \right\rangle\!\!\right\rangle_2 \to \ldots$$

is weakly constant.

Summary

- ⋆ Have operator $\|-\|$ as non-recursive HIT (with side results)
- ⋆ Higher truncation by simply omitting the first steps (?)
- ⋆ There are other constructions (van Doorn, Rijke)

- ⋆ **Obvious question: Which classes of higher inductive types can be constructed non-recursively?**

  Conjecture: "all" apart from inductive-inductive ones.

Many thanks!